CANONICALIZATION BY ABSOLUTENESS

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Abstract. This note will give the argument of Neeman and Norwood to show $\mathbf{\Sigma}_1^1$ ($\mathbf{\Pi}_1^1$ or $\mathbf{\Delta}_1^1$) canonicalization of equivalence relation in $L(\mathbb{R})$ with all $\mathbf{\Sigma}_1^1$ ($\mathbf{\Pi}_1^1$ or $\mathbf{\Delta}_1^1$, respectively) classes using the Neeman-Norwood $L(\mathbb{R})$ embedding theorem for proper forcing. Also it will be shown that the Neeman-Norwood $L(\mathbb{R})$ embedding theorem for proper forcing holds in the $\text{Coll}(\omega, <\kappa)$ forcing extension of L, when κ is a remarkable cardinal. Hence, $\mathbf{\Sigma}_1^1$ ($\mathbf{\Pi}_1^1$ or $\mathbf{\Delta}_1^1$) canonicalization of $L(\mathbb{R})$ equivalence relation with all $\mathbf{\Sigma}_1^1$ ($\mathbf{\Pi}_1^1$ or $\mathbf{\Delta}_1^1$) classes is consistent relative to the consistency of a remarkable cardinal.

1. Introduction

The question of interest has it origin in various forms due to [3], Neeman, [2], and [1]. The most general form of the question considered here will be:

Question 1.1. Let I be a σ-ideal on ${}^\omega\omega$ with $\mathbb{P}_I$ proper. Let R be a binary relation on ${}^\omega\omega$ in $L(\mathbb{R})$ so that for all $x \in {}^\omega\omega$, R_x is $\mathbf{\Sigma}_1^1$ ($\mathbf{\Pi}_1^1$ or $\mathbf{\Delta}_1^1$). Let B be an I^+ $\mathbf{\Delta}_1^1$ set. Is there an I^+ $\mathbf{\Delta}_1^1$ set $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}_1^1$ ($\mathbf{\Pi}_1^1$ or $\mathbf{\Delta}_1^1$, respectively)?

If I is a σ-ideal on ${}^\omega\omega$ then $\mathbb{P}_I = (\mathbf{\Delta}_1^1 \backslash I, \subseteq, {}^\omega\omega)$ consisting on I^+ $\mathbf{\Delta}_1^1$ sets ordered by $\subseteq$ with largest element ${}^\omega\omega$.

If R is a binary relation, then $R_x = \{y : (x, y) \in R\}$. Of course, one could also ask this question with $R^y = \{x : (x, y) \in R\}$ replacing R_x.

R is in $L(\mathbb{R})$ means that $R(x, y)$ if and only if $L(\mathbb{R}) \models \varphi(x, y, \bar{r}, \bar{\alpha})$ where φ is some formula, $\bar{r}$ is a tuple in ${}^\omega\omega$, and $\bar{\alpha}$ is a tuple of ordinals.

This question was answered positively in [1] assuming there exists a measurable cardinal above infinitely many Woodin cardinals. There the main tool is the existence of homogeneous tree representations of various sets. This note will give an argument of Neeman and Norwood using absoluteness to give a positive answer to the question under the same large cardinal assumptions as in [5]. This argument can be applied using a remarkable cardinal to significantly reduce the consistency strength of $\mathbf{\Sigma}_1^1$ ($\mathbf{\Pi}_1^1$ or $\mathbf{\Delta}_1^1$) canonicalization for equivalence relation in $L(\mathbb{R})$ with all $\mathbf{\Sigma}_1^1$ ($\mathbf{\Pi}_1^1$ or $\mathbf{\Delta}_1^1$, respectively) classes. In [4], Neeman and Norwood used variations of the proper forcing embedding theorem to answer much more general forms of the above question under AD^+.

2. Canonicalization

Definition 2.1. The Neeman-Norwood $L(\mathbb{R})$-embedding property for proper forcing is the following statement for all i^*, Ξ, $\mathbb{P}$, and H: Let $i^* : M \to V_\Xi$ be an elementary embedding where $\Xi \in \text{CN}$ and M is a countable transitive set. Let $\mathbb{P} \in V_\Xi$ be a proper forcing so that $\mathbb{P} \in \text{rang}(i^*)$. Let $\mathbb{Q} \in M$ be such that $i^*(\mathbb{Q}) = \mathbb{P}$. Let $H \in V_\Xi$ so that $H \subseteq \mathbb{Q}$ and H is $\mathbb{Q}$-generic over M. Let $i : L(\mathbb{R})^M \to L(\mathbb{R})^{V_\Xi}$ be the elementary embedding induced by i^*. Then there exists an elementary embedding $j : L(\mathbb{R})^M \to L(\mathbb{R})^{M[H]}$ which does not not move reals or ordinals and an elementary embedding $h : L(\mathbb{R})^{M[H]} \to L(\mathbb{R})^{V_\Xi}$ which does not move reals and moves an ordinal α to $i^*(\alpha)$ so that $i = h \circ j$.

Theorem 2.2. *(Neeman-Norwood) Assume that the Neeman-Norwood $L(\mathbb{R})$-embedding property for proper forcing holds. Let I be a σ-ideal on ${}^\omega\omega$ with $\mathbb{P}_I$ proper. Let R be a binary relation on ${}^\omega\omega$ in $L(\mathbb{R})$ with the property that for all $x \in {}^\omega\omega$, R_x is $\mathbf{\Sigma}_1^1$ ($\mathbf{\Pi}_1^1$ or $\mathbf{\Delta}_1^1$). Let $B \subseteq {}^\omega\omega$ be a I^+ $\mathbf{\Delta}_1^1$ set. Then there exists a $C \subseteq B$ which is I^+ and $\mathbf{\Delta}_1^1$ so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}_1^1$ ($\mathbf{\Pi}_1^1$ or $\mathbf{\Delta}_1^1$, respectively).*

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Proof. Suppose R is defined by $R(x,y) \Leftrightarrow L(\mathbb{R}) \models \varphi(x,y,\bar{r},\bar{\alpha})$ for some formula φ, tuple of reals $\bar{r}$, and tuple of ordinals $\bar{\alpha}$. Assume that all R-classes are $\mathbf{\Sigma}^1_1$ (the other cases are similar).

Choose Ξ large enough so that V_Ξ contains $\bar{\alpha}$. Let $N \prec V_\Xi$ be a countable elementary substructure containing $\bar{r}$, $\bar{\alpha}$, $\mathbb{P}_I$, and B. Let $\pi : N \to M$ be the Mostowski collapse. Let $i^* : M \to V_\Xi$ be the $\text{in}_{N,V_\Xi} \circ \pi^{-1}$, where in_{N,V_Ξ} is the inclusion map of N into V_Ξ. Let S be the equivalence relation defined by $S(x,y)$ if and only if $L(\mathbb{R}) \models \varphi(x,y,\bar{r},\pi(\bar{\alpha}))$.

Let $i : L(\mathbb{R})^M \to L(\mathbb{R})^{V_\Xi}$ be the elementary map induced by i^*. Let $g \subseteq \mathbb{P}_I$ be an arbitrary $\mathbb{P}_I$-generic over M. Applying the Neeman-Norwood $L(\mathbb{R})$-embedding property for proper forcing, one has maps $j : L(\mathbb{R})^M \to L(\mathbb{R})^{M[g]}$ and maps $h : L(\mathbb{R})^{M[g]} \to L(\mathbb{R})^{V_\Xi}$ as in Definition 2.1.

Using i and the fact that $L(\mathbb{R})^{V_\Xi} \models$ "All R-sections are $\mathbf{\Sigma}^1_1$", one has that $L(\mathbb{R})^M \models$ "All S-sections are $\mathbf{\Sigma}^1_1$". Using j, one has $L(\mathbb{R})^{M[g]} \models$ "All S-sections are $\mathbf{\Sigma}^1_1$".

So in particular, $L(\mathbb{R})^{M[g]} \models S_g$ is $\mathbf{\Sigma}^1_1$. Since g has an arbitrary generic,

$$M \models 1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} L(\mathbb{R}) \models S_{\dot{x}_{\text{gen}}} \text{ is } \mathbf{\Sigma}^1_1$$

Using fullness, find some $\tau \in M^{\mathbb{P}_I}$ so that

$$M \models 1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} \dot{T} \text{ is a tree on } \check{\omega} \times \check{\omega} \wedge S_{\dot{x}_{\text{gen}}} = p[\dot{T}]$$

Now the claim is that if g is $\mathbb{P}_I$-generic over M and $y \in ({}^\omega\omega)^V$, then

$$R(g,y) \Leftrightarrow y \in p[\dot{T}[g]]$$

This is because for any such g,

$$L(\mathbb{R})^{M[g]} \models S_g = p[\dot{T}[g]]$$

$\dot{T}[g]$ is a tree on $\omega \times \omega$ so it is essentially a real. Hence $h(\dot{T}[g]) = \dot{T}[g]$. So using h, one

$$L(\mathbb{R})^{V_\Xi} \models R_g = p[\dot{T}[g]]$$

So in V, $\dot{T}[y]$ still gives the $\mathbf{\Sigma}^1_1$ definition of R_g. This proves the claim.

Since $\mathbb{P}_I$ is proper, the set $C = \{x \in B : x \text{ is } \mathbb{P}_I\text{-generic over } M\}$ is I^+ $\mathbf{\Delta}^1_1$. The map taking $g \in C$ to $\dot{T}[g]$ is $\mathbf{\Delta}^1_1$. Therefore the statement $y \in p[\dot{T}[g]]$ is $\mathbf{\Sigma}^1_1$. Hence $R \cap (C \cap {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$. This completes the proof. □

3. Consistency Strength of Neeman-Norwood Embedding Property

This section will show that the Neeman-Norwood $L(\mathbb{R})$-embedding property for proper forcing is equiconsistent with a remarkable cardinal. Since the Neeman-Norwood $L(\mathbb{R})$-embedding theorem implies the Neeman-Zapletal $L(\mathbb{R})$-embedding theorem and [6] Corollary 3.6 shows that the Neeman-Zapletal $L(\mathbb{R})$-embedding theorem is equiconsistent with a remarkable cardinal, it suffices to show that the consistency of a remarkable cardinal gives the consistency of the Neeman-Norwood $L(\mathbb{R})$-embedding theorem.

The definition of a remarkable cardinal in given in [6]. The definition will not be needed. Only the following consequences proved in [6] will be used.

Fact 3.1. *Let κ be a remarkable cardinal in L. Let $G \subseteq \text{Coll}(\omega, <\kappa)$ be $\text{Coll}(\omega, <\kappa)$-generic over L. Let $\mathbb{P} \in L[G]$ be a proper forcing in $L[G]$. Let $H \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over $L[G]$. If $x \in ({}^\omega 2)^{L[G][H]}$, then there is a forcing $\mathbb{Q}_x \in L_\kappa$ and a $J_x \subseteq \mathbb{Q}_x$ which is $\mathbb{Q}_x$-generic over L and $J_x \in L[G][H]$ so that $x \in L[J_x]$.*

Proof. See [6], Lemma 2.1. □

Since $\mathbb{Q}_x \in L_\kappa$ and κ is inaccessible in L, there exists some cardinal $\alpha_x < \kappa$ so that $\mathbb{Q}_x \in L_{\alpha_x}$. κ is still inaccessible in $L[J_x]$. $|\mathscr{P}(\text{Coll}(\omega, <\alpha_x))^{L[J_x]}|^{L[J_x]} < \kappa$. Hence $\mathscr{P}(\text{Coll}(\omega, <\alpha_x))^{L[J_x]}$ is countable in $L[G][H]$. So in $L[G][H]$, there exists some $G'_{\alpha_x} \subseteq \text{Coll}(\omega, <\alpha_x)$ in $L[G][H]$ which is $\text{Coll}(\omega, <\alpha_x)$-generic over $L[J_x]$. Let $G_{\alpha_x} \subseteq \text{Coll}(\omega, <\alpha_x)$ be $\text{Coll}(\omega, <\alpha_x)$ be generic over L so that $L[G_{\alpha_x}] = L[J_x][G'_{\alpha_x}]$.

This shows that in Fact 3.1, one find some $\alpha_x < \kappa$ and some $G_{\alpha_x} \subseteq \text{Coll}(\omega, <\alpha_x)$ generic over L with $G_{\alpha_x} \in L[G][H]$ so that $x \in L[G_{\alpha_x}]$.

Fact 3.2. *Let κ be remarkable in L. Let $G \subseteq \text{Coll}(\omega, <\kappa)$ be $\text{Coll}(\omega, <\kappa)$-generic over L. Let $\mathbb{P} \in L[G]$ be a proper forcing in $L[G]$. Let $H \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over $L[G]$. Let $E \subseteq \text{Coll}(\omega, (2^{\aleph_0})^{L[G][H]})$ be $\text{Coll}(\omega, (2^{\aleph_0})^{L[G][H]})$-generic over $L[G][H]$. There is a $G' \subseteq \text{Coll}(\omega, <\kappa)$ with $G' \in L[G][H][E]$ which is $\text{Coll}(\omega, <\kappa)$-generic over L so that $\mathbb{R}^{L[G']} = \mathbb{R}^{L[G][H]}$.*

Proof. See [6], Lemma 2.2. A proof of this will be sketched:

In $L[G][H][E]$, let $(r_n : n \in \omega)$ be an enumeration of $\mathbb{R}^{L[G][H]}$. Construct sequences $(\alpha_i : i \in \omega)$ and $(G_i : i \in \omega)$ so that $i < j$ implies $\alpha_i < \alpha_j < \kappa$ and $G_i \subseteq G_j$, for each $i \in \omega$, $G_i \subseteq \text{Coll}(\omega, < \alpha_i)$ is $\text{Coll}(\omega, < \alpha_i)$-generic over L and in $L[G][H]$, and $r_i \in L[G_i]$. This can be done using the remark above following Fact 3.1.

Note that $\lim_{n\in\omega} \alpha_n = \kappa$ since there is a real in $L[G][H]$ coding each $\alpha < \kappa$. Let $G' = \bigcup_{i\in\omega} G_i$. $\text{Coll}(\omega, < \kappa)$ has the κ-chain condition. Suppose $A \subseteq \text{Coll}(\omega, < \kappa)$ is a maximal antichain in L, then A is a maximal antichain of $\text{Coll}(\omega, < \alpha)$ for some $\alpha < \kappa$. Pick some i so that $\alpha_i > \alpha$. Then $G_i \cap A \neq \emptyset$. This shows that G' is generic for $\text{Coll}(\omega, < \kappa)$ over L. Since the sequence $(G_i : i \in \omega)$ was constructed in $L[G][H][E]$, $G' \in L[G][H][E]$. By construction, $\mathbb{R}^{L[G][H]} \subseteq \mathbb{R}^{L[G']}$. Since $\mathbb{R}^{L[G']} = \bigcup_{\beta<\kappa} \mathbb{R}^{L[G'\restriction\beta]} = \bigcup_{i\in\omega} \mathbb{R}^{L[G_i]}$ and each $G_i \in L[G][H]$, one has $\mathbb{R}^{L[G']} \subseteq \mathbb{R}^{L[G][H]}$. This completes the proof. $\square$

Theorem 3.3. *Let κ be remarkable in L. Let $G \subseteq \text{Coll}(\omega, < \kappa)$ be $\text{Coll}(\omega, < \kappa)$-generic over L. In $L[G]$, the Neeman-Norwood $L(\mathbb{R})$-embedding property for proper forcings holds.*

Proof. Let $G \subseteq \text{Coll}(\omega, < \kappa)$ be generic over L. Let $i : M \to (L[G])_\Xi$ be elementary where M is a countable transitive set. For notational simplicity, Ξ will be dropped in the following. Let α be the ordinal height of M, i.e. $\alpha = M \cap \text{ON}$. Then M has the form $L_\alpha[g]$ where $g \subseteq \text{Coll}(\omega, < \omega_1^M)$ is generic over L_α.

As the map j and h do not move reals, the map is completely determined by how the ordinals are moved. Thus one just need to show that j and h are elementary. The elementarity of the map j is just the Neeman-Zapletal theorem. See [6] Theorem 2.4 for the proof. It is similar to the argument below for showing h is elementary.

Let $\mathbb{P}$ be a proper forcing in $L_\alpha[g]$. Let $H \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over $L_\alpha[g]$. The following will show that h is elementary.

Let φ be a formula. For simplicity, suppose there is only a single real $x \in L_\alpha[g][H]$ and single ordinal $\beta < \alpha$.

Now suppose that

$$L(\mathbb{R})^{L_\alpha[g][H]} \models \varphi(x, \beta)$$

Fix some $E \subseteq \text{Coll}(\omega, (2^{\aleph_0})^{L_\alpha[g][H]})$ be generic over $L_\alpha[g][H]$. Fix some $(r_i : i \in \omega)$ be an enumeration of $\mathbb{R}^{L_\alpha[g][H]}$ in $L_\alpha[g][H][E]$. Let $(\alpha_i : i \in \omega)$, $(r_i : i \in \omega)$, $(g_i : i \in \omega)$, and g' be the objects produced in the proof of Fact 3.2. There is some $i < \omega$ so that $x \in L_\alpha[g_i]$. Now since $\mathbb{R}^{L_\alpha[g][H]} = \mathbb{R}^{L_\alpha[g']}$, one has the above line is equivalent to

$$L(\mathbb{R})^{L_\alpha[g']} \models \varphi(x, \beta)$$

Find some $g'' \subseteq \text{Coll}(\omega, < \omega_1^M)$ be generic over $L[g_i]$ so that $L[g'] = L[g_i][g'']$. So then one has

$$L(\mathbb{R})^{L_\alpha[g_i][g'']} \models \varphi(x, \beta)$$

By homogeneity of $\text{Coll}(\omega, < \omega_1^M)$, this is equivalent to

$$L_\alpha[g_i] \models 1_{\text{Coll}(\omega,<\omega_1^M)} \Vdash_{\text{Coll}(\omega,<\omega_1^M)} L(\mathbb{R}) \models \varphi(\check{x}, \check{\beta})$$

Let $\tau \in L_\alpha^{\text{Coll}(\omega,<\alpha_i)}$ be a name for x. The above is equivalent to

$$\text{There exists } p \in g_i \text{ so that } L_\alpha \models p \Vdash_{\text{Coll}(\omega,<\alpha_i)} 1_{\text{Coll}(\omega,<\check{\omega}_1^M)} \Vdash_{\text{Coll}(\omega,<\check{\omega}_1^M)} L(\mathbb{R}) \models \varphi(\check{\tau}, \check{\beta})$$

Now α_i is countable in $L_\alpha[g]$. Therefore, i does not move $\text{Coll}(\omega, < \alpha_i)$ or any of its conditions. One can also a name for a real τ so that $i(\tau) = \tau$. So by the applying the elementary embedding i, one has the above is equivalent to

$$\text{There exists } p \in g_i \text{ so that } L \models p \Vdash_{\text{Coll}(\omega,<\alpha_i)} 1_{\text{Coll}(\omega,<\check{\omega}_1^{L[G]})} \Vdash_{\text{Coll}(\omega,<\check{\omega}_1^{L[G]})} L(\mathbb{R}) \models \varphi(\check{\tau}, i(\check{\beta}))$$

Recall g_i is $\text{Coll}(\omega, < \alpha_i)$-generic over L_α. Since κ is inaccessible and $\kappa < \alpha$, $\mathscr{P}^{L_\alpha}(\text{Coll}(\omega, < \alpha_i)) = \mathscr{P}^L(\text{Coll}(\omega, < \alpha_i))$. Hence g_i is generic over L. So this implies the above is equivalent to

$$L[g_i] \models 1_{\text{Coll}(\omega,<\omega_1^{L[G]})} \Vdash_{\text{Coll}(\omega,<\omega_1^{L[G]})} L(\mathbb{R}) \models \varphi(\check{x}, i(\check{\beta}))$$

Find some $G' \subseteq \mathrm{Coll}(\omega, <\kappa)$ generic over $L[g_i]$ so that $L[g_i][G'] = L[G]$. Then by homogeneity of $\mathrm{Coll}(\omega, <\kappa)$, the above is equivalent to

$$L[g_i][G'] = L[G] \models L(\mathbb{R}) \models \varphi(x, i(\beta))$$
$$L(\mathbb{R})^{L[G]} \models \varphi(x, i(\beta))$$

It has been shown that

$$L(\mathbb{R})^{L_\alpha[g][H]} \models \varphi(x, \beta) \text{ if and only if } L(\mathbb{R})^{L[G]} \models \varphi(x, i(\beta))$$

This shows h is elementary and completes the proof. □

Theorem 3.4. *Let κ be a remarkable cardinal of L. Let $G \subseteq \mathrm{Coll}(\omega, <\kappa)$ be $\mathrm{Coll}(\omega, <\kappa)$-generic over L. In $L[G]$, if I is a σ-ideal on ${}^\omega\omega$ with $\mathbb{P}_I$ is proper, B is an I^+ $\mathbf{\Delta}^1_1$ set, and $R \in L(\mathbb{R})$ is an equivalence relation so that for all $x \in {}^\omega\omega$, R_x is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$), then there is $C \subseteq B$ which is I^+ $\mathbf{\Delta}^1_1$ so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$, respectively).*

In [1], this same conclusion is proved from the existence of a measurable cardinal with infinitely many Woodin cardinals below it. This reduces the consistency strength of canonicalization for $L(\mathbb{R})$ equivalence relation by showing it can in fact hold in a forcing extension of L given the consistency of a relatively weak large cardinal.

4. Baire Property and Measurability From Neeman-Norwood Embedding Property

This section will show that the Neeman-Norwood $L(\mathbb{R})$-embedding property for proper forcing implies the classical regularity properties for sets in $L(\mathbb{R})$.

Theorem 4.1. *Suppose the Neeman-Norwood $L(\mathbb{R})$-embedding property for proper forcing holds, then all sets in $L(\mathbb{R})$ are Lebesgue measurable and have the Baire property.*

Proof. Suppose U is a set in $L(\mathbb{R})$. This means there is a formula φ, a tuple of reals $\bar{r}$, and a tuple of ordinals $\bar{\alpha}$ so that for all $x \in {}^\omega\omega$

$$x \in U \Leftrightarrow L(\mathbb{R}) \models \varphi(x, \bar{r}, \bar{\alpha})$$

Let $N \prec V_\Xi$ be a countable elementary substructure so that $\bar{r}$ and $\bar{\alpha}$ are elements of N. Let M be the Mostowski collapse of N and let $i^* : M \to V_\Xi$ be the elementary embedding obtained by composition. Let $i : L(\mathbb{R})^M \to L(\mathbb{R})^{V_\Xi}$ be the induced elementary embedding.

Let $I = I_{\text{meager}}$. In M, let A be a maximal antichain of $B \in \mathbb{P}_I$ so that $B \Vdash^M_{\mathbb{P}_I} L(\mathbb{R}) \models \varphi(\dot{x}_{\text{gen}}, \bar{r}, i^{-1}(\bar{\alpha}))$ or $B \Vdash^M_{\mathbb{P}_I} \neg(L(\mathbb{R}) \models \varphi(\dot{x}_{\text{gen}}, \bar{r}, i^{-1}(\bar{\alpha})))$. Let A_0 be the collection of $B \in A$ so that $B \Vdash^M_{\mathbb{P}_I} L(\mathbb{R}) \models \varphi(\dot{x}_{\text{gen}}, \bar{r}, i^{-1}(\bar{\alpha}))$. Let A_1 be the $B \in A$ so that $B \Vdash^M_{\mathbb{P}_I} \neg(L(\mathbb{R}) \models \varphi(\dot{x}_{\text{gen}}, \bar{r}, i^{-1}(\bar{\alpha})))$.

Note that since the conditions of $\mathbb{P}_I$ are $\mathbf{\Delta}^1_1$ sets (coded by reals), the image of each condition of $\mathbb{P}_I$ under i^* does not move. Since A is countable because $\mathbb{P}_I$ satisfies the $\aleph_1$-chain condition, $i^*(A) = A$. Since i^* is elementary, A is a maximal antichain in V_Ξ.

Since A is countable, $\bigcup A$, $\bigcup A_0$, and $\bigcup A_1$ are all $\mathbf{\Delta}^1_1$. $\bigcup A$ is comeager. Let C, C_0, and C_1 be the set of all $\mathbb{P}_I$-generic reals over M inside of $\bigcup A$, $\bigcup A_0$, and $\bigcup A_1$, respectively. The $\aleph_1$-chain condition implies that C, C_1, and C_2 are comeager inside $\bigcup A$, $\bigcup A_0$, and $\bigcup A_1$, respectively. (In particular, C is comeager.) Also $C = C_0 \cup C_1$.

Suppose $g \in C_0$. Then by the forcing theorem, $M[g] \models L(\mathbb{R}) \models \varphi(g, \bar{r}, i^{-1}(\bar{\alpha}))$. Hence $L(\mathbb{R})^{M[g]} \models \varphi(g, \bar{r}, i^{-1}(\bar{\alpha}))$. Applying the elementary embedding h from the Neeman-Norwood embedding property, $L(\mathbb{R})^{V_\Xi} \models \varphi(g, \bar{r}, \bar{\alpha})$. So $g \in U$. This show that $C_0 \subseteq U$. Similarly, one can show $C_1 \subseteq {}^\omega\omega \setminus U$.

This show that $U \setminus C_0$ is meager. U has the property of Baire. Lebesgue measurability is proved similarly. □

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