NON-REDUCIBILITY OF ISOMORPHISM OF COUNTEREXAMPLES TO VAUGHT'S CONJECTURE TO THE ADMISSIBILITY EQUIVALENCE RELATION

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ABSTRACT. Let T be a counterexample to Vaught's conjecture. Let E_T denote the isomorphism equivalence relation of T. Let F_{ω_1} be the countable admissible ordinal equivalence relation defined on ${}^\omega 2$ by $x \; F_{\omega_1} \; y$ if and only if $\omega_1^x = \omega_1^y$. This note will show that in L and set-generic extensions of L, $\neg(E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1})$. Using ZFC, if T is a non-minimal counterexample to Vaught's conjecture, then $\neg(E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1})$.

1. Introduction

An equivalence relation E on a Polish space X is thin if and only if it it does not have a perfect set of E-inequivalent elements.

Burgess showed that $\mathbf{\Sigma}_1^1$ equivalence relations can either have countably many classes, $\aleph_1$-many but not perfectly many classes, or perfectly many classes. Equivalence relations of the second kind are the thin $\mathbf{\Sigma}_1^1$ equivalence relations with uncountably many classes. There are a few notable examples of such equivalence relations.

Let E_{ω_1} be defined in ${}^\omega 2$ be

$$x \; E_{\omega_1} \; y \Leftrightarrow (x, y \notin \text{WO}) \vee (x \cong y)$$

where WO is the Π_1^1 set of reals coding well-orderings and $\cong$ is order-isomorphism. E_{ω_1} is Σ_1^1 thin equvivalence relation with uncountably many classes and has a single Σ_1^1 and not $\mathbf{\Delta}_1^1$ class consisting of the non-well-orderings.

The countable admissible ordinal equivalence relation F_{ω_1} is defined on ${}^\omega 2$ by

$$x \; F_{\omega_1} \; y \Leftrightarrow \omega_1^x = \omega_1^y$$

where ω_1^x is the least ordinal that can not be coded by a real recursive in x. F_{ω_1} is a Σ_1^1 thin equivalence relation with uncountably many classes which are all $\mathbf{\Delta}_1^1$.

If $\mathscr{L}$ is a recursive language and $T \subseteq \mathscr{L}_{\omega_1,\omega}$ is a countable theory, then E_T denotes isomorphism of models of T with a single class consisting of the structures that do not model T. E_T has all $\mathbf{\Delta}_1^1$-classes as a consequence of Scott's isomorphism theorem.

A notable class of thin $\mathbf{\Sigma}_1^1$ equivalence relations that may or may not exist are the isomorphism relations of counterexamples to Vaught's conjecture: $T \subseteq \mathscr{L}_{\omega_1,\omega}$ is a counterexample to Vaught's conjecture if E_T is a thin equivalence relation with uncountably many classes.

The natural task is to compare these thin $\mathbf{\Sigma}_1^1$ equivalence relations. A robust means of comparison is through $\mathbf{\Delta}_1^1$ reductions: Suppose E and F are two equivalence relations on Polish spaces X and Y, respectively. $E \leq_{\mathbf{\Delta}_1^1} F$ if and only if there is a $\mathbf{\Delta}_1^1$ function $\Phi : X \to Y$ so that for all $a, b \in X$, $\Phi(a) \; F \; \Phi(b)$.

First, since E_{ω_1} has a single Σ_1^1 but not $\mathbf{\Delta}_1^1$ class and both F_{ω_1} and E_T have all $\mathbf{\Delta}_1^1$ classes, it is impossible that $E_{\omega_1} \leq_{\mathbf{\Delta}_1^1} F_{\omega_1}$ and $E_{\omega_1} \leq_{\mathbf{\Delta}_1^1} E_T$. Secondly, by the boundedness theorem, $F_{\omega_1} \leq_{\mathbf{\Delta}_1^1} E_{\omega_1}$ and $E_T \leq_{\mathbf{\Delta}_1^1} E_{\omega_1}$ where T is a counterexample to Vaught's conjecture are also not possible.

The second failure is due to a significant global fact. The first failure seems local due to one E_{ω_1} class being non-$\mathbf{\Delta}_1^1$ and this E_{ω_1} class even seems artificially added to ensure E_{ω_1} is defined on all of ${}^\omega 2$.

To compare E_{ω_1} with F_{ω_1} and E_T in a meaningful way which disregards this triviality, Zapletal defined the almost $\mathbf{\Delta}_1^1$ reducibility: $E \leq_{a\mathbf{\Delta}_1^1 F} F$ if only if there is a countable $A \subseteq X$ and a $\mathbf{\Delta}_1^1$ function Φ so that

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for all $x, y \notin [A]_E$, $x \; E \; y$ if and only if $\Phi(x) \; F \; \Phi(y)$. That is, an almost $\mathbf{\Delta}_1^1$ reduction is a reduction that ignores at most a countable set of problematic classes.

Zapletal [7] showed that if there is measurable cardinal and E is an $\mathbf{\Sigma}_1^1$ equivalence relation with infinite pinned cardinal, then $E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} E$. F_{ω_1} has infinite pinned cardinals. If T is a counterexample to Vaught's conjecture then E_T has infinite pinned cardinals. Hence if there is a measurable cardinal, then $E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} E_T$ and $E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} F_{\omega_1}$. In fact, if $0^\sharp$ exists, $E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} F_{\omega_1}$.

The author in [2] showed that in Gödel constructible universe L and set generic extensions of L, $\neg(E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} F_{\omega_1})$. This shows that $E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} F_{\omega_1}$ is not provable in ZFC alone.

It remains to study the relationship between E_T and F_{ω_1} when T is a counterexample to Vaught's conjecture. Sy-David Friedman asked the first such question: If T is a counterexample to Vaught's conjecture, can $E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1}$?

The author in [2] showed that in L and set-generic extensions of L, $\neg(E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1})$ whenever T is a counterexample to Vaught's conjecture. There the author put considerable effort into treating isomorphisms of counterexamples to Vaught's conjecture as merely thin equivalence relations and used only one model-theoretic fact about the Vaught's conjecture ([2] Fact 6.5).

This note will embrace admissible model theory as presented in [6] to give a less technical and more self-contained proof that in L and set-generic extensions of L, $\neg(E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1})$ whenever T is a counterexample to Vaught's conjecture. It will also be shown using just ZFC that if T is a non-minimal counterexample to Vaught's conjecture, then $\neg(E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1})$.

Assume $0^\sharp$ exists, one has that $E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} F_{\omega_1}$. Hence it is a natural question whether it is consistent, using perhap some large cardinal axioms, that $E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1}$. There is some remarks on this at the end of the note, but this question remains open.

2. Infinitary Logic and Admissibility

Definition 2.1. Let $\mathscr{L}$ be a countable recursive first order language. Let $S(\mathscr{L})$ denote the set of $\mathscr{L}$-structures with domain ω.

Under some fixed recursive coding of $\mathscr{L}$-structures by elements of ${}^\omega 2$, $S(\mathscr{L}) = {}^\omega 2$ but the former notation is used to clearly indicate reals are being considered as $\mathscr{L}$-structures on ω.

Definition 2.2. Let $\mathscr{L}$ be a recursive language. Let $T \subseteq \mathscr{L}_{\omega_1,\omega}$ be a countable theory (set of sentences). Define the equivalence relation E_T on $S(\mathscr{L})$ by

$$x \; E_T \; y \Leftrightarrow (x \not\models T \wedge y \not\models T) \vee (x \cong y)$$

where $\models$ is the $\mathscr{L}$-satisfaction relation and $\cong$ is the $\mathscr{L}$-isomorphism relation.

Let z be any real so that T belong to any admissible set containing z. $\models T$ is a $\Delta_1^1(z)$ relation. $\cong$ is a Σ_1^1 relation. E_φ is $\Sigma_1^1(z)$. E_T has all classes $\mathbf{\Delta}_1^1$.

Definition 2.3. Let E be an equivalence relation on a Polish space ${}^\omega 2$. E is a thin equivalence relation if and only if for all perfect subsets $P \subseteq {}^\omega 2$, there exist some $x, y \in P$ so that $x \; E \; y$.

Definition 2.4. Let $\mathscr{L}$ be a recursive language. A countable theory $T \subseteq \mathscr{L}_{\omega_1,\omega}$ is a counterexample to Vaught's conjecture if and only if E_T is a thin equivalence relation with uncountably many classes.

Definition 2.5. Let A be a transitive set. A is an admissible set if and only if $(A, \in) \models \mathsf{KP}$. ($\mathsf{KP}$ is Kripke-Platek set theory.)

Let x be a set. α is an x-admissible ordinal if and only if there is some admissible set A with $x \in A$ and $\alpha = A \cap \mathrm{ON}$. Let $\Lambda(x)$ denote the set of x-admissible ordinals.

Suppose α is an admissible ordinal. Let α^+ denote the least admissible ordinal above α.

If $x \in {}^\omega 2$, then let ω_1^x be the least x-admissible ordinal.

Definition 2.6. Let F_{ω_1} be the equivalence relation on ${}^\omega 2$ defined by $x \; F_{\omega_1} \; y$ if and only if $\omega_1^x = \omega_1^y$.

F_{ω_1} is a Σ_1^1 eqivalence relation with all classes $\mathbf{\Delta}_1^1$ and uncountably many classes.

Definition 2.7. Let $\mathscr{L}$ be a recursive language. A fragment of $\mathscr{L}_{\omega_1,\omega}$ is a collection $\mathcal{F}$ of formulas of $\mathscr{L}_{\omega_1,\omega}$ with some closure properties (see [1] Definition III.2.1).

Let A be a transitive set. Let $\mathscr{L}_A = \mathscr{L}_{\omega_1,\omega} \cap A$.

[4] Chapter 4 gives the following succient definition of a fragment: $\mathcal{F} \subseteq \mathscr{L}_{\omega_1,\omega}$ is a fragment if and only if $\mathcal{F} = \mathscr{L}_A$ for some transitive set A such that
(1) A is closed under pairing, union, and cartesian product. For all $a \in A$, $\text{ON} \cap \text{tc}(a) \in A$, where $\text{tc}(a)$ is the transitive closure of a.
(2) If $\varphi(x_1, ..., x_n) \in \mathscr{L}_{\omega_1,\omega} \cap A$ and $t_1, ..., t_n \in A$ are $\mathscr{L}$-terms, then $\varphi(t_1, ..., t_n) \in A$.

If A is countable, then $\mathscr{L}_A$ is called a countable fragment. If A is admissible, then $\mathscr{L}_A$ is called an admissible fragment.

[6] showed how to enumerate the extensions of a scattered theory in various fragments and determined what admissible sets this enumeration belong to. These ideas are summarized below:

Definition 2.8. Let $\mathscr{L}$ be a recursive language. Let $\mathcal{F}$ be a countable fragment of $\mathscr{L}_{\omega_1,\omega}$. Let $T \subseteq \mathcal{F}$ be a theory in the fragment $\mathcal{F}$.

T is finitarily consistent if not contradiction can be proved using the T and the finitary inference rules.

T is ω-complete in $\mathcal{F}$ if and only for all sentences $\varphi \in \mathcal{F}$, either $\varphi \in T$ or $\neg\varphi \in T$ and whenever a sentence of the form $\bigvee_{i\in\omega} \varphi_i \in T$, then there is some $\varphi_i \in T$.

Fact 2.9. *If T is finitarily consistent and ω-complete in some fragment $\mathcal{F} \subseteq \mathscr{L}_{\omega_1,\omega}$, then T has a model.*

Proof. See [6] Proposition 4.1. □

Fact 2.10. *Let $\mathscr{L}$ be a recursive language. Let γ be an ordinal. Suppose $(\mathcal{F}_\alpha : \alpha < \gamma)$ and $(T_\alpha : \alpha < \gamma)$ are sequences of fragments and theories so that T_α is a finitary consistent and ω-complete theory in the fragment $\mathcal{F}_\alpha$ and whenever $\alpha < \beta < \gamma$, $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ and $T_\alpha \subseteq T_\beta$. Then $\bigcup_{\alpha<\gamma} T_\alpha$ is a finitary consistent and ω-complete theory in the fragment $\bigcup_{\alpha<\gamma} \mathcal{F}_\alpha$.*

Proof. See [6] Proposition 4.2. □

Definition 2.11. Let $\mathscr{L}$ be a recursive language. Let $\mathcal{F}$ be a countable fragment of $\mathscr{L}_{\omega_1,\omega}$ and T be theory in the fragement $\mathcal{F}$. T is scattered if and only

(i) For all countable fragments $\mathcal{F}' \supseteq \mathcal{F}$, there are only countably many finitarily consistent and ω-complete theories $T' \subseteq \mathcal{F}'$ which extend T.

(ii) For all $n > 0$, fragment $\mathcal{F}' \supseteq \mathcal{F}$, and finitarily consistent and ω-complete theory T' in the fragment $\mathcal{F}'$, the set of n-types over T', $S_n(T')$, is countable.

Fact 2.12. *Let T be a theory in some countable fragment. T is a counterexample to Vaught's conjecture if and only if T is scattered and has uncountably many countable models up to isomorphism.*

Definition 2.13. Let $\mathscr{L}$ be a recursive language. Let T be a countable theory in $\mathscr{L}_{\omega_1,\omega}$. The Morley tree of T, denoted $\mathcal{MT}(T)$, is defined as follows:

Level 0 of $\mathcal{MT}(T)$: Let $\mathcal{F}_0$ be the smallest countable fragment of T. The nodes on level 0 are those finitary consistent and ω-complete theories T' in $\mathcal{F}_0$. For each such T', let $\mathcal{F}_{T'}$ be $\mathcal{F}_0$.

Level $\alpha+1$: The nodes of level α have been defined and for each U on level α, the fragment $\mathcal{F}_U$ have been defined. Let U be a node on level α. If U is ω-categorical, then U will not extend to level $\alpha+1$. Suppose U is not ω-categorical. U has a non-isolated n-type for some $n \in \omega$. Let $\mathcal{F}'_U$ be the smallest countable fragment containing $\bigwedge p$ for each non-isolated types p in $S_n(U)$. Every finitarily consistent and ω-complete theory U' extending U in the fragment $\mathcal{F}'_U$ is a node on level $\alpha+1$. Let $\mathcal{F}_{U'} = \mathcal{F}'_U$. The nodes of level $\alpha+1$ are exactly the objects obtained in this way.

Level α where α is a limit: Suppose levels less than α have been defined and for each node U in such level, $\mathcal{F}_U$ have been defined. For each sequence $(T_\beta : \beta < \alpha)$ of nodes with T_β on level β and for all $\gamma < \beta < \alpha$, $T_\gamma \subseteq T_\beta$, then put $T' = \bigcup_{\beta<\alpha} T_\beta$ and let $\mathcal{F}_{T'} = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$. (Note that T' is an ω-complete and finitary consistent theory in the fragment $\mathcal{F}_{T'}$ by Proposition 2.10.) The nodes on level α are exactly those obtained in this way.

Remark 2.14. In the successor case, note that if U is some theory on level α of $\mathcal{MT}(T)$ in the countable fragment $\mathcal{F}_U$ which is not ω-categorical, then there must be a non-isolated type in U. To see this: Suppose $\mathcal{M}$ and $\mathcal{N}$ are two nonisomorphic models. Let $\bar{a}$ in M and $\bar{b}$ in N so that $\text{tp}^{\mathcal{M}}(\bar{a}) = \text{tp}^{\mathcal{N}}(\bar{b})$ (here the type is taken in the fragment $\mathcal{F}$). Thus there is a partial isometry of $\mathcal{M}$ to $\mathcal{N}$ taking $\bar{a}$ to $\bar{b}$. Let $a \in M$.

$\mathrm{tp}^{\mathcal{M}}(\bar{a}a)$ is isolated by some formula $\psi(\bar{x}, x)$. $\mathcal{M} \models (\exists x)(\psi(\bar{a}, x))$. So $\mathcal{N} \models (\exists x)(\psi(\bar{b}, x))$. Let $b \in N$ so that $\mathcal{N} \models \psi(\bar{b}, b)$. Then $\bar{b}b$ realizes $\mathrm{tp}^{\mathcal{M}}(\bar{a}a)$. Hence the map $\bar{a}a$ to $\bar{b}b$ is a partial isometry of $\mathcal{M}$ to $\mathcal{N}$. Using this and a back-and-forth argument, one can show that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

Let p be any nonisolated type of U. Pick any $\varphi(\bar{x}) \in p$. There exists some $q \neq p$ so that $\varphi \in q$. So there is some $\psi(\bar{x})$ so that $\psi(\bar{x}) \in p$ and $\neg\psi(\bar{x}) \in q$. Hence $(\exists x)(\bigwedge p)$ and $(\exists x)(\bigwedge q)$ are consistent sentences in the fragment $\mathcal{F}'_U$. Hence U has at least two incompatible extension on level $\alpha + 1$.

Fact 2.15. *For $\alpha < \omega_1$, level α of $\mathcal{MT}(T)$ is countable.*

The following fact collects some definability properties of the Morley tree of a scattered theory:

Fact 2.16. *Let $\mathscr{L}$ be a recursive language. Let T be a scattered theory in $\mathscr{L}$. Suppose α is so that $L_\alpha(T)$ is admissible. For all $\beta < \alpha$, $\mathcal{MT}(T) \restriction \beta \in L_\alpha(T)$. Moreover, there is a Σ_1 relation $\Xi(x, y, z)$ (with no parameters) so that whenever T is scattered, $L_\alpha(T)$ is admissible, $b \in L_\alpha(T)$, and $\beta \in ON \cap L_\alpha(T)$*

$$(\mathcal{MT}(T) \restriction \beta) = b \Leftrightarrow L_\alpha(T) \models \Xi(T, \beta, b)$$

Proof. See [6] Proposition 4.4. □

Fact 2.17. *Let $\mathscr{L}$ be a recursive language. The statement "T is a counterexample to Vaught's conjecture" is Σ_1 definable in $L_{\omega_1^{L(T)}} = (H_{\aleph_1})^{L(T)}$. By Schoenfield absoluteness, this statement is absolute to inner models containing T with the same ω_1.*

Definition 2.18. Let α be a limit ordinal. Let T' be a node of $\mathcal{MT}(T)$ in some level $\gamma < \alpha$. T' is unbounded below α if and only if for all β with $\gamma < \beta < \alpha$, there is an extension of T' on level β.

A node $T' \in \mathcal{MT}(T)$ on level γ is uniquely-unbounded below α if and only for all β such that $\gamma < \beta < \alpha$, there is a unique unbounded below α node on level β that extends T'.

Fact 2.19. *Let $\mathscr{L}$ be a recursive language. Let T be a counterexample to Vaught's conjecture. Let $\alpha < \omega_1$ be such that $L_\alpha(T)$ is admissible. Let $T' \in \mathcal{MT}(T)$ be unbounded below α. Then there is some extension of T'' of T' on some level below α which is uniquely-unbounded below α.*

Proof. The following argument appears in the proof of [6] Proposition 4.7:

Let T' belong to level β_0 of $\mathcal{MT}(T)$. Note that for each β with $\beta_0 < \beta < \alpha$, T' has an extension on level β which is unbounded below α: Otherwise, consider the relation

$$R(U, \gamma) \Leftrightarrow (\exists b)(\Xi(T, \gamma, b) \wedge (\forall L \in b)(\neg(U \subseteq L))).$$

Ξ is the Σ_1 relation of Fact 2.16. This is a Σ_1 relation in $L_\alpha(T)$. Informally, this states that no nodes on level γ of $\mathcal{MT}(T)$ is an extension of U. $(\mathcal{MT}(T) \restriction \beta + 1) \in L_\alpha(T)$ by Fact 2.16. Let $X = \{U \in \mathcal{MT}(T) \restriction \beta + 1 : U \supseteq T'\}$. $X \in L_\alpha(T)$ by Δ_1-separation. By the failure of the claim, one has that $(\forall U \in X)(\exists \gamma)(R(U, \gamma))$. By Σ_1-replacement, there exists some $\delta < \alpha$ so that $(\forall U \in X)(\exists \gamma < \delta)(R(U, \gamma))$. This then implies T' is not unbounded below α. Contradiction.

Now suppose there is no node U extending T' which is a uniquely-unbounded node below α. Externally in the real universe, define the following map $\Phi : {}^{<\omega}2 \to \mathcal{MT}(T) \restriction \alpha$ as follows with the property that for all s, $\Phi(s) \in \mathcal{MT}(T)$ is an unbounded extension of T' and if $s \subseteq t$ then $\Phi(t)$ extends $\Phi(s)$:

Let $\Phi(\emptyset) = T'$.

Suppose $\Phi(s)$ has been defined for all $s \in {}^n2$. Since T' has no uniquely-unbounded below α extension, neither does $\Phi(s)$. Therefore, find some U_0 and U_1 extending $\Phi(s)$, are both unbounded below α, and are incompatible. Let $\Phi(s\hat{}0) = U_0$ and $\Phi(s\hat{}1) = U_1$.

For each $x \in {}^\omega 2$, let $U_x = \bigcup_{n\in\omega} \Phi(x \restriction n)$. By Fact 2.10, U_x is a node on level α of $\mathcal{MT}(T)$. If $x \neq y$, then U_x and U_y is incompatible and hence different nodes on that level. Hence level α of $\mathcal{MT}(T)$ is not countable. Contradiction. □

Definition 2.20. Let $\mathscr{L}$ be a recursive language. A counterexample to Vaught's conjecture T is a minimal counterexample to Vaught's conjecture if and only if for all $\varphi \in L_{\omega_1,\omega}$, either $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ has only countably many models.

Corollary 2.21. *Let $\mathscr{L}$ be a recursive language. Suppose T is a counterexample to Vaught's conjecture. Then there is an extension T' of T which is a minimal counterexample to Vaught's conjecture.*

Proof. This is well known result of Harnik and Makkai. The following is a proof using the above ideas.

Let $G \subseteq \text{Coll}(\omega, \omega_1^V)$ be generic over the real universe V. In $V[G]$, ω_1^V is countable. By Fact 2.17, T is still a counterexample to Vaught's conjecture. $L_{\omega_1^V}(T)$ is admissible. Applying Fact 2.19 in $V[G]$, let $T' \in \mathcal{MT}(T)$ be an extension of T which is uniquely-unbounded below ω_1^V. Since $T' \in \mathcal{MT}(T) \upharpoonright \omega_1^V \in L_{\omega_1^V}(T)$ by Fact 2.16, $T' \in V$. T' is a minimal counterexample to Vaught's conjecture in V. $\square$

Fact 2.22. *Let $\mathscr{L}$ be a recursive langauge. Let $T \subseteq L_{\omega_1,\omega}$ be a counterexample to Vaught's conjecture. Suppose $\alpha < \beta$ are two T-admissible ordinals. There are infinitely many terminal nodes of $\mathcal{MT}(T)$ on levels above α and cofinal below β.*

Proof. Since T is a counterexample to Vaught's conjecture, there is some unbounded node U of $\mathcal{MT}(T)$ on some level between α and β. By Fact 2.19, there is some R extending U on some level below β so that R is an uniquely-unbounded node.

Suppose R is on level δ. For each γ with $\delta < \gamma < \beta$, let R_γ be the unique unbounded extension of R on level γ.

Fix γ between δ and β. Since R_γ has an extension, it is not ω-categorical. By the Remark 2.14, there are at least two incompatible nodes on level $\gamma + 1$. Since R is uniquely unbounded, one of those nodes is bounded. Let S_γ be some terminal node below level β which extends this bounded node.

Hence $\{S_\gamma : \alpha < \gamma < \beta\}$ is an infinite set of terminal nodes of $\mathcal{MT}(T)$ which belong to some level above α and cofinally below β. $\square$

Infinitary logic in countable admissible fragments will be used to produce useful models of KP.

Fact 2.23. *(Jensen's Model Existence Theorem) Let $\mathcal{A} = (A, \in)$ be a countable admissible set. Let $\mathscr{J}$ be a language which is Δ_1 definable in $\mathcal{A}$ and contains a distinguished binary relation symbol $\dot{\in}$ and constants $\hat{a}$ for each $a \in A$. Let H be a consistent theory in the countable admissible fragment $\mathscr{J}_\mathcal{A}$ which is Σ_1 definable over $\mathcal{A}$ and contains the following sentences:*
(I) $\mathsf{ZFC} - \mathsf{P}$
(II) For each $a \in A$, $(\forall v)(v \in a \Leftrightarrow \bigvee_{z\in a} v = \hat{z})$
Then there is a $\mathscr{J}$-structure $\mathcal{B} \models H$ so that $\text{WF}(\mathcal{B})$ *is transitive, $\mathcal{B}$ end-extends $\mathcal{A}$, and $ON \cap B = ON \cap A$.*

Proof. See [3] Section 4, Lemma 11. $\square$

Fact 2.24. *(Truncation Lemma) Suppose $\mathcal{B} = (B, \in)$ is a model of* KP*. Then* $\text{WF}(\mathcal{B})$ *is also a model of* KP.

Proof. See [1], Lemma II.8.4. $\square$

Fact 2.25. *(Sacks' Theorem) Let $z \in {}^\omega 2$. Let $\alpha \in \Lambda(z) \cap \omega_1$. Then there exists some $y \in {}^\omega 2$ with $z \leq_T y$ and $\omega_1^y = \alpha$.*

Proof. See [5]. See [2] Theorem 2.16 for a proof of this result using Fact 2.23. $\square$

Fact 2.26. *Let α be a countable admissible ordinal and $z \in {}^\omega 2$ be such that $\alpha < \omega_1^z$. Then the set $\{y \in {}^\omega 2 : \omega_1^y = \alpha\}$ is $\Delta_1^1(z)$.*

Proof. See [2] Proposition 2.38 for the computations. $\square$

3. Nonreducibility Results

Fact 3.1. *Let $\mathscr{L}$ be a recursive language. Let $T \subseteq \mathscr{L}_{\omega_1,\omega}$ be a counterexample to Vaught's conjectures. Let $y \in {}^\omega 2$ so that $T \in L_{\omega_1^y}(y)$. Suppose U is a terminal node of $\mathcal{MT}(T)$ on some level above $\omega_1^y + 1$. Then the E_T equivalence class $\{x \in S(\mathscr{L}) : x \models U\}$ is not $\Sigma_1^1(y)$.*

Proof. This result is analogous to [2] Theorem 6.4. The proof is as follows:

Let $B = \{x \in S(\mathscr{L}) : x \models U\}$. Suppose B is $\Sigma_1^1(y)$. Let R be a y-recursive tree on $2 \times \omega$ so that

$$x \in B \Leftrightarrow (\exists f)(f \in [R^x]).$$

Let $\mathscr{J}$ be a language which is Δ_1 in $L_{\omega_1^y}(y)$ consisting of:
(i) A binary relation symbol $\dot{\in}$.
(ii) For each $a \in L_{\omega_1^y}(y)$, a new constant symbol $\hat{a}$.

(iii) Two distinct constant symbols, $\dot{c}$ and $\dot{d}$.

Let H be a theory in the countable admissible fragment $\mathcal{J}_{L_{\omega_1^y}(y)}$ which is Σ_1 definable in $L_{\omega_1^y}(y)$ defined consisting of the following sentences:

(I) $\mathsf{ZFC} - \mathsf{P}$.

(II) For each $a \in L_{\omega_1^y}(y)$, $(\forall v)(v \in \hat{a} \Leftrightarrow \bigvee_{z \in a} v = \hat{z})$.

(III) $\dot{c} \subseteq \hat{\omega}$. $\dot{d} : \hat{\omega} \to \hat{\omega}$.

(IV) $\dot{d} \in [R^{\dot{c}}]$.

H is consistent: Let x be any model of U. Then $x \in B$ so there is some $f : \omega \to \omega$ so that $f \in [R^x]$. Let $\mathcal{B}$ be the $\mathscr{J}$-structure defined by: The domain is $B = H_{\aleph_1}$. $\dot{\in}^{\mathcal{B}} = \in \restriction H_{\aleph_1}$. For each $a \in L_{\omega_1^y}(y)$, let $\hat{a}^{\mathcal{B}} = a$. Let $\dot{c}^{\mathcal{B}} = x$. Let $\dot{d}^{\mathcal{B}} = f$. Then $\mathcal{B} \models H$.

By Fact 2.23, let $\mathcal{C} \models H$ so that $L_{\omega_1^y}(y) \subseteq C$, $\text{WF}(\mathcal{C})$ is transitive, and $\text{WF}(\mathcal{C}) \cap \text{ON} = \omega_1^y$. Let $c = \dot{c}^{\mathcal{C}}$ and $d = \dot{d}^{\mathcal{C}}$. By (III), $c \subseteq \omega$ and $d : \omega \to \omega$. Hence $c, d \in \text{WF}(\mathcal{C})$. By Δ_1-absoluteness from $\mathcal{C}$ down to $\text{WF}(\mathcal{C})$ and then up to the real universe V, one has that $d \in [R^c]$. Hence $c \models U$. By Fact 2.24, $\text{WF}(\mathcal{B})$ is an admissible set. c then belongs to an admissible set of ordinal height ω_1^y. By a result of Nadel (see [6] Section 2), c is a model of a terminal node of $\mathcal{MT}(T)$ (i.e. ω-categorical theory) on level no higher than $\omega_1^y + 1$. But $c \models U$ and U is an ω-categorical theory on level above $\omega_1^y + 1$. Contradiction. ◻

Theorem 3.2. *Let $\mathscr{L}$ be a recursive language. Let $T \subseteq \mathscr{L}_{\omega_1,\omega}$ be a counterexample to Vaught's conjecture. Suppose there is a $\mathbf{\Delta}_1^1$ function $\Phi : S(\mathscr{L}) \to {}^\omega 2$ which witnesses $E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1}$. Then there is some real z so that for all $\alpha \in \Lambda(z)$, $\alpha^+ \notin \Lambda(z)$.*

Proof. This is [2] Theorem 6.9:

Let $z \in {}^\omega 2$ be so that Φ is $\Delta_1^1(z)$ and $T \in L_{\omega_1^z}(z)$.

First, this result will be shown for any $\alpha \in \Lambda(z) \cap \omega_1$. By Schoenfield absoluteness, Φ continues to witness $E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1}$. Hence to conclude this theorem for all $\alpha \in \Lambda(z)$, one should apply the countable case to a $\text{Coll}(\omega, \alpha)$-extension.

So suppose there is some $\alpha \in \Lambda(z) \cap \omega_1$ so that $\alpha^+ \in \Lambda(z)$. Use Fact 2.22 to find three distinct terminal nodes A, B, and C on $\mathcal{MT}(T)$ on levels above $\alpha + 1$ and below α^+. Let M, N, and P be some countable models of A, B, and C, respectively. Since Φ is a reduction of E_T to F_{ω_1}, there is at most one $X \in \{M, N, P\}$ so that $\omega_1^{\Phi(X)} = \alpha$. If such an X exists, then without loss of generality suppose $X = P$.

Claim: $\omega_1^{\Phi(M)} \geq \alpha^+$ and $\omega_1^{\Phi(N)} \geq \alpha^+$.

To prove the claim: Suppose $\omega_1^{\Phi(M)} < \alpha^+$. M and P are not isomorphic since A and C are distinct terminal nodes on the $\mathcal{MT}(T)$. Since Φ is a reduction and $\omega_1^{\Phi(P)} = \alpha$, $\omega_1^{\Phi(M)} \neq \alpha$. Since there are no admissible ordinals between α and α^+, $\omega_1^{\Phi(M)} < \alpha$. Since A is a terminal node, for all $X \in S(\mathscr{L})$,

$$X \models A \Leftrightarrow X \cong M \Leftrightarrow \Phi(X) \in [\Phi(M)]_{F_{\omega_1}}.$$

Using Fact 2.25, find some $y \in {}^\omega 2$ so that $z \leq_T y$ and $\omega_1^y = \alpha$. Then $[\Phi(M)]_{F_{\omega_1}}$ is $\Delta_1^1(y)$ by Fact 2.26. This observation and the above, shows that $\{X \in S(\mathscr{L}) : X \models A\}$ is $\Sigma_1^1(y, z) = \Sigma_1^1(y)$. However, this contradicts Fact 3.1. This proves the claim.

By Fact 2.16, for all $\beta < \alpha^+$, $\mathcal{MT}(T) \restriction \beta \in L_{\alpha^+}(z)$. Hence $A, B \in L_{\alpha^+}(z)$. Since Φ is $\Delta_1^1(z)$, let R be a z-recursive tree on $2 \times 2 \times \omega$ so that

$$\Phi(x) = y \Leftrightarrow (\exists f)(f \in [R^{(x,y)}])$$

Let $\mathscr{J}$ be a language which is Δ_1 in $L_{\alpha^+}(z)$ consisting of:

(i) A binary relation symbol $\dot{\in}$.

(ii) For each $a \in L_{\alpha^+}(z)$, a new constant symbol $\hat{a}$.

(iii) Six distinct constant symbols $\dot{M}$, $\dot{N}$, $\dot{c}$, $\dot{d}$, $\dot{f}$, and $\dot{g}$.

Let H be the theory in the countable admissible fragment $\mathscr{J}_{L_{\alpha^+}(z)}$ which is Σ_1 definable in $L_{\alpha^+}(z)$ consisting of the following sentences:

(I) $\mathsf{ZFC} - \mathsf{P}$.

(II) For each $a \in L_{\alpha^+}(z)$, $(\forall v)(v \dot{\in} \hat{a} \Leftrightarrow \bigvee_{b \in a} v = \hat{b})$.

(III) $\dot{M}, \dot{N}, \dot{c}, \dot{d} \subseteq \hat{\omega}$. $\dot{f}, \dot{g} : \hat{\omega} \to \hat{\omega}$.

(IV) $\dot{f} \in [R^{(\dot{M},\dot{c})}]$ and $\dot{g} \in [R^{(\dot{N},\dot{d})}]$.
(V) $\dot{M} \models \hat{A}$ and $\dot{N} \models \hat{B}$.
(VI) For each $\beta < \alpha^+$, β is not $\dot{c}$-admissible and β is not $\dot{d}$-admissible.

H is consistent: Let $f, g \in {}^\omega\omega$ so that $f \in [R^{(M,\Phi(M))}]$ and $g \in [R^{(N,\Phi(N))}]$. Let $\mathcal{B}$ be the $\mathscr{J}$-structure defined by: The domain is $B = H_{\aleph_1}$. $\dot{\in}^{\mathcal{B}} = \in \restriction H_{\aleph_1}$. $\dot{M}^{\mathcal{B}} = M$, $\dot{N}^{\mathcal{B}} = N$, $\dot{c}^{\mathcal{B}} = \Phi(M)$, $\dot{d}^{\mathcal{B}} = \Phi(N)$, $\dot{f}^{\mathcal{B}} = f$, $\dot{g}^{\mathcal{B}} = g$. $\mathcal{B} \models H$. Note that the claim is used to verify (VI).

By Fact 2.23, let $\mathcal{C} \models H$ so that $L_{\alpha^+}(z) \subseteq C$, $\mathrm{WF}(\mathcal{C})$ is transitive, and $\mathrm{WF}(\mathcal{C}) \cap \mathrm{ON} = \alpha^+$. Let $M' = \dot{M}^{\mathcal{C}}$, $N' = \dot{N}^{\mathcal{C}}$, $c = \dot{c}^{\mathcal{C}}$, $d = \dot{d}^{\mathcal{C}}$, $f' = \dot{f}^{\mathcal{C}}$, and $g' = \dot{g}^{\mathcal{C}}$. Since all the objects above are either subsets of ω or functions from ω to ω, they all belong to $\mathrm{WF}(\mathcal{C})$. By Fact 2.24, $\mathrm{WF}(\mathcal{C})$ is an admissible set of ordinal height α^+. Hence $\omega_1^c, \omega_1^d \leq \alpha^+$. By Δ_1-absoluteness, each $\beta < \alpha^+$ is not c-admissible or d-admissible. Hence $\omega_1^c = \omega_1^d = \alpha^+$. By Δ_1-absoluteness, $f' \in [R^{M',c}]$, $g' \in [R^{N',d}]$, $M' \models A$, and $N' \models B$. In particular $c = \Phi(M')$ and $d = \Phi(N')$. Thus $\omega_1^{\Phi(M')} = \omega_1^{\Phi(N')} = \alpha^+$. Therefore $\Phi(M')\ F_{\omega_1}\ \Phi(N')$. $\Phi(M') \models A$ and $\Phi(N') \models B$ implies that $\neg(M' \cong N')$. Thus $\neg(\Phi(M')\ E_T\ \Phi(N'))$. Φ is not a reduction. Contradiction. ☐

Corollary 3.3. *In L and set-generic extensions of L, $\neg(E_T \leq_{\boldsymbol{\Delta}^1_1} F_{\omega_1})$ whenever T is a counterexample to Vaught's conjecture.*

Proof. For any real $z \in {}^\omega 2$, $z \in L_\beta$ for some $\beta < \omega_1$. Above β, every admissible ordinal is z-admissible. Hence no real satisfying the condition of Theorem 3.2 can exists. ☐

Given that $E_{\omega_1} \leq_{a\boldsymbol{\Delta}^1_1} F_{\omega_1}$ holds if $0^\sharp$ exists and $\neg(E_{\omega_1} \leq_{a\boldsymbol{\Delta}^1_1} F_{\omega_1})$ in set-generic extensions of L, a natural question is whether it is consistent (possible relative to some large cardinal axiom) that $E_T \leq_{\boldsymbol{\Delta}^1_1} F_{\omega_1}$ where T is a counterexample to Vaught's conjecture. The following result is provable in ZFC:

Recall that an ordinal α is x-recursive inaccessible if and only if it is x-admissible and a limit of x-admissible ordinals.

Theorem 3.4. *Let $\mathscr{L}$ be a recursive language. Let $T \subseteq L_{\omega_1,\omega}$ be a counterexample to Vaught's conjecture. Suppose there is a $z \in {}^\omega 2$ so that there is a countable z-recursively inaccessible ordinal α with $T \in L_\alpha(z)$ and $\mathcal{MT}(T)$ has two nodes on level α. Then there is no $\Delta^1_1(z)$ function $\Phi : S(\mathscr{L}) \to {}^\omega 2$ witnessing $E_T \leq_{\boldsymbol{\Delta}^1_1} F_{\omega_1}$.*

Proof. Let A and B be two distinct theory on level α of $\mathcal{MT}(T)$. For $\gamma < \alpha$, let A_γ and B_γ be the unique node on level γ which A and B extends, respectively. For some $\beta < \alpha$, $A_\beta \neq B_\beta$. By Fact 2.16, for each $\gamma < \alpha$, $\mathcal{MT}(T) \restriction \gamma \in L_\alpha(z)$. So $A_\beta, B_\beta \in L_\alpha(T)$. Let $M, N \in S(\mathscr{L})$ be so that $M \models A$ and $N \models B$. In particular, $M \models A_\beta$ and $N \models B_\beta$.

Since Φ is $\Delta^1_1(z)$, let R be a z-recursive tree on $2 \times 2 \times \omega$ so that

$$\Phi(x) = y \Leftrightarrow (\exists f)(f \in [R^{(x,y)}].$$

Let $\mathscr{J}$ be a language which is Δ_1 in $L_\alpha(z)$ consisting of:
(i) A binary relation symbol $\dot{\in}$.
(ii) For each $a \in L_\alpha(z)$, a new constant symbol $\hat{a}$.
(iii) Six distinct constant symbols $\dot{M}$, $\dot{N}$, $\dot{c}$, and $\dot{d}$, $\dot{f}$ and $\dot{g}$.

Let H be the theory in the countable admissible fragment $\mathscr{J}_{L_\alpha(z)}$ which is Σ_1-definable in $L_\alpha(z)$ consisting of the following sentences:
(I) $\mathsf{ZFC} - \mathsf{P}$
(II) For each $a \in L_\alpha(z)$, $(\forall v)(v \dot{\in} \hat{a} \Leftrightarrow \bigvee_{b \in a} v = \hat{b})$.
(III) $\dot{M}, \dot{N}, \dot{c}, \dot{d} \subseteq \hat{\omega}$. $\dot{f}, \dot{g} : \hat{\omega} \to \hat{\omega}$.
(IV) $\dot{f} \in [R^{\dot{M},\dot{c}}]$. $\dot{g} \in [R^{\dot{N},\dot{c}}]$.
(V) $\dot{M} \models \hat{A}_\beta$ and $\dot{N} \models \hat{B}_\beta$.
(VI) For each $\gamma < \alpha$, γ is not $\dot{c}$-admissible and γ is not $\dot{d}$-admissible.

H is shown to be conistent like in Theorem 3.2. In the model used to show the consistency of H, $\dot{c}$ and $\dot{d}$ will be interpreted as M and N. Only the argument to show of (VI) is somewhat different than the situation in Theorem 3.2: (VI) would follow from that fact that $\omega_1^{\Phi(M)} \geq \alpha$ and $\omega_1^{\Phi(N)} \geq \alpha$. Suppose not, then $\omega_1^{\Phi(M)} < \alpha$. Since α is z-recursively inaccessible, there is some z-admissible ordinal γ so that $\omega_1^{\Phi(M)} < \gamma < \alpha$. Using Fact 2.25, find some $y \in {}^\omega 2$ so that $z \leq_T y$ and $\omega_1^y = \gamma$. Let A_M be the terminal node extending A for which M is the unique model of M up to isomorphism. A_M is on level α or higher of $\mathcal{MT}(T)$. Since Φ is a reduction, one has for all $X \in S(\mathscr{L})$

$$X \models A_M \Leftrightarrow X \; E_T \; M \Leftrightarrow \Phi(X) \in [\Phi(M)]_{F_{\omega_1}}.$$

Since $\omega_1^{\Phi(M)} < \gamma = \omega_1^y$, Fact 2.26 implies that $[\Phi(M)]_{F_{\omega_1}}$ is $\Delta_1^1(y)$. Hence $[M]_{E_T}$ is $\Delta_1^1(z,y) = \Delta_1^1(y)$. This contradicts Fact 3.1.

Let $\mathcal{C} \models H$, M', and N' be as in the proof of Theorem 3.2. As in that proof, $\omega_1^{\Phi(M')} = \omega_1^{\Phi(N')} = \alpha$. So $\Phi(M') \; F_{\omega_1} \; \Phi(N')$. However, $M' \models A_\beta$ and $N' \models B_\beta$. Hence M' and N' can not be isomorphic. So $\neg(M' \; E_T \; N')$. This contradicts Φ being a reduction. ☐

Corollary 3.5. *Suppose T is a non-minimal counterexample to Vaught's conjecture. Then $\neg(E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1})$.*

Proof. Suppose Φ witnesses $E_T \leq F_{\omega_1}$. If T is a non-minimal counterexample to Vaught's conjecture. Let $z \in {}^\omega 2$ be so that $T \in L_{\omega_1^z}(z)$ and Φ is $\Delta_1^1(z)$. $\mathcal{MT}(T)$ has two distinct nodes U and V on some level γ which are unbounded below ω_1. Let α be any z-recursive ordinal greater than γ. Since U and V are both unbounded below ω_1, there must be U' and V' on level α which extends U and V. Then this contradicts Theorem 3.4. ☐

A natural question is whether it is consistent assuming large cardinals and there exist counterexamples to Vaught conjecture T and some real z so that all the z-recursively inaccessible levels of $\mathcal{MT}(T)$ has only one node. Montalbán informed the author that this is possible using Turing determinacy. The following is an argument using sharps.

Fact 3.6. *Suppose T is a minimal counterexample to Vaughts conjecture. Suppose $T \in L_\gamma(z)$, for some countable ordinal γ, and $z^\sharp$ exists. Then $\mathcal{MT}(T)$ has a unique node on each $z^\sharp$-admissible level above γ.*

Proof. Let $\Psi(\alpha)$ be the statement

$$1_{\text{Coll}(\omega,\alpha)} \Vdash_{\text{Coll}(\omega,\alpha)} \check{T} \text{ is uniquely-unbounded below } \check{\alpha}.$$

Since T is a minimal counterexample to Vaught's conjecture, $\psi(\omega_1)$ holds. Since $\mathcal{MT}(T) \subseteq L(z)$ by Fact 2.16 and by absoluteness, $L(z) \models \psi(\omega_1^V)$. However ω_1^V is a Silver's indiscernible for $L(z)$. Therefore, for any Silver indiscernible α for $L(z)$ above γ, $L(z) \models \psi(\alpha)$. All $z^\sharp$-admissible ordinals are Silver's indiscernibles for $L(z)$. Hence for all $z^\sharp$-admissible ordinals above γ, $L(z) \models \psi(\alpha)$. Then by absoluteness, T is uniquely-unbounded below α for each countable $z^\sharp$-admissible ordinal above γ. This implies for all $z^\sharp$-admissible ordinal above γ, $\mathcal{MT}(T)$ has a unique node on level α. ☐

This result seems to suggests that perhaps if T is a minimal counterexample in L and $0^\sharp$ exists, then $E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1}$ may be possible.

Question 3.7. If T is a minimal counterexample to Vaught's conjecture in L and $0^\sharp$ exists, then does $E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1}$ hold?

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