SCOTT RANKS OF CLASSIFICATIONS OF THE ADMISSIBILITY EQUIVALENCE RELATION

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Abstract. Let $\mathscr{L}$ be a recursive language. Let $S(\mathscr{L})$ be the set of $\mathscr{L}$-structures with domain ω. Let $\Phi : {}^\omega 2 \to S(\mathscr{L})$ be a Δ^1_1 function with the property that for all $x, y \in {}^\omega 2$, $\omega_1^x = \omega_1^y$ if and only if $\Phi(x) \approx_{\mathscr{L}} \Phi(y)$. Then there is some $x \in {}^\omega 2$ so that $\mathrm{SR}(\Phi(x)) = \omega_1^x + 1$.

1. Introduction

The main equivalence relation of interest here is the countable admissible ordinal equivalence relation, denoted F_{ω_1}. It is an equivalence relation defined on ${}^\omega 2$ by $x \; F_{\omega_1} \; y$ if and only if $\omega_1^x = \omega_1^y$. Recall that if $z \in {}^\omega 2$, then ω_1^z is the supremum of the collection of ordinals which are isomorphic to z-recursive well-orderings on ω. Equivalently, ω_1^z is also the smallest z-admissible ordinal, i.e. the smallest ordinal height of a transitive model of KP containing z. The latter will be the more useful characterization here.

The equivalence relation F_{ω_1} is important and can be meaningfully studied due to its connection with admissibility. A theorem of Sacks [17] states that for any countable admissible ordinal α, there is some $x \in {}^\omega 2$ so that $\omega_1^x = \alpha$. Therefore each equivalence class of F_{ω_1} is associated with a countable admissible ordinal. F_{ω_1} is a thin Σ^1_1 equivalence relation with all equivalence classes $\boldsymbol{\Delta}^1_1$. (An equivalence relation E is thin if and only if there is no perfect set of E-inequivalent elements.)

The topological Vaught's conjecture asserts that if the orbit equivalence relation of a Polish group acting continuously on a Polish space is thin, then it has countably many classes. Marker [14] established a particular instance of this conjecture by showing that F_{ω_1} is not an orbit equivalence relation of a continuous action of a Polish group on ${}^\omega 2$. This answered a question of Kechris. Becker [2] strengthened this by showing that F_{ω_1} is not an orbit equivalence relation of a $\boldsymbol{\Delta}^1_1$ group action on ${}^\omega 2$.

Suppose E and F are equivalence relations on Polish spaces X and Y, respectively. E is $\boldsymbol{\Delta}^1_1$ reducible to F, denoted $E \leq_{\boldsymbol{\Delta}^1_1} F$, if and only if there is a $\boldsymbol{\Delta}^1_1$ function $\Phi : X \to Y$ so that for all $a, b \in X$, $a \; E \; b \Leftrightarrow \Phi(a) \; F \; \Phi(b)$. $\boldsymbol{\Delta}^1_1$ reducibility is a common way of comparing the complexity of equivalence relations.

Despite not being induced by a $\boldsymbol{\Delta}^1_1$ action of a Polish group, F_{ω_1} is however $\boldsymbol{\Delta}^1_1$ reducible to an orbit equivalence relation of a continuous action of the Polish group S_∞. Equivalence relations that are $\boldsymbol{\Delta}^1_1$ reducible to a continuous action of S_∞ have a more model theoretic characterization:

Let $\mathscr{L}$ be a countable language. Let $S(\mathscr{L})$ be the collection of $\mathscr{L}$-structures with domain ω. An equivalence relation E on a Polish space X is classifiable by countable structures if and only if there is a countable language $\mathscr{L}$ so that $E \leq_{\boldsymbol{\Delta}^1_1} \approx_{\mathscr{L}}$, where $\approx_{\mathscr{L}}$ is the $\mathscr{L}$-isomorphism relation defined on $S(\mathscr{L})$. A $\boldsymbol{\Delta}^1_1$ function Φ which witnesses this $\boldsymbol{\Delta}^1_1$ reducibility is called a classification of E by $\mathscr{L}$-structures. F_{ω_1} is classifiable by countable structures by using the isomorphism relation of structures in the language of linear orderings. The following is an example of a classification function witnessing this.

Suppose $x \in {}^\omega 2$. An x-recursive pseudo-wellordering is an x-recursive linear ordering which is not a wellordering but has no x-hyperarithmetic descending sequences. Let η be the order type of $\mathbb{Q}$. It is shown in [7] that x-recursive pseudo-wellorderings have ordertype $\omega_1^x(1 + \eta) + \rho$, for some $\rho < \omega_1^x$. (Also see [19], Lemma III.2.2 (ii).) An x-Harrison linear ordering is an x-recursive linear ordering on ω of ordertype $\omega_1^x(1 + \eta)$. Note that the isomorphism type of an x-Harrison linear ordering is completely determined by ω_1^x. The desired classification of F_{ω_1} by linear orderings will be a map Φ sending x to an x-Harrison linear

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ordering. Such a classification function exists which is $\mathbf{\Delta}_1^1$ follows from existence of a natural and uniform construction of Harrison linear orderings.

Although a recursive pseudo-wellordering can be constructed by a Barwise compactness argument, there is a more concrete natural construction found in [11] which is exposited in [19] Lemma III.2.1: The relation $R(x,y)$ if and only if x is not y-hyperarithmetic is Σ_1^1. As Σ_1^1 sets are projections of recursive trees, one can use the relation R to obtain a recursive tree U in $2 \times \omega$ with the property that for all $y \in {}^\omega 2$, the tree U^y is illfounded but has no y-hyperarithmetic path. It can be shown that the Kleene-Brouwer ordering of U^y is a y-recursive pseudo-wellordering which can be modified to obtain a y-Harrison linear ordering. This procedure is Δ_1^1 and gives the desired Δ_1^1 classification Φ. (See [6] Section 3 for more details.)

Note that the Scott rank of x-Harrison linear orderings is $\omega_1^x + 1$. Therefore, in the example above, for all $x \in {}^\omega 2$, $\mathrm{SR}(\Phi(x)) = \omega_1^x + 1$. It is shown in [6] Theorem 4.2 that any Δ_1^1 classification Φ of F_{ω_1} must send any real x to an x-hyperarithmetic $\mathscr{L}$-structure $\Phi(x)$ with $\mathrm{SR}(\Phi(x)) \geq \omega_1^x$. Recall that any x-hyperarithmetic $\mathscr{L}$-structure M on ω has Scott rank less than or equal to $\omega_1^x + 1$. If an x-hyperarithmetic structure M has $\mathrm{SR}(M) \geq \omega_1^x$, then M is said to have high Scott rank.

[6] Remark 4.3 raised the question of whether every Δ_1^1 classification Φ has the property that for all x, $\mathrm{SR}(\Phi(x)) = \omega_1^x + 1$. An attempt to compel the Scott rank to take the highest possible value in [6] Remark 4.3 fails due to the existence of x-recursive structures of Scott rank ω_1^x. This suggests the following which is the main question of the paper:

Question 1.1. Is there a recursive language $\mathscr{L}$ and Δ_1^1 function $\Phi : {}^\omega 2 \to S(\mathscr{L})$ so that for all $x, y \in {}^\omega 2$, $x \; F_{\omega_1} \; y \Leftrightarrow \Phi(x) \approx_{\mathscr{L}} \Phi(y)$ with the property that for all $x \in {}^\omega 2$, $\mathrm{SR}(\Phi(x)) = \omega_1^x$?

Hyperarithmetic structures of Scott rank $\omega_1^\emptyset$ are more difficult to produce than computable structures of Scott rank $\omega_1^\emptyset + 1$. Makkai [13] constructed the first such hyperarithmetic example. Knight and Millar [4] produced recursive structures of Scott rank $\omega_1^\emptyset$. (Other examples can be found in [8].) These constructions are much more intricate and have some nonuniform aspects due to the use of Barwise or Barwise-Kreisel compactness.

The main result of the paper is the following negative answer to Question 1.1:

Theorem 3.7. Let $\mathscr{L}$ be a recursive language and let $\Phi : {}^\omega 2 \to S(\mathscr{L})$ be a Δ_1^1 function so that $x \; F_{\omega_1} \; y \Leftrightarrow \Phi(x) \approx_{\mathscr{L}} \Phi(y)$. Then there is some $x \in {}^\omega 2$ so that $\mathrm{SR}(\Phi(x)) = \omega_1^x + 1$.

(Becker has informed the authors that this theorem follows from a more general result in [3] which is however proved by different methods in ZFC augmented by $\mathbf{\Sigma}_2^1$-determinacy. All results in this paper are implicitly proved from ZFC.)

Theorem 3.7 can be interpreted to say that there is no construction of a recursive structure of Scott rank $\omega_1^\emptyset$ which is natural enough in the sense that the construction can be relativized to any real x to produce an x-recursive structure of Scott rank ω_1^x whose isomorphism type depends solely on the x-recursive ordinals. The notion of a Δ_1^1 classification function for F_{ω_1} is used to formalize this idea of naturality. This approach is similar to a questin of Martin concerning the unnaturalness of intermediate degrees: A well known result in recursion theory states that there is a degree strictly between $[\emptyset]_T$ and $[\emptyset']_T$. Martin asked whether there are any definable procedures that take an $X \in \mathcal{D}$, the set of Turing degrees, and return a Turing degree between X and its jump. This is formalized by asking in $\mathsf{ZF} + \mathsf{AD}$ whether there is any function $\Phi : \mathcal{D} \to \mathcal{D}$ with the property that for all $X \in \mathcal{D}$, $X <_T \Phi(X) <_T X'$. See [22] and [15] for more on various forms of Martin's conjecture.

The proof of Theorem 3.7 is similar to how [18] Theorem 4.2 (or [20] Corollay 6.2) shows that every counterexample to Vaught's conjecture τ has a model M so that $\mathrm{SR}(M) = \omega_1^M + 1$. This is done by producing an illfounded end-extension of a Σ_2-admissible set which has an appropriate model of τ and has enough Σ_1-absoluteness to conclude that isolating formulas of the original Σ_2-admissible set are still isolating formulas in the illfounded end-extension.

An important feature of the proof of the above result for counterexamples to Vaught's conjecture is access to the complete theory and types of the desired model in small admissible sets even when the model does not exist in that admissible set. In the setting of this paper, access to the fragment, complete theory, and types of the desired model as well as the sufficient definability of these objects within the appropriate admissible set are obtained using the Δ^1_1 classification function and the Solovay product forcing lemma for a suitable class forcing. This seems to be similar to ideas used in [12].

Becker [3] has also considered the unnaturalness of x-recursive structures of Scott rank ω_1^x. [3] shows that under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$, if $\mathscr{L}$ is a countable language and $\mathcal{F}$ is a family of $\aleph_1$ many isomorphism types of $\mathscr{L}$-structures, then there is a $z \in {}^\omega 2$ so that for all $x \geq_T z$, if an x-recursive $\mathscr{L}$-structure M has an isomorphism type in $\mathcal{F}$, then $\mathrm{SR}(M) \neq \omega_1^x$. Given any Δ^1_1 classification Φ of F_{ω_1}, Theorem 3.7 follows by applying Becker's result to the Σ^1_1 family determined by the range of Φ. [3] mentions that it is not known whether the $\mathbf{\Sigma}^1_1$ version of Becker's result holds in ZFC and that the methods of [3] to prove the $\mathbf{\Sigma}^1_1$ version require a determinacy assumption stronger than $\mathbf{\Pi}^1_1$-determinacy but weaker than $\mathbf{\Pi}^1_2$-determinacy.

2. Basics

The results of the paper are proved in ZFC. As customary in set theory, the real universe is denoted by V, which can be understood as some fixed model of ZFC where the results of the paper are being derived. Frequently concepts will be viewed from various different models of set theory. If M is a model of set theory and A is some notion given by a formula, A^M will indicate the relativization of the definition of A within the model M.

Let KP denote Kripke-Platek set theory with the infinity axiom, which can be formulated in any language $\mathscr{J}$ consisting of a distinguished binary relation symbol $\dot\in$ and possibly other symbols. KP is a weak axiom system for set theory. Its distinguishing axiom schemes are Δ_1-separation and Σ_1-collection. Let Σ_2-KP be the axiom system extending KP by the axiom schemes of Δ_2-separation and Σ_2-collection.

An admissible set is a transive model of KP. A Σ_2-admissible set is a transitive model of Σ_2-KP. See [1] and [9] for more information about KP and admissibility.

Let ON denote the class of ordinals. If $\mathcal{A}$ is some $\mathscr{J}$-structure satisfying KP where $\mathscr{J}$ is a language consisting of a distinguish binary relation symbol $\dot\in$, then $\mathrm{WF}(\mathcal{A})$ is the substructure of elements of $\mathcal{A}$ which are $\dot\in^{\mathcal{A}}$-well-founded in the real world V. Via the Mostowski collapse, one can always assume $(\mathrm{WF}(\mathcal{A}), \dot\in^{\mathcal{A}})$ is a transitive set. The ordinal height of $\mathcal{A}$ is $\mathrm{ON} \cap \mathrm{WF}(\mathcal{A}) = \mathrm{ON} \cap \mathcal{A}$, where it is assumed that $\mathrm{WF}(\mathcal{A})$ is a transitive set.

For the rest of the paper, assume that ω belongs to the transitive closure of the well-founded part of any model of KP.

Definition 2.1. Let $x \in {}^\omega 2$. $\alpha \in \mathrm{ON}$ is an x-admissible ordinal if and only if there is an admissible set $\mathcal{A}$ so that $x \in \mathcal{A}$ and $\alpha = \mathrm{ON} \cap \mathcal{A}$. That is, α is the ordinal height of some admissible set containing x.

The least x-admissible ordinal is denoted ω_1^x.

Fact 2.2. *Let $\mathcal{A} \models \mathsf{KP}$. There is a Δ_1 function taking elements of $\alpha \in \mathrm{ON}^{\mathcal{A}}$ to L_α, the segment of Gödel constructible hierarchy. $L^{\mathcal{A}}$ is a Δ_1 class in $\mathcal{A}$ and $L^{\mathcal{A}} \models \mathsf{KP}$.*

These results hold for the relativized Gödel hierarchy.

Definition 2.3. Let $x \in {}^\omega 2$. $y \in {}^\omega 2$ is an x-hyperarithmetic real if and only if y belongs to every admissible set containing x.

A basic fact of descriptive set theory is that y is x-hyperarithmetic if and only if y is $\Delta^1_1(x)$.

Fact 2.4. *Let $x \in {}^\omega 2$. $L_{\omega_1^x}[x]$ is the smallest x-admissible set under $\subseteq$. Hence the x-hyperarithmetic reals are exactly ${}^\omega 2 \cap L_{\omega_1^x}[x]$.*

Fact 2.5. *(Truncation lemma) If $\mathcal{A} \models \mathsf{KP}$, then $\mathrm{WF}(\mathcal{A}) \models \mathsf{KP}$. Therefore assuming that the well-founded part is transitive, $\mathrm{WF}(\mathcal{A})$ is an admissible set.*

Proof. See [1] Lemma II.8.4. □

The following result of Sacks gives an important characterization of countable admissible ordinals.

Fact 2.6. *([17]) If α is a countable admissible ordinal, then there is an $x \in {}^\omega 2$ so that $\omega_1^x = \alpha$.*

This result can be proved using infinitary logic in countable admissible fragments as shown in [9]. These methods were used to study F_{ω_1} in [6] and will again be used in the arguments of this paper. The main tool for this approach is the Jensen's model existence theorem.

Lemma 2.7. *(Jensen's model existence theorem) Let $\mathcal{A}$ be an admissible set. Let $\mathscr{J}$ be a language which is Δ_1 definable in $\mathcal{A}$ and contains a distinguished binary relation symbol $\dot{\in}$ and constant symbols $\hat{a}$ for each $a \in \mathcal{A}$. Let H be a consistent theory in the countable admissible fragment $(\mathscr{J}_{\infty\omega})^{\mathcal{A}}$ associated with $\mathcal{A}$ which is Σ_1 definable in $\mathcal{A}$ and contains the following sentences:*

(I) Extensionality.

(II) For each $a \in \mathcal{A}$, "$(\forall v)(v \dot{\in} \hat{a} \Leftrightarrow \bigvee_{z \in a} v = \hat{z})$".

Then there is a $\mathscr{J}$-structure $\mathcal{B} \models H$ so that $\mathrm{WF}(\mathcal{B})$ is transitive, $\mathcal{B}$ end extends $\mathcal{A}$, and $\mathrm{ON} \cap A = \mathrm{ON} \cap B$.

If $\mathcal{A}$ is a Σ_2-admissible set and the theory H is Σ_2 definable in $\mathcal{A}$, then the same conclusion holds.

Proof. See [9] Section 4, Lemma 11 or [5]. Recall that $\mathcal{B}$ is an end extension of $\mathcal{A}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$ and for all $x \in \mathcal{A}$, $\{y \in \mathcal{A} : y \dot{\in}^{\mathcal{A}} x\} = \{y \in \mathcal{B} : y \dot{\in}^{\mathcal{B}} x\}$. $\square$

Fact 2.6 was originally proved by Sacks using a class forcing over countable admissible sets. For some properties concerning constructibility, the approach by forcing will be useful. A simple class forcing of Steel will be used. The following presents the definitions and basic properties. See [21] for more details.

Definition 2.8. (Steel's forcing with tagged trees; see [21]) Let $\mathcal{A}$ be a countable model of KP. Let ∞ be some symbol formally defined to be larger than all ordinals of $\mathcal{A}$. Let $\mathbb{S}$ be the forcing consisting of (T, h) where T is a finite tree on ω and $h : T \to \mathrm{ON}^{\mathcal{A}} \cup \{\infty\}$, with the property that for all $s, t \in T$ with $s \subseteq t$, $h(t) < h(s)$ or $h(s) = h(t) = \infty$. If $p, q \in \mathbb{S}$ and $p = (T_p, h_p)$ and $q = (T_q, h_q)$, then $p \leq_{\mathbb{P}} q$ if and only if $T_p \supseteq T_q$ and $h_p \supseteq h_q$. Let $1_{\mathbb{S}} = (\emptyset, \emptyset)$. The forcing relation $p \Vdash \varphi$, as a relation ranging over $p \in \mathbb{S}$ and ranked sentences φ (see [21]), is a Δ_1 relation in $\mathcal{A}$.

There are $\mathbb{S}$-names $\dot{T}, \dot{h} \in \mathcal{A}$ so that for any $G \subseteq \mathbb{S}$ which is $\mathbb{S}$-generic over $\mathcal{A}$, $\dot{T}[G] = \bigcup_{p \in G} T_p$ and $\dot{h}[G] = \bigcup_{p \in G} h_p$. Note that $\dot{T}[G]$ is a tree on ω. When $G \subseteq \mathbb{S}$ is $\mathbb{S}$-generic over $\mathcal{A}$, $\mathcal{A}[\dot{T}[G]] \models \mathsf{KP}$. (However $\mathcal{A}[G]$ is not a model of KP.) Therefore, $\omega_1^{\dot{T}[G]} = \mathcal{A} \cap \mathrm{ON}^V$.

Definition 2.9. If $\mathscr{L}$ is a language, then let $\mathscr{L}_{\omega\omega}$ denote the set of first order $\mathscr{L}$-formulas and $\mathscr{L}_{\infty\omega}$ denote the class of infinitary formulas in the language $\mathscr{L}$. In KP, $\mathscr{L}_{\infty\omega}$ is a Δ_1 class. The satisfaction relation between $\mathscr{L}$-structures and formulas of $\mathscr{L}_{\infty\omega}$ is also Δ_1 in KP. A subset $\mathcal{F} \subseteq \mathscr{L}_{\infty\omega}$ is an fragment if it has the closure properties of [1] Definition III.2.1. (See [1] Chapter III for more information about the syntax and semantics of $\mathscr{L}_{\infty\omega}$.)

Definition 2.10. Let $\mathscr{L}$ be a recursive language, $\mathcal{F} \subseteq \mathscr{L}_{\infty\omega}$ be a fragment, $T \subseteq \mathcal{F}$ be a theory, and M be an $\mathscr{L}$-structure. Then $S_n^{\mathcal{F}}(T)$ is the collection of all complete n-types of T in the fragment $\mathcal{F}$.

For each $\varphi \in \mathcal{F}$ with n many free variables, let $[\varphi]_{\mathcal{F}}^T = \{p \in S_n^{\mathcal{F}}(T) : \varphi \in p\}$. The topology on $S_n^{\mathcal{F}}(T)$ is generated by $[\varphi]_{\mathcal{F}}^T$ as basic open sets, where φ ranges over formulas in $\mathcal{F}$ with n free variables. A type $p \in S_n^{\mathcal{F}}(T)$ is an isolated type if $\{p\}$ is an open set. A type which is not isolated is sometimes called a nonprincipal type. A formula φ is an isolating formula if and only $[\varphi]_{\mathcal{F}}^T$ is a singleton. That is, for all $\psi \in \mathcal{F}$ with n free variables, $T \vdash (\forall \bar{x})(\varphi \Rightarrow \psi)$ or $T \vdash (\forall \bar{x})(\varphi \Rightarrow \neg\psi)$.

If $\bar{a}$ is a tuple from M of length n, then $\mathrm{tp}_{\mathcal{F}}^M(\bar{a})$ is the complete n-type consisting of the formulas of $\mathcal{F}$ satisfied by $\bar{a}$.

Definition 2.11. The following definition and properties can be formalized and proved in KP.

Let $\mathscr{L}$ be some language and let $M \in S(\mathscr{L})$. By Σ_1-recursion, the functions $\mathscr{L}_\alpha^M$ and T_α^M are defined as follows:

- Let $\mathscr{L}_0^M = \mathscr{L}_{\omega\omega}$.
- If α is a limit ordinals, then let $\mathscr{L}_\alpha^M = \bigcup_{\beta < \alpha} \mathscr{L}_\beta^M$.
- For any α, let T_α^M be the complete theory of M in the fragment $\mathscr{L}_\alpha^M$.
- Let $\mathscr{L}_{\alpha+1}^M$ be the least fragment $\mathcal{F}$ extending $\mathscr{L}_\alpha^M$ containing $\bigwedge p$ for each nonprincipal $p \in S_n^{\mathscr{L}_\alpha^M}(T_\alpha^M)$ realized by some tuple in M.

The functions $\alpha \mapsto T_\alpha^M$ and $\alpha \mapsto \mathscr{L}_\alpha^M$ are Σ_1-functions on the Δ_1 class of ordinals. Hence these two functions are Δ_1. Note that if $M, N \in S(\mathscr{L})$ and $M \approx_{\mathscr{L}} N$, then $\mathscr{L}_\alpha^M = \mathscr{L}_\alpha^N$ and $T_\alpha^M = T_\alpha^N$ for all ordinals α.

Let $\mathscr{L}_\infty^M = \bigcup_{\alpha \in \mathrm{ON}} \mathscr{L}_\alpha^M$ and $T_\infty^M = \bigcup_{\alpha \in \mathrm{ON}} T_\alpha^M$. For any β, a formula of quantifer rank $\alpha < \beta$ belongs to $\mathscr{L}_\beta^M$ or T_β^M if it already belonged to $\mathscr{L}_\alpha^M$ or T_α^M. This can be used to show that $\mathscr{L}_\infty^M$ and T_∞^M are Δ_1.

The Scott rank of M, denoted $\mathrm{SR}(M)$, is the smallest ordinal α so that M is the atomic model of T_α^M.

Fact 2.12. *Let $\mathscr{L}$ be a recursive language. Let $M \in S(\mathscr{L})$.* $\mathrm{SR}(M) \leq \omega_1^M + 1$.

Proof. See [16]. $\square$

3. Scott Ranks of Classifications

Definition 2.11 shows that T_α^M and $\mathscr{L}_\alpha^M$ can be defined for any structure M within any model $\mathcal{A} \models \mathsf{KP}$ containing M. Suppose σ is a countable admissible ordinal and $x \in {}^\omega 2$ is such that $\omega_1^x = \sigma$. Let Φ be a Δ_1^1 classification of F_{ω_1} by $\mathscr{L}$-structures. Generally, $\Phi(x) \notin L_\sigma$, where L_σ is the σ^{th}-level of Gödel constructible hierarchy and the smallest admissible set of height σ. Later, it will desirable to have $\mathscr{L}_\alpha^{\Phi(x)}$ and $T_\alpha^{\Phi(x)}$ for $\alpha \leq \sigma$ either belong to L_σ or are Δ_1-definable in L_σ, even though the structure $\Phi(x)$ does not belong to L_σ.

The Solovay product forcing lemma for class forcing will be very useful for showing that the relevant theory of a model that does not exist in L_σ is actually definable in L_σ under certain circumstances. The Solovay product lemma states that elements that belong to two mutually generic extensions actually already belong to the ground model.

Lemma 3.1. *(Solovay product lemma) Let $\mathcal{A} \models \mathsf{KP}$. Let $\mathbb{P}$ be a Δ_1-definable forcing in $\mathcal{A}$. (This means that $\mathbb{P}$ and $\leq_\mathbb{P}$ are Δ_1 definable.) Assume that $p \Vdash_\mathbb{P} \varphi$ is a Δ_1 relation in arguments $p \in \mathbb{P}$ and ranked sentences φ. Let $\mathbb{P} \times \mathbb{P}$ denote the product forcing. Let $G, H \subseteq \mathbb{P}$ be $\mathbb{P}$-generic filters over $\mathcal{A}$ such that $G \times H$ is a $\mathbb{P} \times \mathbb{P}$-generic filter over $\mathcal{A}$. Then $\mathcal{A}[G] \cap \mathcal{A}[H] = \mathcal{A}$.*

Proof. Suppose $\mathcal{A}[G] \cap \mathcal{A}[H] \neq \emptyset$. Let $z \in (\mathcal{A}[G] \cap \mathcal{A}[H]) \setminus \mathcal{A}$ be of minimal rank. Hence $z \subseteq \mathcal{A}$. There are $\mathbb{P}$-names σ and τ so that $z = \tau[G] = \sigma[H]$. By the forcing theorem, there is some $(p, q) \in \mathbb{P} \times \mathbb{P}$ so that $(p, q) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau = \sigma$, where here τ and σ are considered as $\mathbb{P} \times \mathbb{P}$-names which yield the result of the original $\mathbb{P}$-names τ and σ, respectively, evaluated using the left and right $\mathbb{P}$-generic filters, respectively, derived from $\mathbb{P} \times \mathbb{P}$-generic filters.

The claim is that for all $x \in \mathcal{A}$, $p \Vdash_\mathbb{P} \check{x} \in \tau$ or $p \Vdash_\mathbb{P} \check{x} \notin \tau$: To see this, assume not. There is some $x \in \mathcal{A}$ and some $p_0, p_1 \leq_\mathbb{P} p$ so that $p_0 \Vdash_\mathbb{P} \check{x} \in \tau$ and $p_1 \Vdash \check{x} \notin \tau$. Without loss of generality, suppose that $\mathcal{A}[H] \models x \notin \sigma[H]$. Then find some $q' \leq_\mathbb{P} q$ so that $q' \Vdash_\mathbb{P} \check{x} \notin \sigma$. Let $G', H' \subseteq \mathbb{P}$ be $\mathbb{P}$-generic filters over $\mathcal{A}$ so that $G' \times H'$ is a $\mathbb{P} \times \mathbb{P}$-generic over $\mathcal{A}$ and $(p_0, q') \in G' \times H'$. By the forcing theorem, $x \in \tau[G']$ and $x \notin \sigma[H']$. Hence $\mathcal{A}[G' \times H'] \models \tau[G'] \neq \sigma[H']$. But $(p_0, q') \leq_{\mathbb{P}\times\mathbb{P}} (p, q)$ and $(p, q) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau = \sigma$. Contradiction.

Let α be the rank of z. Let $\mathcal{A}_\alpha$ denote the elements of $\mathcal{A}$ of rank less than α. Then $z = \{x \in \mathcal{A}_\alpha : p \Vdash_\mathbb{P} \check{x} \in \tau\}$. $z \in \mathcal{A}$ by Δ_1-separation. This contradicts the earlier assumption that $z \notin \mathcal{A}$. $\square$

Corollary 3.2. *Let σ be a countable admissible ordinal.* $\bigcap_{\omega_1^x = \sigma} L_\sigma[x] = L_\sigma$.

Proof. Let $\mathbb{S}$ be the Steel's tagged tree forcing. Let G, H be $\mathbb{S}$-generic filters over L_σ so that $G \times H$ is $\mathbb{S} \times \mathbb{S}$-generic over L_σ. Let $a = \dot{T}[G]$ and $b = \dot{T}[H]$. $\omega_1^a = \omega_1^b = \sigma$. Then $\bigcap_{\omega_1^x = \sigma} L_\sigma[x] \subseteq L_\sigma[a] \cap L_\sigma[b] \subseteq L_\sigma[G] \cap L_\sigma[H] = L_\sigma$ using Lemma 3.1. $\square$

In the following, let σ be an admissible ordinal. Let $\mathscr{L}$ be a recursive language. Let $\Phi : {}^\omega 2 \to S(\mathscr{L})$ be a Δ_1^1 classification of F_{ω_1} by $\mathscr{L}$-structures, i.e. $\omega_1^x = \omega_1^y$ if and only if $\Phi(x) \approx_\mathscr{L} \Phi(y)$.

As mentioned above, since $\Phi(x) \approx_\mathscr{L} \Phi(y)$ if and only if $\omega_1^x = \omega_1^y$, $\mathscr{L}_\alpha^{\Phi(x)} = \mathscr{L}_\alpha^{\Phi(y)}$ and $T_\alpha^{\Phi(x)} = T_\alpha^{\Phi(y)}$ for all ordinals α whenever $\omega_1^x = \omega_1^y$. Therefore, one may define $\mathscr{L}_\alpha^\sigma$ and T_α^σ to be $\mathscr{L}_\alpha^{\Phi(x)}$ and $T_\alpha^{\Phi(x)}$, respectively, where x can be any real so that $\omega_1^x = \sigma$. In general, $\mathscr{L}_\alpha^{\Phi(x)}$ and $T_\alpha^{\Phi(x)}$ are elements of $L_\sigma[\Phi(x)]$ when $\alpha < \sigma$ and is Δ_1 definable in $L_\sigma[\Phi(x)]$ when $\alpha = \sigma$. The Solovay product lemma will indicate that each set belongs to L_σ when $\alpha < \sigma$ and is Δ_1 in L_σ when $\alpha = \sigma$.

The following will give a formal definition of L_α^σ and T_α^σ inside any model of KP and their basic properties.

Definition 3.3. Let $z \in {}^\omega 2$. Let $\mathscr{L}$ be a recursive langauge. Let Φ be a $\Delta_1^1(z)$ classification of F_{ω_1} by $\mathscr{L}$-structures. Let σ be a countable z-admissible ordinal. Let $\mathcal{A}$ be a countable model of KP containing z such that $\mathrm{ON} \cap \mathcal{A} = \sigma$.

Now work in $(L[z])^\mathcal{A}$: Let $\mathbb{S}$ denote Steel's tagged tree forcing defined in $(L[z])^\mathcal{A}$. Next, by Σ_1-recursion in $(L[z])^\mathcal{A}$ (which is a model of KP), define the function taking an ordinal α of $(L[z])^\mathcal{A}$ to $\mathscr{L}_\alpha^\sigma$ and T_α^σ as follows:

- Let $\mathscr{L}^\sigma_0 = \mathscr{L}_{\omega\omega}$.
- If α is limit ordinal, then let $\mathscr{L}^\sigma_\alpha = \bigcup_{\beta<\alpha} \mathscr{L}^\sigma_\beta$.
- For any α, let T^σ_α be that set such that $1_{\mathbb{S}} \Vdash_{\mathbb{S}}$ "$\check{T}^\sigma_\alpha$ is the complete theory of $\Phi(\dot{T})$ in the fragment $\check{\mathscr{L}}^\sigma_\alpha$". Here $\dot{T}$ refers to the canonical name for the generic tree (construed as a real) produced by $\mathbb{S}$ as in Definition 2.8. Such a set exists using (I) of the next lemma to show $T^{\Phi(x)}_\alpha \in (L[z])^{\mathcal{A}}$ for any $x \in ({}^\omega 2)^V$ such that $\omega^x_1 = \sigma$. (Alternatively, T^σ_α is also the set of sentences $\varphi \in \mathscr{L}^\sigma_\alpha$ so that $1_{\mathbb{S}} \Vdash \Phi(\dot{T}) \models \check{\varphi}$.)
- Let $\mathscr{L}^\sigma_{\alpha+1}$ be the smallest fragment $\mathcal{F}$ of $\mathscr{L}_{\infty\omega}$ so that $1_{\mathbb{S}} \Vdash_{\mathbb{S}}$ "$\check{\mathcal{F}}$ is the smallest fragment of $\mathscr{L}_{\infty\omega}$ so that for all $n \in \omega$, $\bigwedge p$ belongs to $\mathcal{F}$ for any nonprincipal $p \in S_n^{\mathscr{L}^\sigma_\alpha}(T^\sigma_\alpha)$ realized by some n-tuple in $\Phi(\dot{T})$".

Lemma 3.4. *Assume the setting of Definition 3.3.*

(I) For any $x \in {}^\omega 2 \cap \mathcal{A}$ such that $\omega^x_1 = \sigma$ (in the real world) and any $\alpha \in \mathrm{ON}^{\mathcal{A}}$, $\mathscr{L}^{\Phi(x)}_\alpha, T^{\Phi(x)}_\alpha \in (L[z])^{\mathcal{A}}$. Any $p \in S_n^{\mathscr{L}^{\Phi(x)}_\alpha}(T^\Phi(x)_\alpha)$ realized by some tuple in $\Phi(x)$ belongs to $(L[z])^{\mathcal{A}}$.

(II) The functions $\alpha \mapsto \mathscr{L}^\sigma_\alpha$ and $\alpha \mapsto T^\sigma_\alpha$ are Δ_1 in $(L[z])^{\mathcal{A}}$.

(III) Let $T^\sigma_\infty = \bigcup_{\alpha\in\mathrm{ON}} T^\sigma_\alpha$ and $\mathscr{L}^\sigma_\infty = \bigcup_{\alpha\in\mathrm{ON}} \mathscr{L}^\sigma_\alpha$. Both are Δ_1 classes in $(L[z])^{\mathcal{A}}$.

(IV) For any $x \in ({}^\omega 2)^V$ with $\omega^x_1 = \sigma$ and $\alpha \le \sigma$, $L^{\Phi(x)}_\alpha = L^\sigma_\alpha$ and $T^{\Phi(x)}_\alpha = T^\sigma_\alpha$.

Proof. (I) is proved using Lemma 3.1.

(II) and (III) are proved much like the corresponding facts for $\mathscr{L}^{\Phi(x)}_\alpha$ and $T^{\Phi(x)}_\alpha$ mentioned in Definition 2.11 and using the definability of the forcing relation $\Vdash_{\mathbb{S}}$.

(IV) is proved by induction. $\square$

Fact 3.5. *Assume the setting of Lemma 3.3. The relation "φ is an isolating formula of $S_n^{\mathscr{L}^\sigma_\infty}(T^\sigma_\infty)$" with free variable φ ranging over $\mathscr{L}^\sigma_\infty$ is a Π_1 relation.*

Proof. Recall that T^σ_∞ and $\mathscr{L}^\sigma_\infty$ are Δ_1 and that $T^\sigma_\infty = \bigcup_{\alpha\in\mathrm{ON}} T^\sigma_\alpha$, where each T^σ_α is a complete theory in $\mathscr{L}^\sigma_\alpha$.

That φ is an isolating formula can be expressed by saying for all $\psi \in \mathscr{L}^\sigma_\infty$, for all $\beta \in \mathrm{ON}$, if $\varphi, \psi \in \mathscr{L}^\sigma_\beta$, then either $(\forall \bar{x})(\varphi \Rightarrow \psi) \in T^\sigma_\beta$ or $(\forall \bar{x})(\varphi \Rightarrow \neg\psi) \in T^\sigma_\beta$. This can be formalized as a Π_1 statement. $\square$

Fact 3.6. *([6] Theorem 4.2) Let $z \in {}^\omega 2$. Let $\mathscr{L}$ be a recursive language. Suppose $\Phi : {}^\omega 2 \to S(\mathscr{L})$ is a $\Delta^1_1(z)$ classification of F_{ω_1} be $\mathscr{L}$-structures. Then for all x such that ω^x_1 is z-admissible,* $\mathrm{SR}(\Phi(x)) \ge \omega^x_1$.

Theorem 3.7. *Let $\mathscr{L}$ be a recursive language. Let $\Phi : {}^\omega 2 \to S(\mathscr{L})$ be a $\Delta^1_1(z)$ classification of F_{ω_1} by $\mathscr{L}$-structures. Let σ be an ordinal so that $L_\sigma[z] \models \Sigma_2$-$\mathsf{KP}$ and $L_\sigma[z] \prec_1 L_{\omega_1^{L[z]}}[z]$. Let $x \in {}^\omega 2$ be such that $\omega^x_1 = \sigma$. Then* $\mathrm{SR}(\Phi(x)) = \sigma + 1$.

Such a countable ordinal σ can be found as follows: By the Löwenheim-Skolem theorem, let $M \prec_\omega L_{\omega_1^{L[z]}}[z]$ be a countable elementary substructure containing z. Since $L_{\omega_1^{L[z]}}[z]$ thinks the transitive closure of all sets are countable, M has this property too. For all $x \in M$, there is some bijection $f : \omega \to \mathrm{tc}(x)^M$, where tc denotes the transitive closure. Since M is elementary, this f really is a bijection of ω with $\mathrm{tc}(x)$. For all $n \in \omega$, $f(n) \in M$ so $\mathrm{tc}(x) \subseteq M$. This shows that M is transitive. Since M is a countable transitive elementary substructure of $L_{\omega_1^{L[z]}}[z]$, there is some countable σ so that $M = L_\sigma[z]$ by condensation. Finally, $L_\sigma[z] \models \mathsf{ZF} - \mathsf{P}$ (and in particular Σ_2-KP) because $L_{\omega_1^{L[z]}}[z] \models \mathsf{ZF} - \mathsf{P}$.

Proof. Before beginning the proof, an outline will be given: For simplicity throughout the proof, suppose Φ is Δ^1_1. By Lemma 3.4, $\langle \mathscr{L}^\sigma_\alpha : \alpha \in \mathrm{ON}\rangle$ and $\langle T^\sigma_\alpha : \alpha \in \mathrm{ON}\rangle$ are Δ_1-classes in the constructible universe of any model of KP whose collection of standard ordinals has ordertype σ. In particular, these two sequences are Δ_1-definable in L_σ. First, one will find an illfounded model of KP, $\mathcal{B}$, so that $\mathcal{B} \cap \mathrm{ON} = \sigma$ and $\mathcal{B}$ contains some real c so that $\omega^c_1 = \sigma$. Lemma 3.4 asserts that each $T^{\Phi(c)}_\alpha = T^\sigma_\alpha$ and $\mathscr{L}^{\Phi(c)}_\alpha = \mathscr{L}^\sigma_\alpha$, and hence they belong to L_σ. Furthermore, $\mathcal{B}$ has the crucial property that any isolating formula for $\mathscr{L}^\sigma_\infty$ in L_σ is an isolating formula for $(\mathscr{L}^\sigma_\infty)^{\mathcal{B}}$ in $\mathcal{B}$. This fact will be accomplished by simply arranging that $L_\sigma \prec_1 \mathcal{B}$. This $\mathcal{B}$ is found using Jensen's model existence theorem with an appropriate theory in a countable admissible fragment of L_σ that attempts to express Σ_1-elementarity. The choice of the ordinal σ will show that the theory to which the Jensen's model existence theorem is applied is consistent since it will be modeled by

$L_{\omega_1^L}$. The purpose of the two sequences, $\langle \mathscr{L}_\alpha^\sigma : \alpha \in \mathrm{ON}\rangle$ and $\langle T_\alpha^\sigma : \alpha \in \mathrm{ON}\rangle$, and the effort to establish that they belong to or are definable in the constructible universe of the relevant models of KP is to be able to express the absoluteness of being an isolating formula within this admissible fragment of L_σ.

By Fact 3.6, $\mathrm{SR}(\Phi(c))$ is σ or $\sigma+1$. Now suppose for a contradiction that $\mathrm{SR}(\Phi(c)) = \sigma$. Within $L_\sigma[\Phi(c)]$, define a function Ψ which assigns each tuple of $\Phi(c)$ to the least ordinal α so that there is some formula $\varphi \in \mathscr{L}_\alpha^\sigma$ which is isolating and is realized by this tuple. This function is well-defined by the assumption that $\mathrm{SR}(\Phi(c)) = \sigma$. Now let $\tilde{\Psi}$ be the function defined in the same way but within the illfounded model $\mathcal{B}$. Using the fact that $L_\sigma \prec_1 \mathcal{B}$, the preservation of isolating formulas implies $\Psi = \tilde{\Psi}$. However, $\mathrm{SR}(\Phi(c)) = \sigma$ implies that the image of $\tilde{\Psi}$ is cofinal within the standard ordinals of $\mathcal{B}$. Then by an overspill argument, $\tilde{\Psi}$ must take on some illfounded ordinal. This contradicts $\Psi = \tilde{\Psi}$. The details of the proof are given below.

Let $\mathscr{J}$ be the language consisting of the following objects:

(i) A binary relation symbol $\dot{\in}$.
(ii) New constant symbol $\hat{a}$ for each element of $a \in L_\sigma$.
(iii) Two new constant symbols $\dot{c}$ and $\dot{d}$.

$\mathscr{J}$ can be considered a Δ_1 definable language in the countable admissible set L_σ.

Let H be the theory in the countable admissible fragment $(\mathscr{J}_{\infty\omega})^{L_\sigma}$ with the following sentences:

(I) All the axioms of $\mathsf{ZF} - \mathsf{P}$.
(II) For each $a \in L_\sigma$, $(\forall v)(v \dot{\in} \hat{a} \Leftrightarrow \bigvee_{z\in a} \hat{z} = v)$.
(III) Add the sentence "$\dot{c} \in 2^{\hat{\omega}}$". For each ordinal $\beta < \sigma$, add "$L_{\hat{\beta}}[\dot{c}] \not\models \mathsf{KP}$".
(IV) For each Π_1 formula $\varphi(x_0, ..., x_{k-1})$ of $\{\dot{\in}\}$ and elements $a_0, ..., a_{k-1} \in L_\sigma$ such that $L_\sigma \models \varphi(a_0, ..., a_{k-1})$, add the sentence "$\varphi(\hat{a}_0, ..., \hat{a}_{k-1})$".
(V) Add the sentence "$\dot{d}$ is an ordinal". For each $\beta < \sigma$, add "$\dot{d} > \hat{\beta}$".

The Σ_1-satisfaction relation of L_σ is a Σ_1 relation in L_σ. (See [10] Corollary 1.13.) Therefore, the Π_1-satisfaction relation of L_σ is Π_1 definable in L_σ. So H is a Π_1 and hence a Σ_2-definable subset of L_σ. Note that (III) states that each $\beta < \sigma$ is a not a $\dot{c}$-admissible ordinal. (IV) states that L_σ will be a Σ_1 elementary substructure of any model of H. (V) states that $\dot{d}$ is an ordinal larger than each $\beta < \sigma$.

(It will be seen below that one will only use the fact that isolating sentences in L_σ remain isolating sentences in models of H. One can rewrite (IV) to express this rather than attempt to obtain full Σ_1-elementarity. Also (I), (II), (III) and an argument similar to the one below constitute Jensen's proof (see [9]) of Sack's theorem.)

H is consistent: Let $\mathcal{B}$ be the $\mathscr{J}$-structure with underlying domain L_{ω_1}. Let $\dot{\in}^{\mathcal{B}} = \in \restriction L_{\omega_1}$. Let c be any real in $L_{\omega_1^L}$ so that $\omega_1^c > \sigma$. Let $\dot{c}^{\mathcal{B}} = c$. Let $\dot{d} = \sigma + 1$. For each $a \in L_\sigma$, let $\hat{a}^{\mathcal{B}} = a$. (I), (II), (III), and (V) are clearly satisfied in $\mathcal{B}$. Note that (IV) is satisfied since $L_\sigma \prec_1 L_{\omega_1^L}$.

By the Σ_2 version of Jensen's model existence theorem (Fact 2.7), there is a $\mathscr{L}$-structure $\mathcal{B}$ so that $\mathcal{B} \models H$, $\mathcal{B}$ end extends L_σ, $\mathrm{WF}(\mathcal{B})$ is transitive, and $\mathrm{ON} \cap \mathcal{B} = \sigma$. Let $c = \dot{c}^{\mathcal{B}}$ and $d = \dot{d}^{\mathcal{B}}$. By (V), d is a nonstandard ordinal. So $\mathcal{B}$ is an illfounded model. Note that $c \in \mathrm{WF}(\mathcal{B})$ since c is a real.

First, to show that $\omega_1^c = \sigma$: For each $\beta < \sigma$, $\mathcal{B} \models L_\beta[c] \not\models \mathsf{KP}$. Satisfaction is Δ_1 so by absoluteness, $\mathrm{WF}(\mathcal{B}) \models L_\beta[c] \not\models \mathsf{KP}$. Again by absoluteness, $V \models L_\beta[c] \not\models \mathsf{KP}$. Hence β is not a c-admissible ordinal. This shows that $\omega_1^c \geq \sigma$. Also by (I), $\mathcal{B} \models \mathsf{ZF} - \mathsf{P}$. In particular, $\mathcal{B} \models \mathsf{KP}$. By the truncation lemma (Fact 2.5), $\mathrm{WF}(\mathcal{B}) \models \mathsf{KP}$. Hence $\mathrm{WF}(\mathcal{B})$ is an admissible set containing c. So $\mathrm{ON} \cap \mathrm{WF}(\mathcal{B}) = \sigma$ is a c-admissible ordinal. This shows σ is the smallest c-admissible ordinal. By definition, $\omega_1^c = \sigma$.

Since Φ is Δ_1^1, $\Phi(c)$ is $\Delta_1^1(c)$. $\Phi(c)$ belong to any admissible set containing c. Thus $\Phi(c) \in \mathrm{WF}(\mathcal{B}) \subseteq \mathcal{B}$.

By Fact 3.6, $\mathrm{SR}(\Phi(c)) \geq \sigma$. Suppose toward a contradiction that $\mathrm{SR}(\Phi(c)) = \sigma$. Then $\Phi(c)$ is an atomic model of $T_\sigma^{\Phi(c)} = (T_\infty^\sigma)^{L_\sigma}$.

Since $\Phi(c) \in S(\mathscr{L})$, $\Phi(c)$ is an $\mathscr{L}$-structure with underlying domain ω. In $L_\sigma[\Phi(c)]$, define the function $\Psi : {}^{<\omega}\omega \to \mathrm{ON}$ by letting $\Psi(\bar{a})$ be the least ordinal α so that there is some $\varphi \in \mathscr{L}_\alpha^\sigma$ with φ an isolating formula for T_∞^σ in the fragment $\mathscr{L}_\infty^\sigma$ realized by the tuple $\bar{a}$.

Note that this is a well-defined function since $\Phi(c)$ is an atomic model of $(T_\infty^\sigma)^{L_\sigma[\Phi(c)]}$. (Also note that since L_σ and $L_\sigma[\Phi(c)]$ have the same ordinals and hence the same constructible universe, T^σ and $\mathscr{L}^\sigma$ are the same class whether relativized in L_σ or relativized in $L_\sigma[\Phi(c)]$.)

Now let $\tilde{\Psi}$ be the function defined in $\mathcal{B}$ by the same formula used to defined Ψ in $L_\sigma[\Phi(c)]$.

(Note that it is not immediately seen that $\tilde{\Psi}$ is the same function as Ψ since $\mathrm{ON}^{\mathcal{B}} \neq \sigma = \mathrm{ON}^{L_\sigma[\Phi(c)]}$. Hence $(T^\sigma_\infty)^{\mathcal{B}} \neq (T^\sigma_\infty)^{L_\sigma[\Phi(c)]}$ and $(\mathscr{L}^\sigma_\infty)^{\mathcal{B}} \neq (\mathscr{L}^\sigma_\infty)^{L_\sigma[\Phi(c)]}$. In particular, it is not immediate that a formula in $(\mathscr{L}^\sigma_\infty)^{L_\sigma[\Phi(c)]} = (\mathscr{L}^\sigma_\infty)^{L_\sigma}$ which isolates a type in L_σ would still isolate a type in the larger fragment $(\mathscr{L}^\sigma_\infty)^{\mathcal{B}}$. Σ_1-elementarity will be used to resolve this.)

The claim is that $\Psi = \tilde{\Psi}$: To see this, let $\bar{a}$ be a finite tuple of natural number understood to be a tuple from $\Phi(c)$. Since $\Phi(c)$ is an atomic model of $(T^\sigma_\infty)^{L_\sigma[\Phi(c)]} = (T^\sigma_\infty)^{L_\sigma}$, there is some $\varphi \in (\mathscr{L}^\sigma_\infty)^{L_\sigma}$ so that $[\varphi]^{T^\sigma_\infty}_{(\mathscr{L}^\sigma)^{L_\sigma}} = \{\mathrm{tp}^{\Phi(c)}_{(\mathscr{L}^\sigma_\infty)^{L_\sigma}}(\bar{a})\}$. $L_\sigma \models$ "φ is a isolating formula for $S_n^{\mathscr{L}^\sigma_\infty}(T^\sigma_\infty)$". By Fact 3.5, this statement is Π_1. Since $L_\sigma \prec_1 \mathcal{B}$. $\mathcal{B} \models$ "φ is an isolating formula of $S_n^{\mathscr{L}^\sigma_\infty}(T^\sigma_\infty)$".

This shows that in $\mathcal{B}$, $[\varphi]^{T^\sigma_\infty}_{\mathscr{L}^\sigma_\infty} = \{\mathrm{tp}^{\Phi(c)}_{\mathscr{L}^\sigma}(\bar{a})\}$. Hence $\mathcal{B} \models \tilde{\Psi}(\bar{a}) \leq \Psi(\bar{a})$. Suppose $\tilde{\Psi}(\bar{a}) < \Psi(\bar{a})$. There is some $\alpha < \Psi(\bar{a})$ and some formula $\psi \in (\mathscr{L}^\sigma_\alpha)^{\mathcal{B}}$ so that $\mathcal{B} \models$ "ψ is an isolating formula which isolates $\mathrm{tp}^{\Phi(c)}_{\mathscr{L}^\sigma}(\bar{a})$". Since $\Psi(\bar{a}) < \sigma$, $(\mathscr{L}^\sigma_\alpha)^{\mathcal{B}} = (\mathscr{L}^\sigma_\alpha)^{L_\sigma}$. Hence $\psi \in L_\sigma$. Then by downward absoluteness of Π_1 statements from $\mathcal{B}$ to $L_\sigma[\Phi(c)]$, $L_\sigma[\Phi(c)] \models$ "ψ is an isolating formula for $\mathrm{tp}^{\Phi(c)}_{\mathscr{L}^\sigma}(\bar{a})$". This contradicts the definition of Ψ in $L_\sigma[\Phi(c)]$.

This shows that $\Psi = \tilde{\Psi}$. Now suppose that $\Psi[{}^{<\omega}\omega] = \sigma$. Since $\mathcal{B} \models \mathsf{ZF} - \mathsf{P}$ (in particular the full replacement axiom), $\tilde{\Psi}[{}^{<\omega}\omega]$ is a set in $\mathcal{B}$. Therefore, $\sup \tilde{\Phi}[{}^{<\omega}\omega] \in \mathrm{ON}^{\mathcal{B}}$. Since $\tilde{\Psi} = \Psi$ and $\Psi[{}^{<\omega}\omega] = \sigma$, one must have that $\sup \tilde{\Psi}[{}^{<\omega}\omega]$ is a nonstandard ordinal greater than each $\beta < \sigma$. Hence there is some $b < \sup \tilde{\Psi}[{}^{<\omega}\omega]$ so that $\mathcal{B} \models \beta < b$ for all standard $\beta < \sigma$ with $b \in \tilde{\Psi}[{}^{<\omega}\omega]$. Thus there is some $\bar{a}$ so that $\Psi(\bar{a}) < b = \tilde{\Psi}(\bar{a})$. This contradicts $\Psi = \tilde{\Psi}$.

This shows that $\Psi[{}^{<\omega}\omega] < \sigma$ which implies $\mathrm{SR}(\Phi(c)) < \sigma$. However, it was already noted that $\mathrm{SR}(\Phi(c)) \geq \sigma$. Contradiction. This shows that $\mathrm{SR}(\Phi(c)) = \sigma + 1$. Since Φ is a classification, for any x with $\omega_1^x = \omega_1^c = \sigma$, $\Phi(x) \approx_{\mathscr{L}} \Phi(c)$. Hence $\mathrm{SR}(\Phi(x)) = \sigma + 1$. This completes the proof of the theorem. ☐

The following is still open.

Question 3.8. If $\mathscr{L}$ is a recursive language and $\Phi : {}^\omega 2 \to S(\mathscr{L})$ is a Δ^1_1 classification of F_{ω_1} by $\mathscr{L}$-structures, then for all $x \in {}^\omega 2$, is $\mathrm{SR}(\Phi(x)) = \omega_1^x + 1$?

Note that the lightface Δ^1_1 is important in the phrasing of the question. The F_{ω_1}-class $\{x : \omega_1^x = \omega_1^\emptyset\}$ is $\Delta^1_1(z)$ for any z such that $\omega_1^z > \omega_1^\emptyset$. Therefore, with access to such a parameter z, one can easily modify a known Δ^1_1 classification of F_{ω_1} to obtain a $\Delta^1_1(z)$ classification that sends all elements of $\{x : \omega_1^x = \omega_1^\emptyset\}$ to some fixed recursive structure of Scott rank $\omega_1^\emptyset$ and leaves the other classes alone.

It seems that if a classification Φ is $\Delta^1_1(z)$, then the relativization of the above question should be to ask the same question but only for those $x \in {}^\omega 2$ such that ω_1^x is z-admissible.

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