THE CLOPEN PARTITION PROPERTY

WILLIAM CHAN AND STEPHEN JACKSON

Abstract. A clopen parition of $[\omega_1]^{\omega_1}$ is a function $\Phi : [\omega_1]^{\omega_1} \rightarrow 2$ so that there exists two sets $A_0, A_1 \subseteq [\omega_1]^{<\omega_1}$ so that for all $f \in [\omega_1]^{\omega_1}$, $\Phi(f) = i$ if and only if there exists some $\alpha < \omega_1$ with $f \upharpoonright \alpha \in A_i$. Using only a simple coding of functions on $[\omega_1]^{<\omega_1}$, it will be shown that for any clopen partition $\Phi : [\omega_1]^{\omega_1} \rightarrow 2$, there is an uncountable $X \subseteq \omega_1$ so that $|\Phi[[X]^{\omega_1}]| = 1$.

1. The Clopen Partition

Definition 1.1. If $\alpha \leq \omega_1$, then $\omega_1 \rightarrow (\omega_1)^\alpha_2$ asserts that for all $\Phi : [\omega_1]^\alpha \rightarrow 2$, there is some $X \subseteq \omega_1$ with $|X| = \aleph_1$ and $|\Phi[[X]^\alpha]| = 1$.

The weak partition property on ω_1 is the statement that $[\omega_1]^\alpha \rightarrow (\omega_1)^\alpha_2$ for all $\alpha < \omega_1$.

The strong partition property on ω_1 is $\omega_1 \rightarrow (\omega_1)^{\omega_1}_2$.

A partition $\Phi : [\omega_1]^{\omega_1} \rightarrow 2$ is clopen if and only if there exist sets $A_0, A_1 \subseteq [\omega_1]^{<\omega_1}$ so that for all $f \in [\omega_1]^{\omega_1}$, $\Phi(f) = i$ if and only if there is an $\alpha < \omega_1$ so that $f \upharpoonright \alpha \in A_i$, for each $i \in 2$.

The clopen partition property on ω_1 states that for all clopen $\Phi : [\omega_1]^{\omega_1} \rightarrow$, there is an uncountable $X \subseteq \omega_1$ so that $|\Phi[[X]^{\omega_1}]| = 1$.

Fact 1.2. *(Martin)* AD *implies for each $\alpha < \omega_1$, $\omega_1 \rightarrow (\omega_1)^\alpha_2$ and $\omega_1 \rightarrow (\omega_1)^{\omega_1}_2$.*

Proof. See [1] and [2]. □

The clopen partition property on ω_1 follows from the strong partition property on ω_1 and implies the weak partition property on ω_1. Proofs of the strong partition property require some more elaborate coding of subsets of ω_1. The proof of the weak partition relation do not require any such coding since the length are fixed countable ordinals. The clopen partition do not involve uncountable subsets of ω_1 but must handle arbitrary long countable sequences. The following proves the clopen partition property for ω_1 using a trivial coding of arbitary countable sequences of countable ordinals.

Definition 1.3. Let $C \subseteq \omega_1$ be a club subset of ω_1. Define

$$\tilde{C} = \{\alpha \in C : (\exists \eta < \omega_1)(\alpha = \sup\{C(\omega \cdot \eta + n) : n \in \omega\})\}$$

where $C(\alpha)$ denotes the α^{th} element of C.

Let $\alpha \in \omega_1$. Let $g : \omega \cdot \alpha \rightarrow \omega_1$. Define $\tilde{g}(\beta) = \sup\{f(\omega \cdot \beta + n) : n \in \omega\}$ for each $\beta < \alpha$.

Let $\alpha \in \omega_1$ and $f \in [\tilde{C}]^\alpha$. A C-witness for f is a function $g \in [C]^{\omega \cdot \alpha}$ so that $\tilde{g} = f$.

Definition 1.4. Fix a recursive coding $\langle \cdot, \cdot \rangle$ of $\omega \times \omega$ by ω throughout.

Let WO denote the reals coding wellorderings on ω. If $X \subseteq \omega_1$, let WO_X denote the subset of WO coding ordinals in X.

Let $x \in \mathbb{R} = {}^\omega\omega$. Let $(x)_n \in \mathbb{R}$ be defined by $(x)_n(m) = x(\langle n, m \rangle)$. For each $n \in \omega$, let $x^0_n, x^1_n \in \mathbb{R}$ be such that $x^i_n(m) = (x)_n(\langle i, m \rangle)$.

For $x \in \mathbb{R}$, let $\mathrm{dom}(x)$ be the largest ordinal α so that the following holds:
(1) For all $\beta < \alpha$, there exists some $n \in \omega$ so that $x^0_n \in \mathrm{WO}$ and $\mathrm{ot}(x^0_n) = \beta$.
(2) For all $\beta < \alpha$, if $\mathrm{ot}(x^0_n) = \beta$, then $x^1_n \in \mathrm{WO}$.
(3) For all $\beta < \alpha$, if $m, n \in \omega$ is such that $\mathrm{ot}(x^0_m) = \mathrm{ot}(x^0_n) = \beta$, then $\mathrm{ot}(x^1_m) = \mathrm{ot}(x^1_n)$.

For $x \in \mathbb{R}$, let $f_x : \mathrm{dom}(x) \rightarrow \omega_1$ be defined by $f_x(\beta) = \mathrm{ot}(x^1_n)$ where n is such that $\mathrm{ot}(x^0_n) = \beta$.

For $x \in \mathbb{R}$, define $\mathrm{dom}^*(x)$ to be the largest ordinal α so that for all $\beta < \alpha$, there exists some $n \in \omega$ with $\mathrm{ot}(x^0_n) = \beta$.

March 8, 2018.

Theorem 1.5. *Suppose $\Phi : [\omega_1]^{\omega_1} \to 2$ is a clopen partition. Then there exists an uncountable $X \subseteq \omega_1$ so that $|\Phi[[X]^{\omega_1}]| = 1$.*

Proof. Let $A_0, A_1 \subseteq \omega_1$ be the subsets of $[\omega_1]^{<\omega_1}$ that witness the clopenness of Φ. Consider the following game where player 1 and 2 takes turn playing integers with Player 1 producing a real x and player 2 producing a real y.

$$\begin{array}{ll} \text{Player 1} & x \\ \text{Player 2} & y \end{array}$$

Let $\zeta_x = \mathrm{dom}(x)$, $\zeta_y = \mathrm{dom}(y)$, $\zeta_x^* = \mathrm{dom}^*(x)$, and $\zeta_y^* = \mathrm{dom}^*(y)$. Let $\zeta = \min\{\zeta_x, \zeta_y\}$ and $\zeta^* = \min\{\zeta_x^*, \zeta_y^*\}$. Let ξ be the largest ordinal such that $\omega \cdot \xi \leq \zeta$. For $\beta < \omega \cdot \xi$, let $h_{x,y}(\beta) = \sup\{f_x(\beta), f_y(\beta)\}$.

The payoff set is described as follows:

(Case a) If $\zeta < \zeta^*$, then Player 1 win if and only if $\zeta_y < \zeta_x$.

(Case b) Suppose that $\zeta = \zeta^*$.

(Subcase b1) Suppose that $\tilde{h}_{x,y}$ has no segment that belong to A_0 or A_1. Player 1 wins if and only if $\zeta_y < \zeta_x$.
(Subcase b2) Suppose some intial segment of $\tilde{h}_{x,y}$ has an initial segment in A_0 or A_1. Player 1 wins if and only if that segment belongs to A_0.

Suppose that τ is a winning strategy for Player 2 in this game. If $x \in \mathbb{R}$, then let $\tau(x)$ denote the Player 2 response that comes from playing τ against x in $x * \tau$.

For $\alpha \leq \beta$, let $A_{\alpha,\beta}$ be the set of $r \in \mathbb{R}$ so that there exists some $x \in \mathbb{R}$ with the following properties:
(i) $\alpha \in \mathrm{dom}(x) \wedge (\forall n)(\mathrm{ot}(x_n^0) \leq \alpha \Rightarrow \mathrm{ot}(x_n^1) \leq \beta)$.
(ii) $\alpha \in \mathrm{dom}^*(\tau(x)) \wedge (\exists n)(\mathrm{ot}((\tau(x))_n^0) = \alpha \wedge r = (\tau(x))_n^1)$.

Although the statement "$\alpha \in \mathrm{dom}(x)$" is not generally $\mathbf{\Sigma}_1^1$, (i) however is $\mathbf{\Sigma}_1^1$ using a code for β because it is essentially stating that $\alpha \in \mathrm{dom}(x)$ is witnessed by the fact that the image of α under f_x is bounded by β. Hence $A_{\alpha,\beta}$ is $\mathbf{\Sigma}_1^1$.

Next, the claim is that $A_{\alpha,\beta} \subseteq \mathrm{WO}$: This amounts to showing that $\alpha \in \mathrm{dom}(\tau(x))$. Suppose not. Note that (i) implies that $\alpha < \zeta_x \leq \zeta_x^*$. If $\alpha \notin \mathrm{dom}(\tau(x))$, then $\zeta_{\tau(x)} \leq \alpha < \zeta_{\tau(x)}^*$. Then $\zeta \leq \alpha < \zeta^*$. However, $\zeta_{\tau(x)} < \zeta_x$. Hence by Case a, Player 1 wins. Contradiction.

By the boundedness principle, let $\delta_{\alpha,\beta}$ be the least ordinal such that for all $r \in \delta_{\alpha,\beta}$, $\mathrm{ot}(r) < \delta_{\alpha,\beta}$. Let $C = \{\eta : (\forall \alpha \leq \beta < \eta)(\delta_{\alpha,\beta} < \eta)\}$. C is a club set.

The claim is that $\Phi[[\tilde{C}]^{\omega_1}] = \{1\}$: Suppose this was not true. Then there is some $f \in [\tilde{C}]^{\omega_1}$ so that $\Phi(f) = 0$. Since Φ is clopen, there is some ϵ so that $f \restriction \epsilon \in A_0$. Let $j \in [C]^{\omega \cdot \epsilon}$ be a C-code for $f \restriction \epsilon$, that is, $\tilde{j} = f \restriction \epsilon$. Let $x \in \mathbb{R}$ code j exactly in the sense that $\mathrm{dom}(x) = \mathrm{dom}^*(x) = \omega \cdot \epsilon$. Play $x * \tau$.

Claim 1: $\zeta_{\tau(x)} \geq \omega \cdot \epsilon$ and $\tilde{h}_{x,\tau(x)} = f \restriction \epsilon$.

To prove this, suppose for a $\gamma < \epsilon$, one has shown that $\tilde{h}_{x,\tau(x)} \restriction \gamma = f \restriction \gamma$.

Subclaim 1.1: $\omega \cdot \gamma + n \in \mathrm{dom}^*(\tau(x))$.

To see this: Suppose not. Then one has that $\zeta = \min\{\zeta_x, \zeta_{\tau(x)}\} = \min\{\omega \cdot \epsilon, \zeta_{\tau(x)}\} = \zeta_{\tau(x)}$. One then must have that $\zeta_{\tau(x)}^* = \zeta_{\tau(x)}$. For otherwise, $\zeta < \zeta^*$ but $\zeta_{\tau(x)} < \zeta_x = \omega \cdot \epsilon$. By Case a, Player 1 has won. Now $\zeta_{\tau(x)} = \zeta$ implies that $\zeta = \zeta^*$. In the construction of $h_{x,y}$, the ordinal ξ is then γ. One has that $\tilde{h}_{x,\tau(x)} = \tilde{h}_{x,\tau(x)} \restriction \gamma = f \restriction \gamma$. If $f \restriction \gamma \in A_0$, then Player 1 has won by Subcase b2. Suppose $f \restriction \gamma \notin A_0$. Since $f \restriction \epsilon \in A_0$, it is impossible that $f \restriction \gamma \in A_1$. So Subcase b1 holds. However, $\zeta_{\tau(x)} < \omega \cdot \gamma + n < \omega \cdot \epsilon = \zeta_x$. Player 1 wins. Contradiction. Subclaim 1.1 has been established.

Now it has been shown that
(A) $\omega \cdot \gamma + n \in \mathrm{dom}(x) \wedge (\forall n)(\mathrm{ot}(x_n^0) \leq \omega \cdot \gamma + n \Rightarrow \mathrm{ot}(x_n^1) \leq j(\omega \cdot \gamma + n))$.
(B) $\omega \cdot \gamma + n \in \mathrm{dom}^*(\tau(x))$.

By the definition of $j(\omega \cdot \gamma + n + 1) \in C$, one has that

$$f_{\tau(x)}(\omega \cdot \gamma + n) < \delta_{\omega \cdot \gamma + n, j(\omega \cdot \gamma + n)} < j(\omega \cdot \gamma + n + 1) = f_x(\omega \cdot \gamma + n + 1).$$

It has been shown that $\tilde{h}_{x,\tau(x)}(\gamma) = \tilde{j}(\gamma) = f(\gamma)$. Claim 1 has been proved.

It has been shown that $\zeta_{\tau(x)} \geq \omega \cdot \epsilon$. Since $\zeta_x = \zeta_x^* = \omega \cdot \epsilon$, one has that $\zeta = \zeta^* = \zeta_x = \omega \cdot \epsilon$. Hence, one is in Case b. The ξ associated with $\tilde{h}_{x,\tau(x)}$ is ϵ and $\tilde{h}_{x,\tau(x)} = f \restriction \epsilon \in A_0$ by Claim 1. By Subcase b2, Player 1 has won. Contradiction.

The argument is similar if Player 1 has a winning strategy. This completes the proof. □

References

1. Steve Jackson, *A new proof of the strong partition relation on ω_1*, Trans. Amer. Math. Soc. **320** (1990), no. 2, 737–745. MR 972702
2. Eugene M. Kleinberg, *Infinitary combinatorics and the axiom of determinateness*, Lecture Notes in Mathematics, Vol. 612, Springer-Verlag, Berlin-New York, 1977. MR 0479903

Department of Mathematics, University of North Texas, Denton, TX 76203
Email address: William.Chan@unt.edu

Department of Mathematics, University of North Texas, Denton, TX 76203
Email address: Stephen.Jackson@unt.edu