CONSTRUCTIBILITY LEVEL EQUIVALENCE RELATION

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Abstract. Assume $\mathsf{ZF} + {}^\omega 2 = ({}^\omega 2)^L$. Define an equivalence relation E_L on ${}^\omega 2$ by $x \ E_L \ y$ if and only if for all admissible levels L_α of Gödel's constructible hierarchy, $x \in L_\alpha$ if and only if $y \in L_\alpha$. E_L is a thin Δ^1_2 equivalence relation which is not $\mathbf{\Pi}^1_1$. E_L has the property that for all $\mathbf{\Sigma}^1_1$ sets B, $[B]_E = \{y \in {}^\omega 2 : (\exists x \in B)(y \ E \ x)\}$ is either a countable or a co-countable set. There is no coloring $c : {}^\omega 2 \to \omega$ of E_L whose graph is $\mathbf{\Sigma}^1_1$.

1. Constructibility Level Equivalence Relation

Assume $\mathsf{ZF} + {}^\omega 2 = ({}^\omega 2)^L$, that is, all reals belong to Gödel's constructible universe L. The following equivalence relation is studied by the author in [1] Section 9.

Definition 1.1. Let KP denote Kripke-Platek set theory. If $x \in {}^\omega 2$, then let $\iota(x)$ be the least ordinal α so that $L_\alpha \models \mathsf{KP}$ and $x \in L_\alpha$. (Any sufficiently strong fragment of ZFC such as $\mathsf{ZFC} - \mathsf{P}$ would suffice in place of KP.) Define an equivalence relation E_L on ${}^\omega 2$ by $x \ E_L \ y$ if and only if $\iota(x) = \iota(y)$.

This short note will collect some facts about E_L which answers some questions of Pikhurko and Tserunyan. E_L is a Π^1_2 equivalence relation and using an idea of Drucker [2], E_L is also Σ^1_2 and hence Δ^1_2.

Fact 1.2. *E_L is a Δ^1_2 equivalence relation.*

Proof. Recall that a Σ^1_2 subset of ${}^\omega 2$ is equivalently a set of reals which is Σ_1 definable over $H_{\aleph_1}$, the collection hereditarily countable sets.

Let $\psi(x, y)$ be the formula which assert that for all transitive set A, if $A \models \mathsf{KP} + V = L$, then $x \in A$ if and only if $y \in A$. Note that φ is a Π_1 formula in the language of set theory. $x \ E_L \ y$ if and only if $H_{\aleph_1} \models \psi(x, y)$. E_L is Π_1 definable in $H_{\aleph_1}$ and thus is Π^1_2.

Also $x \ E_L \ y$ if and only if $H_{\aleph_1} \models$ there is a transitive set M with $x, y \in M$, $M \models \mathsf{KP} + V = L$, and $M \models \psi(x, y)$. Since first order satisfaction is Δ_1 in the language of set theory, E_L is Σ_1 definable in $H_{\aleph_1}$ and hence Σ^1_2. ☐

Fact 1.3. *E_L has all classes countable and has uncountably many classes.*

Proof. For any $x \in {}^\omega 2$, $[x]_{E_L} \subseteq L_{\iota(x)}$. Since $|L_{\iota(x)}| \leq \aleph_0$, $|[x]_{E_L}| \leq \aleph_0$. Since E_L has all classes countable and ${}^\omega 2$ is uncountable, E_L must have uncountably many classes. ☐

The following is the main tool for studying E_L. A perfect tree p on 2 is a subset of ${}^{<\omega}2$ so that for all $t \in p$ if $s \subseteq t$ then $s \in p$ and for all $s \in p$, there is a $t \in p$ so that $t\hat{}0, t\hat{}1 \in p$. A $t \in p$ so that $t\hat{}0, t\hat{}1 \in p$ is called a split node of p.

If p is a perfect tree on 2, let $\Upsilon_p : {}^{<\omega}2 \to p$ be defined by recursion as follows: Let $\Upsilon_p(\emptyset)$ be the shortest split node of p. If $\Upsilon_p(s)$ has been defined, then let $\Upsilon_p(s\hat{}i)$ be the shortest split node of p extending $\Upsilon_p(s)\hat{}i$. Let $\Xi_p : {}^\omega 2 \to [p]$ be defined by $\Xi_p(f) = \bigcup_{n\in\omega} \Upsilon_p(f \upharpoonright n)$. Ξ_p is the canonical homeomorphism between ${}^\omega 2$ and $[p]$.

Lemma 1.4. *For any perfect tree p on 2, there is a $\zeta < \omega_1$ so that for all $x \in {}^\omega 2$ with $\iota(x) \geq \zeta$, $\iota(x) = \iota(\Xi_p(x))$.*

Proof. The set $\{x \in {}^\omega 2 : \iota(\Xi_p(x)) < \iota(p)\}$ is countable. Let $\zeta' = \sup\{\iota(x) + 1 : \iota(\Xi_p(x)) < \iota(p))\}$. Let $\zeta = \max\{\iota(p), \zeta'\}$. Now suppose $\iota(x) \geq \zeta$. Since $p, x \in L_{\iota(x)}$ and $L_{\iota(x)} \models \mathsf{KP}$, $\Xi_p(x) \in L_{\iota(x)}$. Thus $\iota(\Xi_p(x)) \leq \iota(x)$. Since $\iota(x) \geq \zeta$, $\iota(\Xi_p(x)) \geq \iota(p)$. Thus $p, \Xi_p(x) \in L_{\iota(\Xi_p(x))}$. Since $L_{\iota_p(\Xi_p(x))} \models \mathsf{KP}$, one

November 20, 2020.

can recover $x = \Xi_p^{-1}(\Xi_p(x))$ within $L_{\iota(\Xi_p(x))}$ by using $\Xi_p(x)$ and the tree p. Thus $\iota(x) \leq \iota(\Xi_p(x))$. So $\iota(x) = \iota(\Xi_p(x))$. □

Lemma 1.5. *Let p be a perfect tree on 2. There is a $\zeta < \omega_1$ so that for all $\xi \geq \zeta$, there exist countably infinite many $z \in [p]$ so that $\iota(z) = \xi$.*

Proof. By Lemma 1.4, let $\zeta < \omega_1$ be such that for all $x \in {}^\omega 2$, if $\iota(x) \geq \zeta$, then $\iota(x) = \iota(\Xi_p(x))$. Let $\xi \geq \zeta$ and let $A = \{x \in {}^\omega 2 : \iota(x) = \xi\}$ which is a countably infinite set. Since Ξ_p is an injection, if $x \neq y$, then $\Xi_p(x) \neq \Xi_p(y)$. If $x \in A$, then $\Xi_p(x) \in [p]$ and $\iota(\Xi_p(x)) = \iota(x) = \xi$. □

An equivalence relation E on ${}^\omega 2$ is thin if and only if there is no perfect tree p on 2 so that for all $x, y \in [p]$ with $x \neq y$, $\neg(x \; E \; y)$. The following is a different argument from [1] showing that E_L is thin.

Fact 1.6. *([1] Proposition 9.7) E_L is a thin equivalence relation.*

Proof. By Lemma 1.5, every perfect tree p has some $z_0, z_1 \in [p]$ so that $z_0 \neq z_1$ and $\iota(z_0) = \iota(z_1)$. Thus $z_0 \; E_L \; z_1$. □

Fact 1.7. *E_L is not $\mathbf{\Pi}^1_1$.*

Proof. Since Fact 1.3 implies E_L has uncountable many classes, if E_L was $\mathbf{\Pi}^1_1$, then Silver's dichotomy ([3] and [4]) implies that E_L is not thin. This is impossible since Fact 1.6 asserts that E_L is thin. □

Fact 1.8. *If $B \subseteq {}^\omega 2$ is $\mathbf{\Sigma}^1_1$, then $[B]_{E_L} = \{y \in {}^\omega 2 : (\exists x \in B)(y \; E_L \; x)\}$ is either countable or co-countable.*

Proof. If B is countable, then the set $K = \{\iota(x) : x \in B\}$ is countable. So $[B]_{E_L} \subseteq L_{\sup(K)}$. $[B]_{E_L}$ is countable. Suppose B is not countable. By the perfect set property for $\mathbf{\Sigma}^1_1$, there is a perfect tree p on 2 so that $[p] \subseteq B$. By Lemma 1.5, there is a ζ so that for all $\xi \geq \zeta$, there exists a $z \in [p]$ with $\iota(z) = \xi$. Let $H = \{x \in {}^\omega 2 : \iota(x) \geq \zeta\}$ which is a co-countable set and observe that $[B]_{E_L} \supseteq [[p]]_{E_L} \supseteq H$. $[B]_{E_L}$ is co-countable. □

Pikhurko asked whether an equivalence relation E with all classes countable and has the property that for all $\mathbf{\Delta}^1_1$ sets B, $[B]_E$ is $\mathbf{\Delta}^1_1$ must be a $\mathbf{\Delta}^1_1$ equivalence relation. The results above show that it is consistent with ZF that the answer is no.

A relation $G \subseteq {}^\omega 2 \times {}^\omega 2$ is a graph if and only if for all $x, y \in {}^\omega 2$, $\neg(x \; G \; x)$ and $x \; G \; y$ implies $y \; Gx$. If $A \subseteq {}^\omega 2$, let $N_G(A) = \{x \in {}^\omega 2 : (\exists y \in A)(x \; E \; y)\}$ and $N^{\leq 1}_G(A) = A \cup N_G(A)$. The graph G is locally countable if and only if $|N_G(\{x\})| \leq \aleph_0$ for all $x \in {}^\omega 2$. If E is an equivalence relation on ${}^\omega 2$, then the graph G_E associated to E is defined on ${}^\omega 2$ by $x \; G_E \; y$ if and only if $x \neq y \wedge x \; E \; y$. Note that if E is an equivalence relation, then for any $A \subseteq {}^\omega 2$, $N^{\leq 1}_G(A) = [A]_E$. However, $N_G(A)$ may not be $[A]_E$. For example, if $x \in A$ and $[x]_E \cap A = \{x\}$, then $x \notin N_G(A)$.

Pikhurko asked if a locally countable graph has the property for all $\mathbf{\Delta}^1_1$ $B \subseteq {}^\omega 2$, $N^{\leq 1}_G(B)$ is $\mathbf{\Delta}^1_1$, then is G a $\mathbf{\Delta}^1_1$ graph. Since E_L is not $\mathbf{\Pi}^1_1$, G_{E_L} is not $\mathbf{\Pi}^1_1$. Thus the above result implies that it is consistent with ZF that the answer is no. He also asked if G is a locally countable graph with the property that for all $\mathbf{\Delta}^1_1$ $B \subseteq {}^\omega 2$, $N_G(A)$ is $\mathbf{\Delta}^1_1$, then is G a $\mathbf{\Delta}^1_1$ graph. Using the next result, the answer being no is consistent with ZF.

Fact 1.9. *If $B \subseteq {}^\omega 2$ is $\mathbf{\Sigma}^1_1$, then $N_{G_L}(B)$ is countable or co-countable.*

Proof. If B is countable, then $N_G(B) \subseteq N^{\leq 1}_G(B) = [B]_{E_L}$ which is countable as noted in Fact 1.8. Suppose B is uncountable. By the perfect set property for $\mathbf{\Sigma}^1_1$, there is a perfect tree p on 2 so that $[p] \subseteq B$. By Lemma 1.5, there is a $\zeta < \omega_1$ so that for all $\xi \geq \zeta$, there exists at least two elements $z_0 \in [p]$ and $z_1 \in [p]$ so that $\iota(z_0) = \iota(z_1) = \xi$. Fix a $\xi \geq \zeta$ and let $z_0, z_1 \in [p]$ with $z_0 \neq z_1$ and $\iota(z_0) = \iota(z_1) = \xi$. Then $N_G(\{z_0\}) = \{x \in {}^\omega 2 : x \neq z_0 \wedge \iota(x) = \xi\}$ and $N_G(\{z_1\}) = \{x \in {}^\omega 2 : x \neq z_1 \wedge \iota(x) = \xi\}$. So $\{x \in {}^\omega 2 : \iota(x) = \xi\} = N_G(\{z_0\}) \cup N_G(\{z_1\}) \subseteq N_G(B)$. Let $H = \{x \in {}^\omega 2 : \iota(x) \geq \zeta\}$ which is co-countable. Then $H \subseteq N_G(B)$ and so $N_G(B)$ is co-countable. □

A coloring of a graph G is a map $c : {}^\omega 2 \to X$, where X is some Polish space, so that for all $x, y \in {}^\omega 2$, if $x \; G \; y$, then $c(x) \neq c(y)$. Tserunyan asked if G_{E_L} has a $\mathbf{\Delta}^1_1$ coloring $c : {}^\omega 2 \to \omega$. The following result shows that the answer is no.

Fact 1.10. *There is no coloring $c : {}^\omega 2 \to \omega$ of E_L so that graph of c is $\mathbf{\Sigma}^1_1$.*

Proof. Since ${}^\omega 2 = \bigcup_{n\in\omega} c^{-1}[\{n\}]$, there is an $n \in \omega$ so that $c^{-1}[\{n\}]$ is uncountable. Since the graph of c is $\mathbf{\Sigma}^1_1$, $c^{-1}[\{n\}]$ is an uncountable $\mathbf{\Sigma}^1_1$ set. By the perfect set property for $\mathbf{\Sigma}^1_1$, there is a perfect tree p on 2 so that $[p] \subseteq c^{-1}[\{n\}]$. By Lemma 1.5, there exist $z_0, z_1 \in [p]$ with $z_0 \neq z_1$ and $\iota(z_0) = \iota(z_1)$. Hence $z_0 \; G_{E_L} \; z_1$ and $c(z_0) = n = c(z_1)$ which contradict c being a coloring. □

References

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