JENSEN'S MODEL EXISTENCE THEOREM

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Abstract. This expository notes follows and expands on [2] Chapter 4 which provides an introduction to Barwise's theory of infinitary logic in countable admissible fragments, the Jensen's model existence theorem, and a proof of Sacks' theorem as an application.

1. Introduction

This note will give an expanded exposition of Barwise's theory of infinitary logic in countable admissible fragments and the Jensen's model existence theorem as it appears in chapter 4 of Jensen's handwritten notes [2].

Jensen's model existence theorem concerns languages $\mathscr{L}$ and theories T in $\mathscr{L}_{\infty\omega}$ which are sufficiently definable in some countable admissible set $\mathcal{A}$ such that $\mathscr{L}$ contains a distinguished binary relation symbol $\dot{\in}$, T states $\dot{\in}$ behaves like set membership, and T codes $\mathcal{A}$. This result says if T is consistent, then there is a model of T end-extending $\mathcal{A}$ whose wellfounded part has the same ordinal height as the original set $\mathcal{A}$.

This note concludes with an application of Jensen's model existence theorem to prove a result of Sacks which asserts that every countable admissible ordinal α is of the form ω_1^x for some real x. This example begins to show how Jensen's model existence theorem can be combined to prove results about countable model theory while building an admissible set that helps control the admissibility or hyperarithmetic content of the relevant objects.

All results of this notes are due to Barwise, Jensen, or Sacks. This notes follows Chapter 4 of [2].

In conversation with Jensen, the author was informed that Jensen had formulated the Jensen model existence theorem and applied it to prove Sacks' theorem as well as the other applications to set theory that appear in Chapter 4 of [2]. However, he was unable to apply this method to characterize the countable sequences of countable admissible ordinals. Rather, Jensen used proper class forcing over admissible sets to show that if $\langle \alpha_\xi : \xi < \lambda \rangle$, where $\lambda < \omega_1$, is a countable sequence of countable admissible ordinals, then there is some real r such that α_ξ is ξ^{th} r-admissible ordinal. See [2] Chapter 5 and 6.

This introduction ends with the following quote of Jensen from [2] Chapter 4: "In this chapter we give an introduction to this theory, whose potential for set theory has, we feel, been sadly neglected."

2. Syntax and Semantics of Infinitary Logic in Admissible Fragments

This section will provide a brief sketch of the syntax and semantic of infinitary logic.

KP is a weak axiom system of set theory. Besides some basic axioms, the most notable axioms of KP are the axiom schemes of Σ_1-collection and Δ_1-separation. For $n \geq 1$, let Σ_n-KP consists of the basic axioms along with the axiom schemes of Σ_n-collection and Δ_n-separation. An admissible set (or Σ_n-admissible set) is a transitive set that models KP (or Σ_n-KP).

Let X be a set. An ordinal α is X-admissible if and only if α is the ordinal height of an admissible set containing X, i.e. there is some admissible set $\mathcal{A}$ with $X \in \mathcal{A}$ and $\alpha = \mathcal{A} \cap \mathrm{ON}$. If $x \in {}^\omega 2$, let ω_1^x denote the least x-admissible ordinal. If $\mathcal{B} \models \mathsf{KP}$ and $X \in \mathcal{B}$, then $(L[X])^{\mathcal{B}}$, the class of sets constructible from X in the sense of $\mathcal{B}$, is also a model of KP. If $\mathcal{B} \models \mathsf{KP}$, then $\mathrm{WF}(\mathcal{B})$, the wellfounded (in the real world) part of $\mathcal{B}$, is also a model of KP from the truncation lemma. See [1] for more about the basic properties of KP and admissible sets.

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Let $\mathscr{L}$ be a definable class of constant, relation, and function symbols. Fix a Δ_0-definable proper class of symbols for variables. $\mathscr{L}$-terms and $\mathscr{L}$-atomic formulas are defined as usual using the variables mentioned above.

By recursion, the class of infinitary $\mathscr{L}$-formulas, denoted $\mathscr{L}_{\infty\omega}$, is defined to be the smallest class with the following properties.

(I) The atomic $\mathscr{L}$-formulas are infinitary formulas.

(II) If φ is an infinitary formula, then $\neg\varphi$ is an infinitary formula.

(III) If φ is an infinitary formula and v is a variable, then $(\exists v)\varphi$ and $(\forall v)\varphi$ are infinitary formulas.

(IV) If u is a set of infinitary formulas such that only finitely many different free variables appear in any formula of u, then $\bigwedge u$ and $\bigvee v$ are infinitary formulas.

Fact 2.1. (Σ_n-KP) *Suppose $\mathscr{L}$ is a Δ_n-definable language. Then $\mathscr{L}_{\infty\omega}$ is a Δ_n-definable class.*

Proof. See [1] Chapter III Section 1 or [2] Chapter 4. $\square$

The logical axioms of infinitary logic consists of the usual first order axioms and the following:
For all sets $u \subseteq \mathscr{L}_{\infty\omega}$ and $\varphi \in u$, $\bigwedge u \Rightarrow \varphi$ and $\varphi \Rightarrow \bigvee u$ are logical axioms.

Let $T \subseteq \mathscr{L}_{\infty\omega}$. Let $\varphi \in \mathscr{L}_{\infty\omega}$. Let $T \vdash \varphi$ if and only if φ is included in the smallest class $P \subseteq \mathscr{L}_{\infty\omega}$ which contains T, the logical axioms, and closed under the following rules of inference:

(I) If $\varphi \in P$ and $\varphi \Rightarrow \psi \in P$, then $\psi \in P$.

(II) Suppose v is not a free variable of φ. If $\varphi \Rightarrow \psi \in P$, then $\varphi \Rightarrow (\forall v)\psi \in P$, and if $\psi \Rightarrow \varphi \in P$, then $(\exists v)\psi \Rightarrow \varphi \in P$.

(III) Suppose $u \subseteq \mathscr{L}_{\infty\omega}$ and u is a set. Suppose for all $\varphi \in u$, $\psi \Rightarrow \varphi \in P$, then $\psi \Rightarrow \bigwedge u \in P$. Suppose for some $\varphi \in u$, $\varphi \Rightarrow \psi \in P$, then $\bigvee u \Rightarrow \psi \in P$.

Definition 2.2. Let $T \subseteq \mathscr{L}_{\infty\omega}$ be a class. Let $\varphi \in \mathscr{L}_{\infty\omega}$. A proof from T is a sequence $\langle P_\alpha : \alpha < \beta\rangle$ where $\beta \in \mathrm{ON}$ with the property that for all $\alpha < \beta$, for all $\psi \in P_\alpha$, either
(1) $\psi \in T$,
(2) ψ is a logical axiom, or
(3) ψ results from $\bigcup_{\gamma<\alpha} P_\gamma$ by a single rule of inference.

$\langle P_\alpha : \alpha < \beta\rangle$ is a proof of φ from T if and only if $\varphi \in \bigcup_{\alpha<\beta} P_\alpha$.

(The notion of a proof is formulated so that each P_α is a set of infinitary formula rather than a single formula because the axiom of choice is not assumed.)

Fact 2.3. (Σ_n-KP) *Let $\mathscr{L}$ be a Δ_n-definable language. Let $T \subseteq \mathscr{L}_{\infty\omega}$ be a Σ_n-definable set of infinitary sentences. Let $\varphi \in \mathscr{L}_{\infty\omega}$ be a sentence. $T \vdash \varphi$ if and only if there is a proof $\langle P_\alpha : \alpha < \beta\rangle$ of φ from T which is a set.*

Proof. Let X be the set of all sentences $\varphi \in \mathscr{L}_{\infty\omega}$ which has a proof from T. It suffices to show that X is a class containing T, the logical axioms, and closed under the rules of inferences since by definition the class of φ so that $T \vdash \varphi$ is the smallest class with these properties.

X contains T and the logical axioms. It remains to show that X is closed under the rules of inferences. As an example on how Σ_n-KP is used, the following establishes rule of inference (III):

Let $\mathrm{Pr}(p,\gamma)$ assert that p is a proof of γ from T. As T is Σ_n, Pr is Σ_n. Let $u \subseteq X$ be a set such that for all $\psi \in u$, $\varphi \Rightarrow \psi \in X$. Hence $(\forall\psi \in u)(\exists p)\mathrm{Pr}(p, \varphi \Rightarrow \psi)$. By Σ_n-collection and Σ_n-reflection (see [1] Theorem I.4.4), there exists a set v so that $(\forall\psi \in u)(\exists p \in v)\mathrm{Pr}(p, \varphi \Rightarrow \psi)$ and $(\forall p \in v)(\exists\psi \in u)\mathrm{Pr}(p, \varphi \Rightarrow \psi)$. The latter implies v consists entirely of proofs from T. By Σ_1-collection, there is some ordinal α so that α is greater than the length of all proofs in v. Using Δ_1-separation, for $\beta < \alpha$, let $p^*(\beta) = \{\gamma \in \bigcup\bigcup v : (\exists p \in v)(\beta \in \mathrm{dom}(p) \wedge \gamma \in p(\beta)\}$. Let $p^*(\alpha) = \{\varphi \Rightarrow \bigwedge u\}$. Then p^* is a proof of length $\alpha + 1$ of $\varphi \Rightarrow \bigwedge u$ from T. $\varphi \Rightarrow \bigwedge u \in X$. $\square$

Fact 2.4. (Σ_n-KP) *Let $\mathscr{L}$ be a Δ_n-language and T be a Σ_n-definable class of $\mathscr{L}_{\infty\omega}$-sentences. Let φ be an $\mathscr{L}_{\infty\omega}$ sentence. $T \vdash \varphi$ if and only if there is a set $u \subseteq T$ such that $u \vdash \varphi$.*

Proof. By Fact 2.3, there is a set proof p of φ from T. Using Δ_n-separation, let u be the set of $\psi \in \bigcup_{\alpha<\mathrm{dom}(p)} p(\alpha)$ such that there exists some $\alpha < \mathrm{dom}(p)$ so that $\psi \in p(\alpha)$, ψ is not a logical axiom, ψ does not follow from $\bigcup_{\beta<\alpha} p(\beta)$ by one application of a rule of inference. From the definition of a proof from T, $u \subseteq T$ and p is a proof of φ from u. $\square$

Fact 2.5. (Σ_n-KP) *Let $\mathscr{L}$ be a Δ_n-language and T be a Σ_n-definable class (both using a set p as a parameter).*

The set $\{\varphi : T \vdash \varphi\}$ is Σ_n (in the parameter p).

The class $\{(u, \varphi) : u \subseteq \mathscr{L}_{\infty\omega} \wedge \varphi \in \mathscr{L}_{\infty\omega} \wedge u \vdash \varphi\}$ is Σ_n (in the parameter p).

The statement "T is consistent" is Π_n (in the parameter p).

Proof. By Fact 2.3, $T \vdash \varphi$ if and only if there exists a proof P of φ from T. $\square$

Let $\mathscr{L}$ be a language. A $\mathscr{L}$-structure $\mathcal{M}$ is defined exactly as in the usual first order case. The satisfaction relation on the first order formulas are defined exactly as usual. Suppose u is a set of formulas in $\mathscr{L}_{\infty\omega}$ in which only finitely many free variables appear in any formula of u. Let $\bar{a}$ be some assignments of elements of $\mathcal{M}$ to the free variables that appear in u. Define by recursion, $\mathcal{M} \models \bigwedge u[\bar{a}]$ if and only if for all $\varphi \in u$, $\mathcal{M} \models \varphi[\bar{a}]$, and $\mathcal{M} \models \bigvee u[\bar{a}]$ if and only if there exists some $\varphi \in u$, $\mathcal{M} \models \varphi[\bar{a}]$. By the usual recursion, this defines the satisfaction relation on all $\mathscr{L}_{\infty\omega}$-formulas. (Note these concepts are, at the moment, formalized in ZFC and there is currently no demand concerning admissibility.)

Let T be a collection of $\mathscr{L}_{\infty\omega}$-sentences. If there is a $\mathscr{L}$-structure $\mathcal{M}$ so that $\mathcal{M} \models A$, then T is consistent. Next, it will be shown that if T is definable in a countable admissible set $\mathcal{A}$ which thinks T is consistent in the infinitary logic in the sense of $\mathcal{A}$, then (externally) T has a model.

Fact 2.6. *Let $\mathbb{B}$ be a Boolean algebra. Let $\langle X_n : n \in \omega\rangle$ be a countable collection of subsets of $\mathbb{B}$ such that $\bigvee X_n$ exists in $\mathbb{B}$. Then there is an ultrafilter $U \subseteq \mathbb{B}$ so that for all $n \in \omega$, $\bigvee X_n \in U$ if and only if $X_n \cap U \neq \emptyset$.*

Proof. Define a sequence $\langle c_n : n \in \omega\rangle$ be recursion as follows:

Let $c_0 = 1$ (the largest element of $\mathbb{B}$).

Suppose c_n has been defined. $X_n \cup \{\neg \bigvee X_n\}$ is a dense subset of $\mathbb{B}$. There is some $b \in X_n \cup \{\neg \bigvee X_n\}$ so that $c_n \wedge b \neq 0$. Let $c_{n+1} = c_n \wedge b$ for such a b.

Let $F = \{b \in \mathbb{B} : (\exists n)(c_n \leq b)\}$. F is a filter. Let U be an ultrafilter extending F.

Suppose $\bigvee X_n \in U$. It is not possible that $c_{n+1} \leq \neg \bigvee X_n$. From the construction, it must be the case there is some $b \in X_n$ so that $c_{n+1} \leq b$. Since $c_{n+1} \in U$, one has $b \in U$. $X_n \cap U \neq \emptyset$. Suppose that $X_n \cap U \neq \emptyset$. There is some $b \in U$ so that $b \in X_n \cap U$. It is not possible that $c_{n+1} \leq \neg \bigvee X_n$. By construction there must be some $b' \in X_n$ so that $c_{n+1} \leq b'$. Then $\bigvee X_n \in U$. $\square$

Theorem 2.7. *(Barwise Completeness Theorem) Let $\mathcal{A}$ be a countable Σ_n-admissible set. Let $\mathscr{L}$ be a language which is Δ_n-definable in $\mathcal{A}$. Let $T \subseteq \mathscr{L}^{\mathcal{A}}_{\infty\omega}$ be a Σ_n-definable class of sentences. If $\mathcal{A} \models$ "T is consistent", then T has a model.*

Proof. Let C be a Δ_1-definable proper class of new constant symbols which do not appear in $\mathscr{L}$. Let $\mathscr{L}' = \mathscr{L} \cup C$.

For sentences $\varphi, \psi \in (\mathscr{L}'_{\infty\omega})^{\mathcal{A}}$, define $\varphi \sim \psi$ if and only if $\mathcal{A} \models T \vdash \varphi \Leftrightarrow \psi$. Let $\mathbb{B}$ be the Lindenbaum algebra of the equivalence classes of sentences in $(\mathscr{L}'_{\infty\omega})^{\mathcal{A}}$ under $\sim$.

Define $[\varphi] \leq [\psi]$ if and only if $\mathcal{A} \models T \vdash \varphi \Rightarrow \psi$. This induces the Boolean operations:

$$[\varphi] \wedge [\psi] = [\varphi \wedge \psi], [\varphi] \vee [\psi] = [\varphi \vee \psi], \neg[\varphi] = [\neg\varphi]$$

One can show that for all $u \in \mathcal{A}$ which is a collection of $(\mathscr{L}'_{\infty\omega})^{\mathcal{A}}$ sentences,

$$\bigwedge\{[\varphi] : [\varphi] \in u\} = [\bigwedge u], \bigvee\{[\varphi] : \varphi \in u\} = [\bigvee u].$$

Also, for any $\mathscr{L}'_{\infty\omega}$ formula $\varphi(v)$ in the single free variable v:

$$\bigwedge\{[\varphi(c)] : c \in C\} = [(\forall v)\varphi(v)], \bigvee\{[\varphi(c)] : c \in C\} = [(\exists v)\varphi(v)].$$

To see this, let ψ be such that for all $c \in C$, $[\psi] \leq [\varphi(c)]$. As C is a proper class, there is some $c \in C$ such that c does not appear in ψ. Fix such a c. Hence $\mathcal{A} \models T \vdash \psi \Rightarrow \varphi(c)$. Since c does not appear in the $\mathscr{L}$-theory T, the infinitary version of the generalization of constant can be use to show that one can choose some variable v not appear in $\psi \Rightarrow \varphi(c)$ so that $\mathcal{A} \models T \vdash \psi \Rightarrow (\forall v)\varphi(c)$. This implies that $[\psi] \leq [(\forall v)\varphi(v)]$. This shows that $[(\forall v)\varphi(c)]$ is the greatest lower bound of $\{[\varphi(c)] : c \in C\}$.

Note that since T is consistent in $\mathcal{A}$, $\mathbb{B}$ is a Boolean algebra containing at least two elements.

For each $\varphi(v) \in (\mathscr{L}'_{\infty\omega})^{\mathcal{A}}$ with a single free variable v, let $A_{\varphi(v)} = \{\varphi(c) : c \in C\}$. Since $\mathcal{A}$ is countable, let $\langle X_n : n \in \omega\rangle$ enumerate all sets $u \in \mathcal{A}$ of $\mathscr{L}'_{\infty\omega}$-sentences and all $A_{\varphi(v)}$ where $\varphi(v)$ is a $\mathscr{L}'_{\infty\omega}$-formula in the single free variable v. By the discussion above, the supremum of each X_n exists in $\mathbb{B}$. Using Fact 2.6, let U be the ultrafilter given by the fact with respect to the sequence $\langle X_n : n \in \omega\rangle$.

Note that if u is a set of $\mathscr{L}'_{\infty\omega}$-sentences then $[\bigvee u] \in U$ if and only if there exists some $\varphi \in u$ such that $[\varphi] \in U$ by the property of the ultrafilter. The ultrafilter also respects $\bigwedge$: Suppose $[\bigwedge u] \in U$. Then $\neg[\bigwedge u] \notin U$ since U is an ultrafilter. $\neg[\bigwedge u] = [\neg \bigwedge u] = [\bigvee \tilde{u}]$, where $\tilde{u} = \{\neg\varphi : \varphi \in u\}$. Thus $[\bigvee \tilde{u}] \notin U$ implies that for all $\varphi \in u$, $[\neg\varphi] \notin U$. So for all $\varphi \in u$, $[\varphi] \in U$. A similar argument shows that if $[\varphi] \in U$ for all $\varphi \in u$, then $[\bigwedge u] \in U$.

Also if $\bigvee\{[\varphi(c)] : c \in C\} \in U$ if and only if there exists a $c \in C$, $[\varphi(c)] \in U$. An analogous statement holds for $\bigwedge$.

Define an equivalence relation E on C by $c \mathrel{E} d$ if and only $[c = d] \in U$. Let M be the set of E-equivalence classes. A $\mathscr{L}$-structure $\mathcal{M}$ with domain M will be defined as follows:

For each constant t of $\mathscr{L}$, let $t^{\mathcal{M}} = [c]_E$ if and only if $[t = c] \in U$.

For each n-ary relation $R \in \mathscr{L}$ and $c_1, ..., c_n \in C$, $R^{\mathcal{M}}([c_1]_E, ..., [c_n]_E)$ if and only if $[R(c_1, ..., c_n)] \in U$.

For each n-ary function $f \in \mathscr{L}$ and $c_1, ..., c_n, d \in C$, $f^{\mathcal{M}}([c_1]_E, ..., [c_n]_E) = [d]_E$ if and only if $[f(c_1, ..., c_n) = d] \in U$.

By induction, one can show that for any $\mathscr{L}$-formula $\varphi(v_1, ..., v_n)$, $\mathcal{M} \models \varphi([c_1]_E, ..., [c_n]_E)$ if and only if $[\varphi(c_1, ..., c_n)] \in U$.

If $\varphi \in T$, then $[\varphi] = 1 \in U$. Hence $\mathcal{M} \models T$. □

Note that the Barwise completeness theorem and its proof does not say that the model exists in $\mathcal{A}$. The fact that $\mathcal{A}$ is countable was very important. Except to formalize infinitary logic in $\mathcal{A}$, admissibility played a very minor role. The Barwise completeness theorem is stated with additional definability condition on the language and theory for safety. In the following, admissibility and the definability of the language and theory will be very important.

Theorem 2.8. *(Barwise Compactness) Let $\mathcal{A}$ be a countable Σ_n-admissible set. Let $\mathscr{L}$ be a language which is Δ_n-definable in $\mathcal{A}$. Let $T \subseteq \mathscr{L}^{\mathcal{A}}_{\infty\omega}$ be a Σ_n-definable class of sentences. If every $u \subseteq T$ with $u \in \mathcal{A}$ has a model, then T has a model.*

Proof. If every $u \subseteq T$ with $u \in \mathcal{A}$ has a model, then $\mathcal{A} \models$ "u is consistent". By Fact 2.4, $\mathcal{A} \models$ "T is consistent". Theorem 2.7 implies T has a model. □

3. Jensen's Model Existence Theorem

Fact 3.1. *Let $\mathcal{A}$ be an admissible set. Let $\mathscr{L}$ be a language which is a class in $\mathcal{A}$ and contains a binary relation symbol $\dot{\in}$ and constant symbols $\dot{a}$ for each $a \in \mathcal{A}$. Let $T \subseteq \mathscr{L}^{\mathcal{A}}_{\infty\omega}$ be a theory which is a class in $\mathcal{A}$ and contains the following sentences:*

(I) (Extensionality) $(\forall x)(\forall y)(\forall z)((z\dot{\in}x \Leftrightarrow z\dot{\in}y) \Rightarrow x = y)$.

(II) For each $a \in \mathcal{A}$, $(\forall v)(v\dot{\in}\dot{a} \Leftrightarrow \bigvee_{z\in a} \dot{z} = v)$.

Suppose $\mathcal{M}$ is an $\mathscr{L}$-structure such that $\mathcal{M} \models T$ and $\mathrm{WF}(\mathcal{M})$ is transitive, then for all $a \in \mathcal{A}$, $\dot{a}^{\mathcal{M}} = a$. $\mathrm{WF}(\mathcal{M})$ and $\mathcal{M}$ both end-extends $\mathcal{A}$.

Proof. This is proved by induction. □

Theorem 3.2. *(Jensen's Model Existence Theorem) Let $\mathcal{A}$ be a countable Σ_n-admissible set. Let $\mathscr{L}$ be a Δ_n-definable language in $\mathcal{A}$ which contains a binary relation symbol $\dot{\in}$ and constant symbols $\dot{a}$ for each $a \in \mathcal{A}$. Let $T \subseteq \mathscr{L}^{\mathcal{A}}_{\infty\omega}$ be a consistent Σ_n-definable theory in $\mathcal{A}$ which contains the following sentences:*

(I) Extensionality.

(II) For each $a \in \mathcal{A}$, $(\forall v)(v\dot{\in}\dot{a} \Leftrightarrow \bigvee_{z\in a} \dot{z} = v)$.

Then there is an $\mathscr{L}$-structure $\mathcal{M} \models T$ so that $\mathrm{WF}(\mathcal{M})$ is transitive, $\mathcal{M}$ end-extends $\mathcal{A}$, and $\mathrm{ON} \cap \mathcal{M} = \mathrm{ON} \cap \mathcal{A}$.

Proof. Let E be a Δ_1-definable proper class of new constant symbols. For each $e \in E$, let A_e consists of the sentences:

(i) "$e \in \mathrm{ON}$", where ON is an abbreviation for the appropriate $\dot{\in}$-formula.

(ii) "$e > \dot{\beta}$" for each $\beta \in \mathcal{A} \cap \mathrm{ON}$, where $>$ is an abbreviation for $e \dot{\in} \dot{\beta}$.

Let $\mathscr{J} = \mathscr{L} \cup E$. Let $U = T \cup \bigcup_{e \in E} A_e$. $\mathscr{J}$ is a Δ_n definable language in $\mathcal{A}$ and U is a Σ_n definable $\mathscr{J}_{\infty\omega}$-theory in $\mathcal{A}$.

Let $F \in \mathcal{A}$ be such that $F \subseteq U$. Since T is consistent, let $\mathcal{N} \models T$. By Mostowski collapsing $\mathrm{WF}(\mathcal{N})$, one may assume that $\mathrm{WF}(\mathcal{N})$ is transitive. By Fact 3.1, $\mathcal{A}$ is end-extended of $\mathrm{WF}(\mathcal{N})$. As $F \in \mathcal{A}$, there is some ordinal $\delta \in \mathcal{A}$ such that all β such that $\dot{\beta}$ appears in any sentences of F are below δ. Also since $F \in \mathcal{A}$, there is a set $u \in \mathcal{A}$ such that all constants of E that appear in F belong to u. It is clear now that F can be modeled by an expansion of the $\mathscr{L}$-structure $\mathcal{N}$ to an $\mathscr{J}$-structure where the constants of u are interpreted as ordinals in $\mathcal{A} \subseteq \mathrm{WF}(\mathcal{N})$ which are above δ. By Barwise compactness (Theorem 2.8), U is consistent.

Having shown U is a consistent $\mathscr{J}_{\infty\omega}$-theory, one can now complete the proof by a slightly modified version of the argument in the Barwise completeness theorem (Theorem 2.7):

As in the proof of Theorem 2.7, let C be a proper class of new constant symbols (which ultimately will name the elements of the model to be constructed). Let $\mathscr{J}' = \mathscr{J} \cup C$. Consider U as an $\mathscr{J}'$-theory. Let $\mathbb{B}$ be the Lindenbaum algebra for U as defined in the proof of Theorem 2.7.

For each $c \in C$, let B_c consists of the following $\mathscr{J}'_{\infty\omega}$-sentences:

(1) "$c \notin \mathrm{ON}$".

(2) "$c \leq \dot{\beta}$" for each $\beta \in \mathrm{ON} \cap \mathcal{A}$.

(3) "$e < c$" for each $e \in E$.

The claim is for each $c \in C$, $\bigvee B_c = 1$.

To see this, suppose that $\bigvee B_c \neq 1$. Let $\neg B_c = \{\neg[\varphi] : \varphi \in B_c\}$. Then there is some ψ so that $[\psi] \neq 0$ such that $[\psi] \leq [\varphi]$ for all $\varphi \in \neg B_c$. Note that $[\psi] \neq 0$ implies that $U \cup [\psi]$ is consistent. As $\psi \in M$ and E is a proper class, there is some $e \in E$ that is not mentioned in ψ. Since $c \leq e \in \neg B_c$, $U \cup \{\psi\} \vdash c \leq e$. Let $U' = U \setminus A_e$. Note that $U' \cup \{\psi\}$ does not mention the constant e. $U \cup \{\psi\} = U' \cup A_e \cup \{\psi\} \vdash c \leq e$. By Fact 2.4 and the fact that T contains the sentences described in (II), one can find some $\delta \in \mathcal{A} \cap \mathrm{ON}$ so that $U' \cup \{\psi\} \cup \{\dot{\delta} \leq e\} \vdash c \leq e$. By the deduction theorem, $U' \cup \{\psi\} \vdash \dot{\delta} \leq e \Rightarrow c \leq e$. Since e does not appear in $U' \cup \{\psi\}$, $U' \cup \{\psi\} \vdash (\forall v)(\dot{\delta} \leq v \Rightarrow c \leq v)$. Hence $U' \cup \{\psi\} \vdash c \leq \dot{\delta}$. But $U \cup \{\psi\} \vdash \dot{\delta} < c$ since $U \cup \{\psi\} \vdash \varphi$ for all $\varphi \in \neg B_c$. As $U' \subseteq U$, this shows that $U \cup \{\psi\}$ is inconsistent. Contradiction.

Add to the countable sequence of subset of $\mathbb{B}$, considered in the proof of Theorem 2.7, the set B_c for all $c \in C$. The result is still a countable sequence of subset of $\mathbb{B}$ since $\mathcal{A}$ is countable. Use Fact 2.6 on this countable sequence to obtain an ultrafilter V on $\mathbb{B}$. As in the proof of the Theorem 2.7, use V to construct a model $\mathcal{M}$ of T using the constants of C. For all φ which are $(\mathscr{J}'_{\infty\omega})^{\mathcal{A}}$-sentences, $\mathcal{M} \models \varphi \Leftrightarrow [\varphi] \in V$.

By Mostowski collapsing $\mathrm{WF}(\mathcal{M})$, one may assume that $\mathrm{WF}(\mathcal{M})$ is transitive. Hence Fact 3.1 implies that $\mathcal{M}$ and $\mathrm{WF}(\mathcal{M})$ end-extends $\mathcal{A}$. To show that $\mathrm{ON} \cap \mathcal{M} = \mathrm{ON} \cap \mathcal{A}$, its suffices to show that $X = \{x \in \mathrm{ON}^{\mathcal{M}} : (\forall \beta \in \mathcal{A} \cap \mathrm{ON})(\mathcal{M} \models (x > \beta))\}$ has no least element. Fix any $x \in X$. By construction, this x is represented by some constant $c \in C$. By the property of the ultrafilter V, the fact that $\bigvee B_c = 1 \in V$, $x \in \mathrm{ON}$, and $x > \beta$ for all $\beta \in \mathcal{A}$, one must have that $[e < c] \in V$ for some $e \in E$. But $e > \dot{\beta} \in U$ for all $\beta \in \mathrm{ON} \cap \mathcal{A}$. So $[e > \dot{\beta}] = 1 \in V$ for all $\beta \in \mathrm{ON} \cap \mathcal{A}$. Therefore, the element represented by e is less than x and belongs to X.

The proof is complete. $\square$

Jensen's model existence theorem can be applied to prove a theorem of Sacks which characterizes the countable admissible ordinals.

Theorem 3.3. *([3]) Let $x \in {}^{\omega}2$. Suppose α is a countable x-admissible ordinal. Then there is some $y \geq_T x$ so that $\omega_1^y = \alpha$.*

Proof. Since α is a countable x-admissible, this means that there is some countable admissible set $\mathcal{A}$ such that $x \in \mathcal{A}$ and $\mathrm{ON} \cap \mathcal{A} = \alpha$.

Let $\mathscr{L}$ consists of a binary relation symbol $\dot{\in}$, $\dot{a}$ for each $a \in \mathcal{A}$, and a new constant symbol c. $\mathscr{L}$ can be coded as a Δ_1-definable proper class in $\mathcal{A}$. Let T be a theory in $\mathscr{L}^{\mathcal{A}}_{\infty\omega}$ consisting of

(I) All of the axioms of KP (which includes extensionality).

(II) For all $a \in \mathcal{A}$, $(\forall v)(v \in \dot{a} \Leftrightarrow \bigvee_{z \in a} \dot{z} = v)$.

(III) $c : \dot{\omega} \to \dot{2}$. $\dot{x} \leq_T c$.

(IV) For each $\beta < \alpha$, $L_{\dot{\beta}}(c) \not\models \mathsf{KP}$.

T can be considered a Σ_1-definable theory in $\mathcal{A}$. T is consistent: Let r be any real which codes α under some fix recursive coding of pairs of integers by integers. For any $\beta < \alpha$, $L_\beta(d) \not\models \mathsf{KP}$ since under KP the Mostowski collapse of r which is α would belong to $L_\beta(r)$. However $\alpha \notin L_\beta(r)$ when $\beta < \alpha$. A model $\mathcal{N}$ of T is defined as follows: Let $\mathcal{N}$ have domain $H_{\aleph_1}$, the hereditarily countable sets. Since $\mathcal{A}$ is a countable admissible set, $\mathcal{A} \subseteq H_{\aleph_1}$, the set of hereditarily countable sets. For each $a \in \mathcal{A}$, let $\dot{a}^{\mathcal{N}} = a$. Let $\dot{\in}^{\mathcal{N}} = \in \restriction H_{\aleph_1}$. Let $c^{\mathcal{N}} = r$. $\mathcal{N} \models T$ so T is consistent.

Apply Theorem 3.2 to obtain $\mathcal{M} \models T$ so that $\mathcal{M}$ end-extends $\mathcal{A}$, $\mathrm{WF}(\mathcal{M})$ is transitive, and $\mathrm{ON} \cap \mathcal{M} = \mathrm{ON} \cap \mathcal{A} = \alpha$. By the truncation lemma ([1] II.8.4), the wellfounded part of a model of KP is a model of KP. Hence $\mathrm{WF}(\mathcal{M})$ is an admissible set. Let $y = c^{\mathcal{M}}$. Since $\mathcal{M}$ thinks $c^{\mathcal{M}}$ is a function from $\omega \to 2$, $y \in \mathrm{WF}(\mathcal{M})$. Since y belongs to the admissible set $\mathrm{WF}(\mathcal{M})$ which has ordinal height $\mathrm{WF}(\mathcal{M}) \cap \mathrm{ON} = \alpha$, $\omega_1^y \leq \alpha$. However for each $\beta < \alpha$, $\mathcal{M} \models L_\beta(y) \not\models \mathsf{KP}$. By the absoluteness of satisfaction, $L_\beta(y) \not\models \mathsf{KP}$ in the real world. If $\mathcal{A}$ is a y-admissible set, then $(L[y])^{\mathcal{A}}$ is also an admissible set. This implies that no transitive set of ordinal height β is a model of KP. Hence α is the smallest ordinal so that α is the ordinal height of an admissible set containing y. $\omega_1^y = \alpha$. $x \leq_T y$ by the absoluteness of Turing reduciblity. $\square$

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