

EXPOSITION ON NO INFINITE MAXIMAL ALMOST DISJOINT FAMILIES

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ABSTRACT. This note will provide a self-contained exposition of a result of He, Luo, and Zhang ([2]) that ZF , $\text{DC}_{\mathbb{R}}$, and $\omega \rightarrow (\omega)_2^{\omega}$ imply there are no infinite maximal almost disjoint families which removed the assumption of Ramsey Uniformization from the analog result of Schritterser and Törnquist. The key insight of He, Luo, and Zhang is to use the fact that every relation is Ramsey almost everywhere “continuous” rather than using an almost everywhere continuous uniformization for the relation.

(This note is based on an early draft of [2] provided by Su Gao. The author would like to thank Yipu Li and Jonathan Schilhan for helpful comments. Any error here is due to the author and may not appear in He-Luo-Zhang [2]. Since the writing of this note, He, Luo, and Zhang ([3]) have further strengthen their result to show that ZF , $\text{AC}_{\omega}^{\mathbb{R}}$, and $\omega \rightarrow (\omega)_2^{\omega}$ imply there are no infinite maximal almost disjoint families with an even shorter proof. A few people have expressed that this expository note has been helpful and hopefully this may still be of some use.)

ZF will denote the Zermelo-Frankel axioms without the axiom of choice. ZF will be assumed throughout.

Definition 1. $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is an almost disjoint family if and only if for all $A, B \in \mathcal{A}$ with $A \neq B$, $A \cap B$ is finite. An almost disjoint family is maximal if and only if for any infinite $B \subseteq \omega$, there is some $A \in \mathcal{A}$ so that $A \cap B$ is infinite.

$\text{DC}_{\mathbb{R}}$ is dependent choice for \mathbb{R} which states that for every relation $R \subseteq \mathbb{R} \times \mathbb{R}$ with the property that for all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ so that $R(x, y)$, then there exists a function $F : \omega \rightarrow \mathbb{R}$ so that $R(F(n), F(n+1))$ for all $n \in \omega$.

If $A \subseteq \omega$ and $N \leq \omega$, let $[A]^N$ be the set of all increasing functions $\ell : N \rightarrow A$. For all $N \leq \omega$, let $\omega \rightarrow (\omega)_2^N$ be the assertion that for all $P : [\omega]^N \rightarrow 2$, there exists an infinite $A \subseteq \omega$ and $i \in 2$ so that for all $\ell \in [A]^N$, $P(\ell) = i$. The Ramsey theorem states that for all $1 \leq n < \omega$, $\omega \rightarrow (\omega)_2^n$ holds under ZF . The prominent hypothesis of this note is $\omega \rightarrow (\omega)_2^{\omega}$ which is sometimes called the Ramsey property.

Ramsey uniformization is the statement that for all $R \subseteq [\omega]^{\omega} \times {}^{\omega}2$, there exists an infinite $A \subseteq \omega$ and a function $\Phi : [A]^{\omega} \rightarrow {}^{\omega}2$ so that for all $f \in [A]^{\omega} \cap \text{dom}(R)$, $R(f, \Phi(f))$.

Schritterser and Törnquist [4] showed that there are no infinite maximal almost disjoint families under ZF , $\text{DC}_{\mathbb{R}}$, $\omega \rightarrow (\omega)_2^{\omega}$, and Ramsey uniformization. In the course of their argument, there is an important relation $R \subseteq [\omega]^{\omega} \times {}^{\omega}2$. Ramsey uniformization is used to find an infinite $B \subseteq \omega$ and function $\Phi : [B]^{\omega} \rightarrow {}^{\omega}2$ so that Φ uniformizes R . A theorem of Mathias (which will be presented in a slightly different context in Fact 3) shows that $\text{DC}_{\mathbb{R}}$ and $\omega \rightarrow (\omega)_2^2$ provide an infinite $C \subseteq B$ so that $\Phi : [C]^{\omega} \rightarrow {}^{\omega}2$ is continuous. This continuous uniformization of R is used to find finite initial segments with certain properties. He, Luo, and Zhang [2] removed the use of Ramsey uniformization. Their key insight is that those finite initial segments can also be found using a “continuous restriction” of R rather than a continuous uniformization of R .

Definition 2. Let $R \subseteq [\omega]^{\omega} \times {}^{\omega}2$. If $B \subseteq \omega$, then let $R \upharpoonright [B]^{\omega} = R \cap ([B]^{\omega} \times {}^{\omega}2)$. $R \upharpoonright [B]^{\omega}$ is a continuous relation on $[B]^{\omega}$ if and only if for all $f \in [B]^{\omega}$, $g \in {}^{\omega}2$, and $n \in \omega$ such that $R(f, g)$ holds, then there is an $m \in \omega$ so that for all $f' \in [B]^{\omega}$ with $f' \upharpoonright m = f \upharpoonright m$, there exists a $g' \in {}^{\omega}2$ with $g' \upharpoonright n = g \upharpoonright n$ and $R(f', g')$ holds.

Mathias showed that under $\text{DC}_{\mathbb{R}}$ and $\omega \rightarrow (\omega)_2^{\omega}$, every function $\Phi : [\omega]^{\omega} \rightarrow {}^{\omega}2$ is continuous Ramsey almost everywhere: that is, there is an infinite $B \subseteq \omega$ so that for all $f \in [B]^{\omega}$ and $n \in \omega$, there is an $m \in \omega$ so that for all $g \in [B]^{\omega}$ with $f \upharpoonright m = g \upharpoonright m$, $\Phi(f) \upharpoonright n = \Phi(g) \upharpoonright n$. The proof of this continuity result (for example from [1] Fact 5.3) can be adapted to show every relation is almost everywhere a continuous relation in the sense of Definition 2.

Fact 3. Assume $\text{DC}_{\mathbb{R}}$ and $\omega \rightarrow (\omega)_2^2$. If $R \subseteq [\omega]^\omega \times {}^\omega 2$, then there is an infinite $B \subseteq \omega$ so that $R \upharpoonright [B]^\omega$ is a continuous relation on $[B]^\omega$.

Proof. Let $B_0 = \emptyset$ and $C_0 = \omega$. Let $t_0^0 = \emptyset$ and $\ell_0^0 = \emptyset$. Let $p_0^0 = (t_0^0, \ell_0^0) = (\emptyset, \emptyset)$. Define $P^{p_0^0} : [C_0]^\omega \rightarrow 2$ by $P^{p_0^0}(g) = 0$ if and only if there exists $h \in {}^\omega 2$ such that $h \upharpoonright 0 = \ell_0^0$ and $R(t_0^0 \hat{\ } g, h)$. By $\omega \rightarrow (\omega)_2^2$, pick an infinite $C^{p_0^0} \subseteq C_0$ which is homogeneous for $P^{p_0^0}$ which means that $P^{p_0^0}$ is constant on $[C^{p_0^0}]^\omega$. Let $B_1 = B_0 \cup \{\min C^{p_0^0}\}$ and $C_1 = C^{p_0^0} \setminus B_1$.

Suppose B_n and C_n have been defined. Let $p_0^n, \dots, p_{K_n}^n$, for some $K_n \in \omega$, enumerate $[B_n]^{<\omega} \times {}^n 2$. For each $k \leq K_n$, let $t_k^n \in [B_n]^{<\omega}$ and $\ell_k^n \in {}^n 2$ be such that $p_k^n = (t_k^n, \ell_k^n)$. Let $P^{p_k^n} : [C_n]^\omega \rightarrow 2$ be defined by $P^{p_k^n}(g) = 0$ if and only if there exists $h \in {}^\omega 2$ such that $R(t_k^n \hat{\ } g, h)$ and $h \upharpoonright n = \ell_k^n$. By $\omega \rightarrow (\omega)_2^2$, pick an infinite $C^{p_k^n} \subseteq C_n$ which is homogeneous for $P^{p_k^n}$. Suppose for $k < K_n$, $C^{p_k^n}$ has been defined. Define $P^{p_{k+1}^n} : [C^{p_k^n}]^\omega \rightarrow 2$ by $P^{p_{k+1}^n}(g) = 0$ if and only if there exists $h \in {}^\omega 2$ such that $R(t_{k+1}^n \hat{\ } g, h)$ and $h \upharpoonright n = \ell_{k+1}^n$. By $\omega \rightarrow (\omega)_2^2$, let $C^{p_{k+1}^n} \subseteq C^{p_k^n}$ be infinite and homogeneous for $P^{p_{k+1}^n}$. Let $B_{n+1} = B_n \cup \{\min C^{p_{K_n}^n}\}$. Let $C_{n+1} = C^{p_{K_n}^n} \setminus B_{n+1}$.

By $\text{DC}_{\mathbb{R}}$, there are sequences $\langle B_n : n \in \omega \rangle$, $\langle C_n : n \in \omega \rangle$, and $\langle C^{p_k^n} : n \in \omega \wedge k \leq K_n \rangle$ compatible with the above construction.¹ Let $B = \bigcup_{n \in \omega} B_n$. The claim is that $R \upharpoonright [B]^\omega$ is a continuous relation: Let $f \in [B]^\omega$, $n \in \omega$, $h \in {}^\omega 2$ so that $R(f, h)$. Let $k \leq K_n$ be such that $h \upharpoonright n = \ell_k^n$ and t_k^n is such that $f = t_k^n \hat{\ } g$ for some $g \in [C_{n+1}]^\omega$. Since $\ell_k^n = h \upharpoonright n$ and $g \in [C_{n+1}]^\omega$, one has that $C_{n+1} \subseteq C^{p_k^n}$ must be homogeneous for $P^{p_k^n}$ taking value 0 as witnessed by h . Let $m' = |t_k^n|$. Let $m = m' + 1$. Suppose $w \in [B]^\omega$ is such that $w \upharpoonright m = f \upharpoonright m$. Write w as $w = (w \upharpoonright m') \hat{\ } u$ for some $u \in [B]^\omega$. Since $u(0) = w(m') = f(m') \in C_{n+1}$, $u \in [C_{n+1}]^\omega$ by the construction of B . Thus $P^{p_k^n}(u) = 0$ which implies that there is an $\bar{h} \in {}^\omega 2$ such that $R(t_k^n \hat{\ } u, \bar{h})$ and $\bar{h} \upharpoonright n = \ell_k^n = h \upharpoonright n$. However since $w \upharpoonright m = f \upharpoonright m$, $m = m' + 1$ where $m' = |t_k^n|$, and $f = t_k^n \hat{\ } g$, one has that $t_k^n \hat{\ } u = (f \upharpoonright m') \hat{\ } u = (w \upharpoonright m') \hat{\ } u = w$. So $R(w, \bar{h})$ and $\bar{h} \upharpoonright n = h \upharpoonright n$. This establishes that $R \upharpoonright [B]^\omega$ is a continuous relation. \square

Fact 4. Assume $\text{DC}_{\mathbb{R}}$. For every infinite $A \subseteq \mathbb{R}$, there is an injection of ω into A .

Proof. Let $A \subseteq \mathbb{R}$ be infinite. Let $\mathcal{I}(A)$ be the set of all injection $\sigma : n \rightarrow A$ where $n \in \omega$. Let $\Phi : \mathbb{R} \rightarrow \mathcal{I}(A)$ be a surjection. Define $R \subseteq \mathbb{R} \times \mathbb{R}$ by $R(x, y)$ if and only if $\Phi(x) \subsetneq \Phi(y)$. (That is, $\Phi(y)$ is an injection which properly extends $\Phi(x)$.) Since A is infinite, for all $x \in \mathbb{R}$, there is a $y \in \mathbb{R}$ so that $R(x, y)$. By $\text{DC}_{\mathbb{R}}$, there is a function $F : \omega \rightarrow \mathbb{R}$ so that $R(F(n), F(n+1))$ for all $n \in \omega$. Let $g = \bigcup_{n \in \omega} \Phi(F(n))$. Then $g : \omega \rightarrow A$ is an injection. \square

Next, the argument of Schrittemser and Törnquist will be modified using the key insight of He, Luo, and Zhang that every relation is almost everywhere a continuous relation. Since one will only have a continuous relation in this argument, one can only apply this continuity one time in this setting. In contrast, the argument of Schrittemser and Törnquist uses continuity of a uniformization function ω -many times. (He, Luo, Zhang [3] have strengthened their result to show that ZF , $\text{AC}_{\omega}^{\mathbb{R}}$, and $\omega \rightarrow (\omega)_2^2$ imply there are no infinite maximal almost disjoint family. Fact 4 that every infinite $A \subseteq \mathbb{R}$ has an injection of ω into it can be proved under $\text{AC}_{\omega}^{\mathbb{R}}$ and this is their only need for $\text{AC}_{\omega}^{\mathbb{R}}$.)

In the following, one will identify $\mathcal{P}(\omega)$ with ${}^\omega 2$ by identifying $U \in \mathcal{P}(\omega)$ with its characteristic function $U \in {}^\omega 2$.

Theorem 5. (He-Luo-Zhang; [2]) Assume ZF , $\text{DC}_{\mathbb{R}}$, and $\omega \rightarrow (\omega)_2^2$. There are no infinite maximal almost disjoint families.

Proof. Suppose \mathcal{A} is an infinite maximal almost disjoint family. By $\text{DC}_{\mathbb{R}}$ and Fact 4, there is an injection $\langle A_n : n \in \omega \rangle$ of ω into \mathcal{A} . For each $i < j < \omega$, let $\eta_{i,j} \in \omega$ be the least element $\eta \in A_i$ so that $\eta > j$ and $\eta \notin \bigcup_{m < i} A_m$. For each $f \in [\omega]^\omega$, let $B_f = \{\eta_{f(2n), f(2n+1)} : n \in \omega\}$. Note that B_f is an infinite subset of ω since $f(2n+1) < \eta_{f(2n), f(2n+1)}$. Define $R \subseteq [\omega]^\omega \times {}^\omega 2$ by $R(f, A)$ if and only if $A \in \mathcal{A}$ and $A \cap B_f$ is infinite. Since \mathcal{A} is maximal, $\text{dom}(R) = [\omega]^\omega$. By Fact 3, let $C \subseteq \omega$ be infinite so that $R \upharpoonright [C]^\omega$ is a continuous relation.

¹Each step depends on the previous B_n and C_n where C_n is an infinite subset of ω . $\text{DC}_{\mathbb{R}}$ is needed to produce the infinite sequence of objects with the desired relation to each other. The reader can supply the straightforward but rather tedious application of $\text{DC}_{\mathbb{R}}$.

Claim:² There is no $\hat{A} \in \mathcal{A}$ so that for all $f \in [C]^\omega$, $R(f, \hat{A})$. To see this, suppose there is an $\hat{A} \in \mathcal{A}$ so that for all $f \in [C]^\omega$, $R(f, \hat{A})$. First, note that $\hat{A} \neq A_m$ for any $m \in \omega$. To see this, suppose for the sake of contradiction that $\hat{A} = A_m$ for some $m \in \omega$. Pick any $f \in [C]^\omega$ so that $f(0) > m$. For all $n < \omega$, $m < f(2n)$ and thus $\eta_{f(2n), f(2n+1)} \notin A_m$. Thus $B_f \cap \hat{A} = B_f \cap A_m = \emptyset$ which contradicts $R(f, \hat{A})$. A function $h \in [C]^\omega$ will be constructed by recursion as follows. Let $h(0) = \min(C)$. Suppose $h(2n)$ has been defined. As shown above $\hat{A} \neq A_{h(2n)}$. So $\hat{A} \cap A_{h(2n)}$ is bounded since $\hat{A}, A_{h(2n)} \in \mathcal{A}$ and \mathcal{A} is an almost disjoint family. Let $h(2n+1)$ be the least element of C greater than $\sup(\hat{A} \cap A_{h(2n)})$. Note that one has $\eta_{h(2n), h(2n+1)} \notin \hat{A}$ since $\eta_{h(2n), h(2n+1)} > h(2n+1) > \sup(\hat{A} \cap A_{h(2n)})$ and $\eta_{h(2n), h(2n+1)} \in A_{h(2n)}$. Let $h(2n+2)$ be the least element of C greater than $h(2n+1)$. This completes the construction of h . Note that $\hat{A} \cap B_h = \emptyset$ since $\eta_{h(2n), h(2n+1)} \notin \hat{A}$ for all $n \in \omega$ by the construction above. However this contradicts $R(h, \hat{A})$. The claim has been established.

Pick any $f \in [C]^\omega$. Since \mathcal{A} is a maximal almost disjoint family, there is an $\bar{A} \in \mathcal{A}$ so that $B_f \cap \bar{A}$ is infinite. Let $K = \{n \in \omega : \eta_{f(2n), f(2n+1)} \in \bar{A}\}$ which is an infinite set. Let $\text{enum}_K : \omega \rightarrow K$ be the increasing enumeration of K . For each $n \in \omega$, let $g(2n) = f(2\text{enum}_K(n))$ and $g(2n+1) = f(2\text{enum}_K(n)+1)$. Note that $g \in [C]^\omega$ and $B_g = \bar{A} \cap B_f$. By the claim, there is some $h \in [C]^\omega$ so that $\neg R(h, \bar{A})$. Since $h \in [C]^\omega \subseteq \text{dom}(R)$, there is some $\check{A} \in \mathcal{A}$ with $\check{A} \neq \bar{A}$ and $R(h, \check{A})$.³ Let $\bar{j} \in \omega$ be such that $\bar{A}(\bar{j}) \neq \check{A}(\bar{j})$. Since R is a continuous relation on $[C]^\omega$, there is an $\bar{i} \in \omega$ so that for all $p \in [C]^\omega$ with $p \upharpoonright 2\bar{i} = h \upharpoonright 2\bar{i}$, there exists some $A \in \mathcal{A}$ such that $R(p, A)$ and $A(\bar{j}) = \bar{A}(\bar{j})$.⁴ Let $\bar{k} \in \omega$ be such that $g(2\bar{k}) > \sup(h \upharpoonright 2\bar{i})$. Let $\text{drop}(g, 2\bar{k}) \in [C]^\omega$ be defined by $\text{drop}(g, 2\bar{k})(n) = g(2\bar{k}+n)$. Let $\bar{p} = (h \upharpoonright 2\bar{i}) \hat{\ } \text{drop}(g, 2\bar{k})$ and note that $\bar{p} \in [C]^\omega$ and $\bar{p} \upharpoonright 2\bar{i} = h \upharpoonright 2\bar{i}$. Thus by the property of $h \upharpoonright 2\bar{i}$, there is an $\check{A} \in \mathcal{A}$ so that $R(\bar{p}, \check{A})$ and $\check{A}(\bar{j}) = \bar{A}(\bar{j})$. Since $\bar{A}(\bar{j}) \neq \check{A}(\bar{j})$, one has that $\bar{A}(\bar{j}) \neq \check{A}(\bar{j})$ and thus $\bar{A} \neq \check{A}$. $R(\bar{p}, \check{A})$ implies that $B_{\bar{p}} \cap \check{A}$ is infinite. Since $\bar{p} = (h \upharpoonright 2\bar{i}) \hat{\ } \text{drop}(g, 2\bar{k})$, one has that $\{\eta_{\text{drop}(g, 2\bar{k})(2n), \text{drop}(g, 2\bar{k})(2n+1)} : n \in \omega\} \cap \check{A} = \{\eta_{g(2n), g(2n+1)} : \bar{k} \leq n < \omega\} \cap \check{A}$ is infinite. Let $T = \{n \in \omega : \bar{k} \leq n < \omega \wedge \eta_{g(2n), g(2n+1)} \in \check{A}\}$ which is an infinite set. Note that $B_g = \{\eta_{g(2n), g(2n+1)} : n \in \omega\} \subseteq \bar{A}$ (since $B_g = \bar{A} \cap B_f$). Thus $\{\eta_{g(2n), g(2n+1)} : n \in T\} \subseteq \bar{A} \cap \check{A}$. Hence $\bar{A} \cap \check{A}$ is infinite and $\bar{A} \neq \check{A}$. Since $\bar{A}, \check{A} \in \mathcal{A}$, this contradicts \mathcal{A} being an almost disjoint family. \square

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²The author would like to thank Yipu Li for reading over the note, pointing out typos, and making a suggestion which made the proof of this claim more concise.

³One can show that $h \neq g$ but this seems to be unnecessary: Suppose $h = g$. Then $R(g, \bar{A})$ implies $B_g \cap \bar{A} \subseteq \bar{A} \cap \bar{A}$ is infinite since $B_g \subseteq \bar{A}$. This violates $\bar{A} \cap \bar{A}$ is finite since $\bar{A}, \bar{A} \in \mathcal{A}$ and \mathcal{A} is an almost disjoint family.

⁴Recall $B_f = \{\eta_{f(2n), f(2n+1)} : n \in \omega\}$. Thus if f' is the increasing enumeration of the range of f with an odd number of elements removed, then $B_{f'}$ could be infinitely different from B_f . However if the range of f' only lost an even number of elements from the range of f , then B_f and $B_{f'}$ will only differ by finitely many elements. It is for this reason one need to choose an even length to witness the continuity of the relation R which is possible since extensions still witness the desired continuity property.