ALMOST DISJOINT FAMILIES UNDER DETERMINACY

WILLIAM CHAN, STEPHEN JACKSON, AND NAM TRANG

Abstract. For each cardinal κ, let $\mathscr{B}(\kappa)$ be the ideal of bounded subsets of κ and $\mathscr{P}_\kappa(\kappa)$ be the ideal of subsets of κ of cardinality less than κ. Under determinacy hypothesis, this paper will completely characterize for which cardinals κ there is a nontrivial maximal $\mathscr{B}(\kappa)$ almost disjoint family. Also, the paper will completely characterize for which cardinals κ there is a nontrivial maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family when κ is not an uncountable cardinal of countable cofinality. More precisely, the following will be shown.

Assuming AD^+, for all $\kappa < \Theta$, there are no maximal $\mathscr{B}(\kappa)$ almost disjoint families $\mathcal{A}$ such that $\neg(|\mathcal{A}| < \text{cof}(\kappa))$. For all $\kappa < \Theta$, if $\text{cof}(\kappa) > \omega$, then there are no maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint families $\mathcal{A}$ so that $\neg(|\mathcal{A}| < \text{cof}(\kappa))$.

Assume AD and $V = L(\mathbb{R})$ (or more generally, AD^+ and $V = L(\mathscr{P}(\mathbb{R}))$). For any cardinal κ, there is a maximal $\mathscr{B}(\kappa)$ almost disjoint family $\mathcal{A}$ so that $\neg(|\mathcal{A}| < \text{cof}(\kappa))$ if and only if $\text{cof}(\kappa) \geq \Theta$. For any cardinal κ with $\text{cof}(\kappa) > \omega$, there is a maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family if and only if $\text{cof}(\kappa) \geq \Theta$.

1. Introduction

The classical almost disjoint family on ω is a set $\mathcal{A} \subseteq \mathscr{P}(\omega)$ so that for each element $A \in \mathcal{A}$, $|A| = \omega$ and for any $A, B \in \mathcal{A}$ with $A \neq B$, $|A \cap B| < \omega$. An almost disjoint family $\mathcal{A}$ on ω is maximal if for all almost disjoint families $\mathcal{B}$ on ω, if $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A} = \mathcal{B}$. Maximal almost disjoint families on ω exists using the axiom of choice. Mathias had asked whether maximal almost disjoint families on ω could exists in choiceless universes satisfying certain regularity properties. Possible settings include extensions of the axiom of determinacy, AD, such as Woodin's theory AD^+, where all sets of reals have the perfect set property, have the Baire property, are Lebesgue measurable, and the Ramsey property holds for all partitions. Neeman and Norwood [15] showed that there are no maximal almost disjoint families on ω under AD^+ using forcing absoluteness results. Schrittesser and Törnquist [16] resolved a question of Mathias by showing that there are no maximal almost disjoint families on ω if dependent choice for the reals $\mathsf{DC}_\mathbb{R}$, the Ramsey property, and a Ramsey almost everywhere uniformization principle hold. (These combinatorial principles hold in AD^+).

Under the axiom of choice AC, maximal almost disjoint families exist on all cardinals κ. There has been much work on the possible cardinalities of maximal almost disjoint families on uncountable singular cardinals. (For example, see results by Erdős and Heckler [5] and Kojman, Kubiś, and Shelah [11].) More recently, Müller and Lücke ([14]) showed under AC that there are no nontrivial maximal almost disjoint families which are Σ_1 definable with parameters from $H_\kappa \cup \{\kappa\}$ when κ is an iterable cardinal which is a limit of measurable cardinals.

Müller asked the first author whether there exist uncountable maximal almost disjoint families on ω_1 under the axiom of determinacy. More generally, when do maximal almost disjoint families on other cardinals exist under determinacy assumptions? There are also two candidates for the definition of an almost disjoint family for singular cardinals. Let κ be a cardinal. The more classical $\mathscr{P}_\kappa(\kappa)$ almost disjoint family is a set $\mathcal{A} \subseteq \mathscr{P}(\kappa)$ so that for all $A \in \mathcal{A}$, $|A| = \kappa$ and for all $A, B \in \mathcal{A}$, if $A \neq B$, then $|A \cap B| < \kappa$. A $\mathscr{B}(\kappa)$ almost disjoint family is an $\mathcal{A} \subseteq \mathscr{P}(\kappa)$ so that for all $A \in \mathcal{A}$, $\sup(A) = \kappa$ and for all $A, B \in \mathcal{A}$, if $A \neq B$, then $\sup(A \cap B) < \kappa$. If κ is a regular cardinal, then these two notions are the same.

The main result of this paper, Theorem 6.14, will completely characterize, under AD^+ and $V = L(\mathscr{P}(\mathbb{R}))$, for which cardinals κ there exist nontrivial maximal $\mathscr{B}(\kappa)$ and $\mathscr{P}_\kappa(\kappa)$ almost disjoint families (except when κ is uncountable of countable cofinality for the $\mathscr{P}_\kappa(\kappa)$ case). This paper is fairly self-contained and all necessary consequences of determinacy will be precisely stated and/or proved.

November 7, 2023. The first author was supported by NSF grant DMS-1703708. The second author was supported by NSF grant DMS-1800323. The third author was supported by NSF grant DMS-1855757 and DMS-1945592.

For any cardinal κ, if $\lambda < \text{cof}(\kappa)$, there is always a $\mathscr{P}_\kappa(\kappa)$ and a $\mathscr{B}(\kappa)$ maximal almost disjoint family of size λ (Fact 2.4). Thus the natural maximal $\mathcal{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family problem is whether there exists a maximal $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family $\mathscr{A}$ so that $\neg(|\mathscr{A}| < \text{cof}(\kappa))$. (Cardinalities are ordered by the injection comparison relation. Thus $|\mathscr{A}| < \text{cof}(\kappa)$ is the assertion that $\mathscr{A}$ injects into $\text{cof}(\kappa)$ but $\text{cof}(\kappa)$ does not inject into $\mathscr{A}$ or in other words, $\mathscr{A}$ is wellorderable and in bijection with a cardinal strictly less than $\text{cof}(\kappa)$. Without the axiom of choice, cardinalities are not wellordered and thus its negation is not equivalent to $\text{cof}(\kappa) \leq |\mathscr{A}|$ which abbreviates $\text{cof}(\kappa)$ injects into $\mathscr{A}$.) Since the axiom of choice fails, non-wellorderable families must be considered. An immediate observation is that every $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family of size exactly $\text{cof}(\kappa)$ cannot be maximal (Fact 2.5).

In many settings, if κ is a cardinal of uncountable cofinality, then $\mathscr{P}_\kappa(\kappa)$ and $\mathscr{B}(\kappa)$ almost disjoint families are wellorderable sets. Generalizing the perfect set property for $\mathbb{R}$, the perfect set dichotomy for a set X is the assertion that X is either wellorderable or $\mathbb{R}$ injects into X. Using category arguments, one has the following result concerning wellorderability.

Theorem 3.12. Assume all sets of reals have the Baire property. If κ is a cardinal with $\text{cof}(\kappa) > \omega$ and the perfect set dichotomy holds for all subsets of $\mathscr{P}(\kappa)$, then every $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family is wellorderable.

Alternatively, measures with wellfounded ultrapowers can also be used to establish wellorderability.

Proposition 3.17. Suppose κ is a cardinal and there exists a countably complete ultrafilter μ on κ so that every subset $A \subseteq \kappa$ with $|A| < \kappa$ does not belong to μ and the ultrapower ${}^\kappa\kappa/\mu$ is a wellordering. Then every $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family is wellorderable.

The category and ultrapower approaches yield slightly different conclusion in certain extensions of AD. For instance, using the methods of measures and wellfounded ultrapower, one has the following result using only AD and $\mathsf{DC}_\mathbb{R}$.

Theorem 3.18. Assume AD and $\mathsf{DC}_\mathbb{R}$. If $\kappa < \Theta$ and $\text{cof}(\kappa) > \omega$, then every $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family is wellorderable.

Woodin showed under AD^+ that all sets which are surjective images of $\mathbb{R}$ satisfy the perfect set dichotomy. Caicedo and Ketchersid [1] showed that in natural models of AD^+ satisfying $V = L(\mathscr{P}(\mathbb{R}))$, all sets satisfy the perfect set dichotomy. The category approach yields the following wellorderability result which holds for all cardinals κ of uncountable cofinality in natural models of the form $V = L(\mathscr{P}(\mathbb{R}))$ (even if $\kappa \geq \Theta$ where Θ is the supremum of the ordinals which $\mathbb{R}$ surjects onto).

Theorem 3.13. Assume AD^+. If $\kappa < \Theta$ and $\text{cof}(\kappa) > \omega$, then every $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family is wellorderable.

Assume $\mathsf{AD}^+ + V = L(\mathscr{P}(\mathbb{R}))$. If κ is a cardinal with $\text{cof}(\kappa) > \omega$, then every $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family is wellorderable.

Given that when κ has uncountable cofinality, $\mathscr{P}_\kappa(\kappa)$ and $\mathscr{B}(\kappa)$ almost disjoint families are wellorderable, knowledge of the possible sizes of such families will be useful for the maximal almost disjoint family problem at these cardinals. The boldface GCH at κ is the assertion that there are no injections of κ^+ into $\mathscr{P}(\kappa)$. Under AD, boldface GCH holds at ω since there are no uncountable wellorderable sets of reals (Fact 3.3). Boldface GCH holds at ω_1 since ω_2 is measurable under AD (Fact 3.20). More generally, results of Steel ([20]) and Woodin show that boldface GCH holds below Θ under AD^+.

Using the boldface GCH at ω_1 and the fact that the ultrapower of the club measure on ω_1 is wellfounded under AD, the measure and ultrapower approach gives a solution to Müller's question under AD alone. Moreover since branches of Kurepa trees are $\mathscr{P}_{\omega_1}(\omega_1)$ almost disjoint families, these methods also yield the failure of the Kurepa hypothesis under AD.

Theorem 3.23. (AD) There are no uncountable maximal $\mathscr{P}_{\omega_1}(\omega_1) = \mathscr{B}(\omega_1)$ almost disjoint families. There are no Kurepa trees and thus the Kurepa hypothesis KH fails. There are no maximal $\mathscr{P}_{\omega_1}(\omega_1) = \mathscr{B}(\omega_1)$ almost disjoint families.

More generally,

Theorem 4.1. Suppose κ is a cardinal and boldface GCH holds at $\text{cof}(\kappa)$. If $\mathcal{A}$ is a wellorderable $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family of cardinality greater than $\text{cof}(\kappa)$, then $\mathcal{A}$ is not maximal.

Under AD^+, this resolves the maximal $\mathscr{P}_\kappa(\kappa)$ and $\mathscr{B}(\kappa)$ almost disjoint family problem at cardinals κ with $\omega < \text{cof}(\kappa) < \Theta$.

Theorem 4.3. Assume AD^+. If $\kappa < \Theta$ is a cardinal and $\text{cof}(\kappa) > \omega$, then there are no maximal $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint families $\mathcal{A}$ so that $\neg(|\mathcal{A}| < \text{cof}(\kappa))$.

Assume $\mathsf{AD}^+ + V = L(\mathscr{P}(\mathbb{R}))$. If κ is a cardinal so that $\omega < \text{cof}(\kappa) < \Theta$, then there are no maximal $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint families $\mathcal{A}$ so that $\neg(|\mathcal{A}| < \text{cof}(\kappa))$.

The results concerning wellorderability requires that $\text{cof}(\kappa) > \omega$. For any cardinal κ of countably cofinality, there are always nonwellorderable $\mathscr{P}_\kappa(\kappa)$ and $\mathscr{B}(\kappa)$ almost disjoint families. An interesting consequence of Theorem 4.1 is that any potential maximal almost disjoint families on ω must be nonwellorderable. The maximal almost disjoint family problem is more challenging at cardinals of countable cofinality since one necessarily needs to handle nonwellorderable objects.

$\omega \to (\omega)^\omega_2$ is the strong partition property for ω or the Ramsey property which asserts for every partition $P : [\omega]^\omega \to 2$, there is an $i \in \{0, 1\}$ and an infinite $A \subseteq \omega$ so that for all $f \in [A]^\omega$, $P(f) = i$. Under AD^+, Mathias forcing over an inner model of the axiom of choice containing an ∞-Borel code (a highly absolute definition) for a partition will create infinite homogeneous sets for the partition. Thus $\omega \to (\omega)^\omega_2$ holds in AD^+ models although it is open if AD proves $\omega \to (\omega)^\omega_2$. Ramsey Uniformization for κ is the assertion that every relation $R \subseteq [\omega]^\omega \times \mathscr{P}(\kappa)$ has a Ramsey almost everywhere uniformization: there is an infinite $C \subseteq \omega$ and $\Lambda : \text{dom}(R) \cap [C]^\omega \to \mathscr{P}(\kappa)$ so that for all $f \in \text{dom}(R) \cap [C]^\omega$, $R(f, \Lambda(f))$. By a reflection into scales, Ramsey Uniformization for ω holds in $L(\mathbb{R}) \models \mathsf{AD}$ and more generally in AD^+. The additivity of the Ramsey null ideal and almost everywhere continuity for functions on $[\omega]^\omega$ will be investigated in Section 5. The following almost everywhere weak continuity for functions of the form $\Phi : [\omega]^\omega \to \mathscr{P}(\kappa)$ will be established.

Theorem 5.10. Assume AD, $\mathsf{DC}_\mathbb{R}$, $\omega \to (\omega)^\omega_2$, and Ramsey Uniformization for ω. Suppose $\kappa \in \text{ON}$ and $\Phi : [\omega]^\omega \to {}^\kappa 2$. Then there is an infinite $B \subseteq \omega$ so that $\Phi : [B]^\omega \to {}^\kappa 2$ is continuous in the following sense: for all $f \in [B]^\omega$, for all $\alpha < \kappa$, there exists an $n \in \omega$ so that for all $g \in [B]^\omega$ such that $f \restriction n = g \restriction n$, $\Phi(f)(\alpha) = \Phi(g)(\alpha)$.

The arguments of Schritteser and Törnquist can be adapted to show the following.

Theorem 5.13. Assume AD, $\mathsf{DC}_\mathbb{R}$, and $\omega \to (\omega)^\omega_2$. Suppose κ is an infinite cardinal, $\text{cof}(\kappa) = \omega$, Ramsey Uniformization for κ holds, and ω-injects into every infinite subset of $\mathscr{P}(\kappa)$. Then there are no infinite maximal $\mathscr{B}(\kappa)$ almost disjoint families.

This yields the following solution to the $\mathscr{B}(\kappa)$ almost disjoint family problem when $\text{cof}(\kappa) = \omega$.

Theorem 5.17. Under AD^+, there are no infinite maximal $\mathscr{B}(\kappa)$ almost disjoint families for $\kappa < \Theta$ with $\text{cof}(\kappa) = \omega$.

Theorem 6.2. Assume AD and $V = L(\mathbb{R})$. For any κ with $\text{cof}(\kappa) = \omega$, there are no infinite $\mathscr{B}(\kappa)$ almost disjoint families.

The maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family problem for singular cardinals κ of countable cofinality is still open. In the constructions, it appears to be much more challenging to make intersections have large cardinality than to make intersections unbounded.

Woodin's analysis of $L(\mathbb{R})$ satisfying AD implies that when $\kappa \geq \Theta$, the boldface GCH fails at κ since $\kappa^+ = (\kappa^+)^{\mathrm{HOD}}$. Moreover, there are maximal $\mathscr{B}(\kappa)$ and $\mathscr{P}_\kappa(\kappa)$ almost disjoint families of size κ whenever $\mathrm{cof}(\kappa) \geq \Theta$.

Theorem 6.3. Assume AD and $V = L(\mathbb{R})$. For any cardinal κ with $\mathrm{cof}(\kappa) \geq \Theta$, there exists a wellorderable maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family and a wellorderable maximal $\mathscr{B}(\kappa)$ almost disjoint family which does not inject into $\mathrm{cof}(\kappa)$ and in fact has cardinality κ^+.

Combining the earlier results, one has, in $L(\mathbb{R})$ satisfying AD, the following complete answer to the $\mathscr{B}(\kappa)$ almost disjoint family problem for all cardinals κ and a solution to the $\mathscr{P}_\kappa(\kappa)$ almost disjoint family problem when κ is not a singular cardinal of countable cofinality.

Theorem 6.4. Assume AD and $V = L(\mathbb{R})$. For every cardinal κ, there is a maximal $\mathscr{B}(\kappa)$ almost disjoint family $\mathcal{A}$ so that $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$ if and only if $\mathrm{cof}(\kappa) \geq \Theta$.

Theorem 6.5. Assume AD and $V = L(\mathbb{R})$. Suppose κ is a cardinal which is not a singular cardinal of countable cofinality. There is a maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family $\mathcal{A}$ such that $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$ if and only if $\mathrm{cof}(\kappa) \geq \Theta$.

These arguments can be generalized to the following main result of this paper which gives an analogous answer in all natural models of AD^+ of the form $V = L(\mathscr{P}(\mathbb{R}))$.

Theorem 6.14. Assume AD^+ and $V = L(\mathscr{P}(\mathbb{R}))$.

- There is a maximal $\mathscr{B}(\kappa)$ almost disjoint family $\mathcal{A}$ so that $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$ if and only if $\mathrm{cof}(\kappa) \geq \Theta$.
- If κ is not a singular cardinal of countable cofinality, then there is a maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family $\mathcal{A}$ so that $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$ if and only if $\mathrm{cof}(\kappa) \geq \Theta$.

2. Almost Disjoint Families

Definition 2.1. An ideal on a set X is a set $\mathcal{I} \subseteq \mathscr{P}(X)$ so that $\emptyset \in \mathcal{I}$, $X \notin \mathcal{I}$, for all $A, B \in \mathscr{P}(X)$, if $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$, and for all $A, B \in \mathcal{I}$, $A \cup B \in \mathcal{I}$. A filter on a set X is a set $\mathcal{F} \subseteq \mathscr{P}(X)$ so that $X \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$, for all $A, B \in \mathscr{P}(X)$, if $A \subseteq B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$, and for all $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$.

If $\mathcal{I}$ is an ideal on X, then let $\mathcal{F}_\mathcal{I} = \{A \in \mathscr{P}(X) : X \setminus A \in \mathcal{I}\}$ which is the filter dual to $\mathcal{I}$. Let $\mathcal{I}^+ = \mathscr{P}(X) \setminus \mathcal{I}$ which is the collection of $\mathcal{I}$-positive sets.

The ideal $\mathcal{I}$ is countably complete if and only if for any sequence $\langle A_n : n \in \omega \rangle$ in $\mathcal{I}$, $\bigcup_{n\in\omega} A_n \in \mathcal{I}$. A filter $\mathcal{F}$ is countably complete if and only for any sequence $\langle A_n : n \in \omega \rangle$ in $\mathcal{F}$, $\bigcap_{n\in\omega} A_n \in \mathcal{F}$.

Let $\mathscr{P}_\kappa(\kappa) = \{A \subseteq \kappa : |A| < \kappa\}$. $\mathscr{P}_\kappa(\kappa)$ is an ideal and is countably complete if and only if $\mathrm{cof}(\kappa) > \omega$. Let $\mathscr{B}(\kappa) = \{A \subseteq \kappa : \sup(A) < \kappa\}$. $\mathscr{B}(\kappa)$ is an ideal and is countably complete if and only if $\mathrm{cof}(\kappa) > \omega$. If κ is regular, then $\mathscr{P}_\kappa(\kappa) = \mathscr{B}(\kappa)$.

Definition 2.2. Suppose $\mathcal{I}$ is an ideal on a set X. An $\mathcal{I}$ almost disjoint family is a set $\mathcal{A} \subseteq \mathcal{I}^+$ so that for all $A, B \in \mathcal{A}$ with $A \neq B$, $A \cap B \in \mathcal{I}$. An $\mathcal{I}$ almost disjoint family is called a maximal almost disjoint family if and only if if $\mathcal{B} \supseteq \mathcal{A}$ and $\mathcal{B}$ is a $\mathcal{I}$ almost disjoint family, then $\mathcal{B} = \mathcal{A}$.

If κ is a cardinal, a $\mathscr{P}_\kappa(\kappa)$ almost disjoint family is traditionally called an almost disjoint family on κ.

When κ is regular, $\mathscr{P}_\kappa(\kappa)$ almost disjointness and $\mathscr{B}(\kappa)$ almost disjointness coincide.

Fact 2.3. (AC) *For any ideal $\mathcal{I}$, there exists a maximal $\mathcal{I}$ almost disjoint family.*

Fact 2.4. *For any cardinal κ and $\lambda < \mathrm{cof}(\kappa)$, there is a $\mathcal{A} \subseteq \mathscr{P}(\kappa)$ which is both a maximal $\mathscr{P}_\kappa(\kappa)$ and $\mathscr{B}(\kappa)$ almost disjoint family of size λ.*

Proof. Since $|\kappa \times \lambda| = |\kappa|$, let $\pi : \kappa \times \lambda \to \kappa$ be a bijection. For each $\alpha < \lambda$, let $A_\alpha = \pi[\kappa \times \{\alpha\}]$. Let $\mathcal{A} = \{A_\alpha : \alpha < \lambda\}$. Note that $\kappa = \bigcup \mathcal{A}$. For all $\alpha < \beta < \lambda$, $A_\alpha \cap A_\beta = \emptyset$. Let $B \subseteq \kappa$ be a set with $|B| = \kappa$

and for all $\alpha < \lambda$, $|B \cap A_\alpha| < \kappa$. Let $\delta_\alpha = |B \cap A_\alpha|$. Since $B = \bigcup_{\alpha<\lambda} B \cap A_\alpha$, $\sup\{\delta_\alpha : \alpha < \lambda\} = \kappa$. This contradicts $\lambda < \mathrm{cof}(\kappa)$. $\mathcal{A}$ is a maximal $\mathscr{P}_\kappa(\kappa)$-almost disjoint family. $\mathcal{A}$ is also a maximal $\mathscr{B}(\kappa)$-almost disjoint family. □

Fact 2.5. *For any cardinal κ, there is no $\mathcal{A} \subseteq \mathscr{P}(\kappa)$ with $|\mathcal{A}| = \mathrm{cof}(\kappa)$ which is a maximal $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family.*

Proof. Let $\rho : \mathrm{cof}(\kappa) \to \kappa$ be an increasing cofinal map.

Let $\mathcal{A}$ be a $\mathscr{B}(\kappa)$ almost disjoint family such that $|\mathcal{A}| = \mathrm{cof}(\kappa)$. Let $\langle A_\alpha : \alpha < \mathrm{cof}(\kappa)\rangle$ be a bijection of $\mathrm{cof}(\kappa)$ with $\mathcal{A}$. For each $\alpha < \beta < \mathrm{cof}(\kappa)$, let $\delta_{\alpha,\beta} = \sup(A_\alpha \cap A_\beta)$. For each $\beta < \mathrm{cof}(\kappa)$, let $\epsilon_\beta = \sup\{\delta_{\alpha,\beta} : \alpha < \beta\}$. Let ξ_β be the least element of A_β greater than $\max\{\epsilon_\beta, \rho(\beta)\}$. Let $E = \{\xi_\alpha : \alpha < \mathrm{cof}(\kappa)\}$. Note that E is unbounded in κ and thus $E \in \mathscr{B}(\kappa)^+$. Let $\alpha < \mathrm{cof}(\kappa)$. For all β with $\alpha < \beta < \mathrm{cof}(\kappa)$, $\xi_\beta \notin A_\alpha$. Thus $A_\alpha \cap E \subseteq \{\xi_\gamma : \gamma \leq \alpha\}$ and thus $A_\alpha \cap E$ is bounded below κ. $A \cup \{E\}$ is a strictly larger $\mathscr{B}(\kappa)$ almost disjoint family.

Let $\mathcal{A}$ be a $\mathscr{P}_\kappa(\kappa)$ almost disjoint family such that $|\mathcal{A}| = \mathrm{cof}(\kappa)$. Let $\langle A_\alpha : \alpha < \mathrm{cof}(\kappa)\rangle$ be a bijection of $\mathrm{cof}(\kappa)$ with $\mathcal{A}$. For each $\beta < \mathrm{cof}(\kappa)$, let $\Sigma(\beta) = \bigcup_{\alpha<\beta} A_\alpha \cap A_\beta$ and observe $|\Sigma(\beta)| < \kappa$ since $\beta < \mathrm{cof}(\kappa)$ and $|A_\alpha \cap A_\beta| < \kappa$ for each $\alpha < \beta$. Therefore for each $\beta < \mathrm{cof}(\kappa)$, $|A_\beta \setminus \Sigma(\beta)| = \kappa$. Let ν_β be the least $\nu < \kappa$ so that $(A_\beta \setminus \Sigma(\beta)) \cap \nu$ has ordertype $\rho(\beta)$. Let $E_\beta = (A_\beta \setminus \Sigma(\beta)) \cap \nu_\beta$ which is a set bounded below κ. Let $E = \bigcup_{\beta<\mathrm{cof}(\kappa)} E_\beta$. Observe that $|E| = \kappa$ and thus $E \in \mathscr{P}_\kappa(\kappa)^+$. Note that for all $\alpha < \beta < \mathrm{cof}(\kappa)$, $A_\alpha \cap E_\beta \subseteq A_\alpha \cap (A_\beta \setminus \Sigma(\alpha)) = \emptyset$. Thus $A_\alpha \cap E \subseteq A_\alpha \cap \bigcup_{\gamma\leq\alpha} E_\gamma$. Thus $A_\alpha \cap E$ is bounded by $\sup\{\nu_\gamma : \gamma \leq \alpha\} < \kappa$ and in particular, $|A_\alpha \cap E| < \kappa$. Thus $\mathcal{A} \cup \{E\}$ is a strictly larger $\mathscr{P}_\kappa(\kappa)$ almost disjoint family than $\mathcal{A}$. □

3. Concerning Wellorderability of Almost Disjoint Families

Fact 3.1. *Assume all sets of reals have the Baire property, then a wellordered union of meager subsets of $\mathbb{R}$ is a meager subset of $\mathbb{R}$. That is, if $\zeta \in \mathrm{ON}$ and $\langle A_\alpha : \alpha < \zeta\rangle$ is a sequence of meager subsets of $\mathbb{R}$, then $\bigcup_{\alpha<\zeta} A_\alpha$ is a meager subset of $\mathbb{R}$.*

Proof. Consider $\mathbb{R}$ as ${}^\omega\omega$. Suppose there is a wellordered sequence of meager sets whose union is not meager. Then there is a sequence $\langle A_\alpha : \alpha < \delta\rangle$ of meager subsets of ${}^\omega\omega$ (where δ is an ordinal) such that $\bigcup_{\alpha<\delta} A_\alpha$ is nonmeager but for all $\gamma < \delta$, $\bigcup_{\alpha<\gamma} A_\alpha$ is meager. By the property of Baire, there is an $s \in {}^{<\omega}\omega$ so that $\bigcup_{\alpha<\delta} A_\alpha$ is comeager in the basic neighborhood $N_s = \{f \in {}^\omega\omega : s \subseteq f\}$. Let $\Pi : N_s \to {}^\omega\omega$ be a homeomorphism. For each $\alpha < \delta$, let $B_\alpha = \Pi[A_\alpha \cap N_s]$. Then $\langle B_\alpha : \alpha < \delta\rangle$ has the property that $\bigcup_{\alpha<\delta} B_\alpha$ is comeager and for all $\gamma < \delta$, $\bigcup_{\alpha<\gamma} B_\alpha$ is meager. For $x \in \bigcup_{\alpha<\delta} B_\alpha$, let $\iota(x)$ be the least $\alpha < \delta$ so that $x \in B_\alpha$. Define $R(x, y)$ by $x, y \in \bigcup_{\alpha<\delta} B_\alpha$ and $\iota(x) \leq \iota(y)$. For any $x \in \bigcup_{\alpha<\delta} B_\alpha$, $R_x = \{y \in {}^\omega\omega : R(x, y)\} = \bigcup_{\iota(x)\leq\alpha<\delta} B_\alpha$ is comeager. So $\{x \in {}^\omega\omega : R_x \text{ is comeager}\} = \bigcup_{\alpha<\delta} B_\alpha$ is comeager. By the Kuratowski-Ulam theorem, R is comeager in ${}^\omega\omega \times {}^\omega\omega$. By the Kuratowski-Ulam theorem again, $\{y \in {}^\omega\omega : R^y \text{ is comeager}\}$ is comeager, where $R^y = \{x \in {}^\omega\omega : R(x, y)\}$. However, for all $y \in \bigcup_{\alpha<\delta} B_\alpha$, $R^y = \bigcup_{\alpha\leq\iota(y)} B_\alpha$ is meager. Thus $\{y \in {}^\omega\omega : R^y \text{ is meager}\}$ is comeager. Contradiction. □

Fact 3.2. *Assume all sets of reals have the Baire property. Then there is no injection of $\mathbb{R}$ into* ON.

Proof. Suppose $\Phi : \mathbb{R} \to \mathrm{ON}$ is an injection. By replacement, there is a $\delta < \mathrm{ON}$ so that $\Phi[\mathbb{R}] \subseteq \delta$. Then $\mathbb{R} = \bigcup_{\alpha<\delta} \Phi^{-1}[\{\alpha\}]$. Since Φ is an injection, this shows that $\mathbb{R}$ is a wellordered union of singletons (which are meager sets). By Fact 3.1, $\mathbb{R}$ is meager which is impossible. □

Fact 3.3. *If all sets of reals have the Baire property and the perfect set property, then there are no uncountable wellorderable sets of reals.*

Proof. Let X be an uncountable wellorderable set of reals. Thus let $\Psi : X \to \delta$ be a bijection into some ordinal δ. By the perfect set property, there is an injection $\Phi : \mathbb{R} \to X$. Then $\Psi \circ \Phi : \mathbb{R} \to \delta$ is an injection of $\mathbb{R}$ into the ordinals which is impossible by Fact 3.2. □

Fact 3.4. *Assume all sets of reals have the Baire property. Suppose $\delta \in \mathrm{ON}$ and $\langle T_\alpha : \alpha < \delta\rangle$ is a sequence of pairwise disjoint subsets of $\mathbb{R}$ so that $\bigcup_{\alpha<\delta} T_\alpha$ is a nonmeager subset of $\mathbb{R}$. Then $E = \{\alpha : T_\alpha \text{ is nonmeager}\}$ is countable.*

Proof. Suppose E as above is uncountable. Let $\Phi : \omega_1 \to E$ be an injection. Consider $\mathbb{R}$ as ${}^\omega 2$ and fix a recursive wellordering of ${}^{<\omega}2$. For each $s \in {}^{<\omega}2$, let $N_s = \{f \in {}^\omega 2 : s \subseteq f\}$. Since all sets of reals have the Baire property and each $T_{\Phi(\alpha)}$ is nonmeager, let s_α be the least $s \in {}^{<\omega}2$ so that $T_{\Phi(\alpha)}$ is comeager in N_s. Since Φ is an injection and $\langle T_\alpha : \alpha < \delta\rangle$ consists of pairwise disjoint sets, if $\alpha < \beta < \omega_1$, then $s_\alpha \neq s_\beta$. Thus $\langle s_\alpha : \alpha < \omega_1\rangle$ is an injection of ω_1 into the countable set ${}^{<\omega}2$. Contradiction. □

The following result asserts that for all ordinals κ and function $\Phi : \mathbb{R} \to \mathscr{P}(\kappa)$, Φ is well behaved on a comeager set: there is a countable set $\mathcal{E}$ and a comeager set C so that for all $x \in C$, $\Phi(x)$ is a union of elements from $\mathcal{E}$.

Proposition 3.5. *Assume all sets of reals have the Baire property and there are no uncountable wellorderable sets of reals. Suppose κ is an ordinal and $\Phi : \mathbb{R} \to \mathscr{P}(\kappa)$. Then there is a countable set $\mathcal{E} \subseteq \mathscr{P}(\kappa)$ and a comeager subset $C \subseteq \mathbb{R}$ so that $\mathcal{E}$ consists of pairwise disjoint subsets of κ, $\bigcup \mathcal{E} = \kappa$, and for all $x \in C$, there is an $\mathcal{F} \subseteq \mathcal{E}$ so that $\Phi(x) = \bigcup \mathcal{F}$.*

Proof. For each $\alpha < \kappa$, let $B_\alpha = \{x \in {}^\omega 2 : \alpha \in \Phi(x)\}$. For each $\alpha < \kappa$, let $U_\alpha = \{s \in {}^{<\omega}2 : B_\alpha \text{ is comeager in } N_s\}$. Let $K_\alpha = \bigcup_{s \in U_\alpha} N_s$. Since all sets of reals have the Baire property, $B_\alpha \triangle K_\alpha$ is meager. Let $C_\alpha = {}^\omega 2 \setminus (B_\alpha \triangle K_\alpha) = (B_\alpha \cap K_\alpha) \cup ({}^\omega 2 \setminus (B_\alpha \cup K_\alpha))$, which is a comeager subset of ${}^\omega 2$. Observe that for all $x \in C_\alpha$, $x \in B_\alpha$ if and only if $(\exists s \in U_\alpha)(s \subseteq x)$. By Fact 3.1, $C = \bigcap_{\alpha<\kappa} C_\alpha$ is comeager. Since for each $\alpha < \kappa$, U_α is a subset of ${}^{<\omega}2$ and there are no uncountable wellorderable set of reals, $\{U_\alpha : \alpha < \kappa\}$ is a countable set. Let $\langle \hat{U}_n : n \in \omega\rangle$ be a surjection onto this countable set. For each $\alpha \in \kappa$, let $\mathfrak{n}(\alpha)$ be the least $n \in \omega$ so that $\hat{U}_n = U_\alpha$. Let $E_n = \{\alpha \in \kappa : \mathfrak{n}(\alpha) = n\}$. Note that if $m \neq n$, then $E_m \cap E_n = \emptyset$. Suppose $n \in \omega$, $x \in C$, and $\alpha, \beta \in E_n$. Note $\alpha \in \Phi(x)$ if and only if $x \in B_\alpha$ if and only if $(\exists s \in U_\alpha)(s \subseteq x)$ if and only if $(\exists s \in \hat{U}_n)(s \subseteq x)$ if and only if $(\exists s \in U_\beta)(s \subseteq x)$ if and only if $x \in B_\beta$ if and only if $\beta \in \Phi(x)$. It has been shown that for all $n \in \omega$ and $x \in C$, $E_n \subseteq \Phi(x)$ or $E_n \cap \Phi(x) = \emptyset$. Let $\mathcal{E} = \{E_n : n \in \omega\}$. Since $\bigcup_{n\in\omega} E_n = \kappa$, $\Phi(x) = \bigcup \mathcal{F}$ where $\mathcal{F} = \bigcup\{E_n : E_n \subseteq \Phi(x)\}$. □

Definition 3.6. A set $p \subseteq {}^{<\omega}2$ is a tree if and only if for all $s, t \in {}^{<\omega}2$, if $s \subseteq t$ (s is an initial segment of t) and $t \in p$, then $s \in p$. If p is a tree, then let $[p] = \{f \in {}^\omega 2 : (\forall n)(f \restriction n \in p)\}$. A tree p is a perfect tree if and only if $p \neq \emptyset$ and for all $s \in p$, there is a $t \in p$ with $s \subseteq t$ and $t\hat{\ }0, t\hat{\ }1 \in p$. If p is pefect, then $|[p]| = |{}^\omega 2|$.

Fact 3.7. *(Mycielski's theorem) Suppose $R \subseteq {}^\omega 2 \times {}^\omega 2$ is a comeager subset of $\mathbb{R}^2$, then there is a perfect tree $p \subseteq {}^{<\omega}2$ so that $\{(x, y) \in [p] \times [p] : x \neq y\} \subseteq R$.*

Proof. Since R is comeager, there is a sequence $\langle D_n : n \in \omega\rangle$ of comeager subset of ${}^\omega 2 \times {}^\omega 2$ with the property that for all $n \in \omega$, $D_{n+1} \subseteq D_n$ and $\bigcap_{n\in\omega} D_n \subseteq R$. A sequence $\langle u_s : s \in {}^{<\omega}2\rangle$ will be defined with the following properties:

1. For all $s \in {}^{<\omega}2$ and $i \in \omega$, $u_s\hat{\ }i \subseteq u_{s\hat{\ }i}$.
2. For all $s, t \in {}^{<\omega}2$ with $s \neq t$ and $|s| = |t|$, if $n = |s| - 1$, then $N_{u_s} \times N_{u_t} \subseteq D_n$.

Let p be the $\subseteq$-downward closure of $\{u_s : s \in {}^{<\omega}2\}$. p is a perfect tree with the desired properties. It remains to construct $\langle u_s : s \in {}^{<\omega}2\rangle$. Let $u_\emptyset = \emptyset$. Suppose $n \in \omega$ and u_s has been defined for all $s \in {}^n 2$. Let $\langle (e_i, f_i) : i < K\rangle$ for some $K \in \omega$ enumerate $\{(e, f) : e, f \in {}^{n+1}2 \wedge e \neq f\}$. For each $s \in {}^n 2$, let $v^0_{s\hat{\ }i} = u_s\hat{\ }i$. This defines $\langle v^0_s : s \in {}^{n+1}2\rangle$. Suppose $i < K$ and $\langle u^i_s : s \in {}^{n+1}2\rangle$ has been defined. Since D_n is dense open, there is some $a, b \in {}^{<\omega}2$ so that $u_{e_i} \subseteq a$, $u_{f_i} \subseteq b$ and $N_a \times N_b \subseteq D_n$. Let $v^{i+1}_{e_i} = a$, $v^{i+1}_{f_i} = b$, and $v^{i+1}_s = v^i_s$ for all $s \in {}^{n+1}2$ with $s \neq e_i$ and $s \neq f_i$. Let $u_s = v^K_s$ for all $s \in {}^{n+1}2$. This defines $\langle u_s : s \in {}^{n+1}2\rangle$. This completes the construction. □

Lemma 3.8. *Assume all sets of reals have the Baire property. Suppose κ is a cardinal with $\mathrm{cof}(\kappa) > \omega$. If $\mathcal{A}$ is a $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family, then there is no injection of $\mathbb{R}$ into $\mathcal{A}$.*

Proof. Suppose there is a cardinal κ with $\mathrm{cof}(\kappa) > \omega$, a $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family $\mathcal{A}$, and an injection $\Phi : \mathbb{R} \to \mathcal{A}$. For each $r \in \mathbb{R}$, let $A_r = \Phi(r)$. Let $\mathbb{R}^2_* = \{(r, s) \in \mathbb{R}^2 : r \neq s\}$ which is a comeager subset of $\mathbb{R}^2$. Define $\Psi : \mathbb{R}^2_* \to \kappa$ by $\Psi(r, s) = \mathrm{ot}(A_r \cap A_s)$. Let $E = \{\gamma \in \kappa : \Psi^{-1}[\{\gamma\}] \text{ is nonmeager}\}$ which is a countable set by Fact 3.4. For each $\gamma \in E$ and $\eta < \gamma$, let $\Sigma^\gamma_\eta : \Psi^{-1}[\{\gamma\}] \to \kappa$ be defined by $\Sigma^\gamma_\eta(r, s)$ is the η^{th}-element of $A_r \cap A_s$. Again by Fact 3.4, the set $F^\gamma_\eta = \{\alpha \in \kappa : (\Sigma^\gamma_\eta)^{-1}[\{\alpha\}] \text{ is nonmeager}\}$ is countable. Let $D^\gamma_\eta = \bigcup_{\alpha \in F^\gamma_\eta} (\Sigma^\gamma_\eta)^{-1}(\{\alpha\})$ which is comeager in the nonmeager set $\Psi^{-1}[\{\gamma\}]$. Let $D^\gamma = \bigcap_{\eta<\gamma} D^\gamma_\eta$ which is a comeager subset of $\Psi^{-1}[\{\gamma\}]$ by Fact 3.1. Let $D = \bigcup_{\gamma \in E} D^\gamma$ which is a comeager subset of $\mathbb{R}^2$. Since for each $\gamma \in E$ and $\eta < \gamma$, F^γ_η is countable, $|\bigcup_{\eta<\gamma} F^\gamma_\eta| < \kappa$. Since E is countable and $\mathrm{cof}(\kappa) > \omega$, $G = \bigcup_{\gamma\in E}\bigcup_{\eta<\gamma} F^\gamma_\eta$ has cardinality strictly less than κ. Let $(r,s) \in D$. There is a $\gamma \in E$ so that $(r,s) \in D^\gamma \subseteq \Psi^{-1}[\{\gamma\}]$. So $\mathrm{ot}(A_r \cap A_s) = \gamma$. For each $\eta < \gamma$, $(r,s) \in D^\gamma_\eta = \bigcup_{\alpha \in F^\gamma_\eta} (\Sigma^\gamma_\eta)^{-1}[\{\alpha\}]$. For each $\eta < \gamma$, there is an $\alpha \in F^\gamma_\eta$ so that $\Sigma^\gamma_\eta(r,s) = \alpha$ which means that the η^{th}-element of $A_r \cap A_s$ is α. Since $\eta < \gamma$ was arbitrary, this shows that $A_r \cap A_s \subseteq \bigcup_{\eta<\gamma} F^\gamma_\eta \subseteq G$. Let $H = \kappa \setminus G$. Observe that since for all $(r,s) \in D$, $A_r \cap A_s \subseteq G$, $(A_r \cap H) \cap (A_s \cap H) = \emptyset$. Also since $|G| < \kappa$ and $|A_r| = |A_s| = \kappa$, one has that $A_r \cap H \neq \emptyset$ and $A_s \cap H \neq \emptyset$. By Fact 3.7, there is a perfect tree $p \subseteq {}^{<\omega}2$ so that $\{(r,s) \in [p] \times [p] : r \neq s\} \subseteq D$. Let $\Lambda : \mathbb{R} \to [p]$ be a bijection. Let $\Upsilon : \mathbb{R} \to \kappa$ be defined by $\Upsilon(r) = \min(A_{\Lambda(r)} \cap H)$. If $r, s \in \mathbb{R}$ and $r \neq s$, then $(\Lambda(r), \Lambda(s)) \in D$ and thus $(A_{\Lambda(r)} \cap H) \cap (A_{\Lambda(s)} \cap H) = \emptyset$. Hence $\Upsilon(r) \neq \Upsilon(s)$. This shows that $\Upsilon : \mathbb{R} \to \kappa$ is an injection. This is impossible by Fact 3.2. □

Proof. (The following is another proof of Lemma 3.8 under the additional assumption that there are no uncountable wellorderable sets of reals.) Suppose κ is a cardinal with $\mathrm{cof}(\kappa) > \omega$, $\mathcal{A}$ is a $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family, and $\Phi : \mathbb{R} \to \mathcal{A}$ is an injection. By Proposition 3.5, there is a countable set $\mathcal{E}$ and a comeager $C \subseteq \mathbb{R}$ such that $\mathcal{E}$ consists of pairwise disjoint subsets of κ, $\bigcup \mathcal{E} = \kappa$, and for every $x \in C$, $\Phi(x)$ is a countable union of elements from $\mathcal{E}$. Let $\mathcal{E}' = \{E \in \mathcal{E} : |E| = \kappa\}$. For each $E \in \mathcal{E}'$, let $K_E = \{x \in C : E \subseteq \Phi(x)\}$. Since $\mathcal{E}'$ is countable, there is some $E \in \mathcal{E}'$ so that K_E has at least two elements. Let $x, y \in K_E$ with $x \neq y$. Then $E \subseteq \Phi(x) \cap \Phi(y)$ and $|E| = \kappa$. Thus $|\Phi(x) \cap \Phi(y)| = \kappa$. Hence $\mathcal{A}$ is not a $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family. Contradiction. □

Definition 3.9. For $x, y \in {}^\omega 2$, let $x \leq_{\text{Turing}} y$ denote x is Turing reducible to y. Let $x \equiv_{\text{Turing}} y$ if and only if $x \leq_{\text{Turing}} y$ and $y \leq_{\text{Turing}} x$. $\equiv_{\text{Turing}}$ is an equivalence relation on ${}^\omega 2$. For any $x \in {}^\omega 2$, let $[x]_{\text{Turing}}$ be the $\equiv_{\text{Turing}}$ equivalence class of x and is called the degree of x. Let $\mathcal{D}_{\text{Turing}}$ be the set of $\equiv_{\text{Turing}}$-degrees. If $X, Y \in \mathcal{D}_{\text{Turing}}$, then let $X \leq_{\text{Turing}} Y$ if $x \leq_{\text{Turing}} y$ for all $x \in X$ and $y \in Y$. If $X \in \mathcal{D}_{\text{Turing}}$, then let $\mathcal{C}_X = \{Y \in \mathcal{D}_{\text{Turing}} : X \leq_{\text{Turing}} Y\}$ which is the Turing cone above X.

Let the Martin filter μ_{Turing} on $\mathcal{D}_{\text{Turing}}$ be defined by $A \in \mu_{\text{Turing}}$ if and only if there exists an $X \in \mathcal{D}_{\text{Turing}}$ so that $\mathcal{C}_X \subseteq A$. Under AD, Martin showed that μ_{Turing} is a countably complete ultrafilter.

Definition 3.10. Let X be a set. The perfect set dichotomy holds for X if and only if either X is wellorderable or $|\mathbb{R}| \leq |X|$ (that is, there is an injection of $\mathbb{R}$ into X).

Fact 3.11. *The following are some settings in which the perfect set dichotomy holds.*

1. *(Woodin) Assume AD, all sets of reals have ∞-Borel codes, and $\prod_{\mathcal{D}_{\text{Turing}}} \omega_1/\mu_{\text{Turing}}$ is wellfounded. (These assumptions hold under AD^+.) For every set X which is a surjective image of $\mathbb{R}$, the perfect set dichotomy holds for X.*
2. *(Caicedo-Kechersid; [1]) Assume $\mathsf{AD}^+ + V = L(\mathscr{P}(\mathbb{R}))$. Then every set X satisfies the perfect set dichotomy.*

Proof. For the argument of (1) see [2] Theorem 8.5 or [9]. For (2), under these assumptions, Woodin showed $V = L(J, \mathbb{R})$ where J is a set of ordinals or $V \models \mathsf{AD}_{\mathbb{R}}$. If $V = L(J, \mathbb{R})$, then every set is uniformly (in the ordinals, reals, and formula witnessing that the original set is ordinal definable from J and a real) a wellorderable union of sets which are surjective images of $\mathbb{R}$ coming from the structure of $L(J, \mathbb{R})$. If $V \models \mathsf{AD}_{\mathbb{R}}$, then every set is also a uniform wellorderable union of sets which are surjective images of $\mathbb{R}$ using certain supercompactness measures. By (1), each piece either contains an injective copy of $\mathbb{R}$ or is wellorderable. By the uniformity of the proof of (1), if all pieces are wellorderable, then there is uniformly a wellordering for each pieces. These wellorderings can be patched together to provide a wellordering of the original set. The details of a similar argument can be found in [3]. □

Theorem 3.12. *Assume all sets of reals have the Baire property. If κ is a cardinal with $\mathrm{cof}(\kappa) > \omega$ and the perfect set dichotomy holds for all subsets of $\mathscr{P}(\kappa)$, then every $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family is wellorderable.*

Proof. Suppose $\mathcal{A}$ is a $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family. By Lemma 3.8, there is no injection of $\mathbb{R}$ into $\mathcal{A}$. Since the perfect set dichotomy holds for $\mathcal{A}$, $\mathcal{A}$ must be wellorderable. □

Theorem 3.13. *Assume AD^+. If $\kappa < \Theta$ and $\mathrm{cof}(\kappa) > \omega$, then every $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family is wellorderable.*

Assume $\mathsf{AD}^+ + V = L(\mathscr{P}(\mathbb{R}))$. If κ is a cardinal with $\mathrm{cof}(\kappa) > \omega$, then every $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family is wellorderable.

Proof. Lemma 3.8 implies that there is no injection of $\mathbb{R}$ into any $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family for any κ with $\mathrm{cof}(\kappa) > \omega$. The first statement follows since the perfect set dichotomy holds for all sets which are surjective images of $\mathbb{R}$ by Fact 3.11 (1). The second statement follows since all sets now satisfy the perfect set dichotomy by Fact 3.11 (2). □

Fact 3.14. *(Moschovakis Coding Lemma;* [6] *Theorem 2.12) Assume* AD*. Let $P \subseteq \mathbb{R}$, $\preceq$ is a prewellordering on P, and $\varphi : P \to \kappa$ is the unique surjective norm associated to $\preceq$. Let Γ be a nonselfdual pointclass closed $\exists^{\mathbb{R}}$ and $\wedge$ and contains P and $\preceq$. For all $R \subseteq P \times \mathbb{R}$, there is an $S \in \Gamma$ so that $S \subseteq R$ and for all $\alpha < \kappa$, $R \cap (\varphi^{-1}[\{\alpha\}] \times \mathbb{R}) \neq \emptyset$ if and only if $S \cap (\varphi^{-1}[\{\alpha\}] \times \mathbb{R}) \neq \emptyset$.*

The following consequence of the Moschovakis coding lemma is very useful.

Fact 3.15. *(Moschovakis) If $\kappa < \Theta$, then there is a surjection of $\mathbb{R}$ onto $\mathscr{P}(\kappa)$.*

Proof. If $\kappa < \Theta$, then there is a surjection $\varphi : \mathbb{R} \to \kappa$. Let $\preceq$ be a prewellordering on $\mathbb{R}$ defined by $x \preceq y$ if and only if $\varphi(x) \leq \varphi(y)$. Let Γ be any nonselfdual pointclass closed under $\exists^{\mathbb{R}}$ and $\wedge$ and $\preceq \in \Gamma$ (for example, $\Gamma = \boldsymbol{\Sigma}^1_1(\preceq)$). Since Γ is nonselfdual, the Wadge lemma implies there is a $U \subseteq \mathbb{R}^3$ which is universal for subsets of $\mathbb{R} \times \mathbb{R}$ in Γ. If $e \in \mathbb{R}$, define $A_e = \{\alpha < \kappa : (\exists x, y \in \mathbb{R})(\varphi(x) = \alpha \wedge U(e, x, y))\}$. Define $\Pi : \mathbb{R} \to \mathscr{P}(\kappa)$ by $\Pi(e) = A_e$. Fix $A \subseteq \kappa$. Define $R_A \subseteq \mathbb{R} \times \mathbb{R}$ by $R_A(x, y) \Leftrightarrow \varphi(x) \in A$. By the Moschovakis coding lemma (Fact 3.14), there is an $S \in \Gamma$ so that $S \subseteq R_A$ and for all $\alpha < \kappa$, $R_A \cap (\varphi^{-1}[\{\alpha\}] \times \mathbb{R}) \neq \emptyset$ if and only if $S \cap (\varphi^{-1}[\{\alpha\}] \times \mathbb{R}) \neq \emptyset$. Since U is universal, there is some $e \in \mathbb{R}$ so that $S = U_e$. Then $A = A_e = \Pi(e)$. Π has been shown to be a surjection. □

Fact 3.16. *(Kunen) Assume* AD*. Every countably complete filter on an ordinal $\kappa < \Theta$ can be extended to a countably complete ultrafilter on κ.*

Proof. Let $\mathcal{F}$ be a countably complete filter. Since $\kappa < \Theta$, the Moschovakis coding lemma (Fact 3.15) implies there is a surjection $\tilde{\pi} : \mathbb{R} \to \mathscr{P}(\kappa)$ and thus a surjection $\pi : \mathbb{R} \to \mathcal{F}$. For each Turing degree X, let $E_X = \{\pi(y) : [y]_{\mathrm{Turing}} \leq_{\mathrm{Turing}} X\}$ which is a countable set. Since $\mathcal{F}$ is countably complete, $\bigcap E_X \in \mathcal{F}$ and thus $E_X \neq \emptyset$. Define $\Phi : \mathcal{D}_{\mathrm{Turing}} \to \kappa$ by $\Phi(X) = \min(\bigcap E_X)$. Let $\mathcal{U} = \{A \subseteq \kappa : \Phi^{-1}[A] \in \mu_{\mathrm{Turing}}\}$ which is a countably complete ultrafilter on κ. Suppose $A \in \mathcal{F}$. There is an $x \in \mathbb{R}$ with $\pi(x) = A$. Let $X = [x]_{\mathrm{Turing}}$. For all $Y \geq_{\mathrm{Turing}} X$, $A = \pi(x) \in E_Y$. Thus $\bigcap E_Y \subseteq A$ and so $\Phi(Y) = \min(\bigcap E_Y) \in A$. Thus the Turing cone above X lies inside of $\Phi^{-1}[A]$ and hence $A \in \mathcal{U}$. This shows $\mathcal{F} \subseteq \mathcal{U}$. □

Proposition 3.17. *Suppose κ is a cardinal and there exists a countably complete ultrafilter μ on κ so that every subset $A \subseteq \kappa$ with $|A| < \kappa$ does not belong to μ and the ultrapower ${}^\kappa\kappa/\mu$ is a wellordering. Then every $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family is wellorderable.*

Proof. Let $\mathcal{A}$ be a $\mathscr{P}_\kappa(\kappa)$ almost disjoint family. For each $A \in \mathcal{A}$, since $A \in \mathscr{P}_\kappa(\kappa)$, let $f_A : \kappa \to A$ be the strictly increasing enumeration of A. Define $\Phi : \mathcal{A} \to {}^\kappa\kappa/\mu$ by $\Phi(A) = [f_A]_\mu$. Suppose $A, B \in \mathcal{A}$ and $A \neq B$. Let $E = \{\alpha \in \kappa : f_A(\alpha) = f_B(\alpha)\}$. Then $|E| \leq |A \cap B| < \kappa$. By the hypothesis on μ, $E \notin \mu$. Thus $\Phi(A) = [f_A]_\mu \neq [f_B]_\mu = \Phi(B)$. $\Phi : \mathcal{A} \to {}^\kappa\kappa/\mu$ is an injection of $\mathcal{A}$ into a wellordering and therefore $\mathcal{A}$ is wellorderable.

Now suppose $\mathcal{A}$ is a $\mathscr{B}(\kappa)$ almost disjoint family. For each $A \in \mathcal{A}$ and $\alpha < \kappa$, let $g_A(\alpha)$ be the least element of A strictly greater than α which exists since $A \in \mathscr{B}(\kappa)^+$ is unbounded in κ. Suppose $A, B \in \mathcal{A}$ and $A \neq B$. Let $\delta = \sup(A \cap B) < \kappa$. For all $\alpha > \delta$, $g_A(\alpha) \neq g_B(\alpha)$ and so $[g_A]_\mu \neq [g_B]_\mu$. The map $\Psi : \mathcal{A} \to {}^\kappa\kappa/\mu$ defined by $\Psi(A) = [g_A]_\mu$ is an injection and so $\mathcal{A}$ is wellorderable. □

Theorem 3.18. *Assume* AD *and* $\mathsf{DC}_{\mathbb{R}}$*. If $\kappa < \Theta$ and $\mathrm{cof}(\kappa) > \omega$, then every $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family is wellorderable.*

Proof. Since $\mathrm{cof}(\kappa) > \omega$, the filter $\mathcal{F}$ consisting of $A \subseteq \kappa$ so that $|\kappa \setminus A| < \kappa$ is a countably complete filter on κ. By Fact 3.16, let μ be a countably complete ultrafilter extending $\mathcal{F}$. Every $A \subseteq \kappa$ with $|A| < \kappa$ does not belong to μ since $\kappa \setminus A \in \mathcal{F} \subseteq \mu$. Since $\kappa < \Theta$, the Moschovakis coding lemma (Fact 3.15) implies there is a surjection of $\mathbb{R}$ onto $\mathscr{P}(\kappa)$. Thus $\mathsf{DC}_\mathbb{R}$ is sufficient to provide $\mathsf{DC}_{{}^\kappa\kappa}$ and hence ${}^\kappa\kappa/\mu$ is a wellordering. Proposition 3.17 implies that every $\mathscr{P}_\kappa(\kappa)$ and $\mathscr{B}(\kappa)$ almost disjoint family on κ is wellorderable. □

Definition 3.19. If κ is a cardinal, then boldface GCH at κ is the statement that there is no injection of κ^+ into $\mathscr{P}(\kappa)$. One says that boldface GCH holds below λ if and only if boldface GCH holds for all $\kappa < \lambda$.

Fact 3.20. *If κ is a cardinal so that κ^+ is measurable, then boldface GCH holds at κ.*

Proof. Let μ be a κ^+-complete nonprincipal ultrafilter on κ^+. Suppose there is an injection $\Phi : \kappa^+ \to \mathscr{P}(\kappa)$. For each $\beta < \kappa$, let $E^0_\beta = \{\alpha < \kappa^+ : \beta \notin \Phi(\alpha)\}$ and $E^1_\beta = \{\alpha < \kappa^+ : \beta \in \Phi(\alpha)\}$. Since $E^0_\beta \cup E^1_\beta = \kappa^+$ and μ is an ultrafilter, there is an $i_\beta \in 2$ so that $E^{i_\beta}_\beta \in \mu$. Let $A^* = \{\beta \in \kappa : i_\beta = 1\}$. Since μ is κ^+-complete, $E = \bigcap_{\beta<\kappa} E^{i_\beta}_\beta \in \mu$. Since μ is nonprincipal, let $\delta, \epsilon \in E$ with $\delta \neq \epsilon$. Then $\Phi(\delta) = A^* = \Phi(\epsilon)$. This contradicts Φ being an injection. □

Under AD, Martin showed that ω_2 is measurable (by being a weak partition cardinal). Thus boldface GCH holds at ω_1. Under $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$, Jackson and Kunen showed for all $n \in \omega$, $\boldsymbol{\delta}^1_{2n+2} = \boldsymbol{\delta}^+_{2n+1}$ is a measurable cardinal (by being a weak partition cardinal). Thus under $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$, boldface GCH holds at $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ and more generally $\boldsymbol{\delta}^1_{2n+1}$ for all $n \in \omega$.

Definition 3.21. If κ is an ordinal, a κ-tree $\mathcal{T}$ is a partial order $(\kappa, \prec)$ so that for all $\gamma \in T$, $\{\beta \in \kappa : \beta \prec \gamma\}$ is a wellordering under $\prec$ and for each $\alpha < \kappa$, the α^{th} level $\mathcal{L}^\mathcal{T}_\alpha = \{\beta \in T : \text{rk}_\mathcal{T}(t) = \alpha\}$ has cardinality less than κ. A branch of $\mathcal{T}$ is a maximal subset of κ which is linearly ordered by $\prec$. If $\mathcal{T}$ is a tree, then let $\mathcal{B}_\mathcal{T}$ be the set of branches of $\mathcal{T}$.

A Kurepa tree is a ω_1-tree $\mathcal{T} = (\omega_1, \prec)$ so that $\neg(|\mathcal{B}_\mathcal{T}| \leq \omega_1)$, i.e $\mathcal{B}_\mathcal{T}$ does not inject into ω_1. For a cardinal κ, a κ-Kurepa tree is a κ-tree $\mathcal{T} = (\kappa, \prec)$ so that $\neg(|\mathcal{B}_\mathcal{T}| \leq \kappa)$.

The Kurepa hypothesis KH is the assertion that there is a Kurepa tree. The κ-Kurepa hypothesis is the assertion that there is a κ-Kurepa tree.

Fact 3.22. *If κ is a cardinal and $\mathcal{T}$ is a κ-tree on κ, then $\mathcal{B}_\mathcal{T}$ is a $\mathscr{P}_\kappa(\kappa)$ almost disjoint family.*

The following result resolves the Kurepa hypothesis and the maximal almost disjoint family problem for ω_1, which answers a question of Müller.

Theorem 3.23. (AD) *There are no uncountable maximal $\mathscr{P}_{\omega_1}(\omega_1) = \mathscr{B}(\omega_1)$ almost disjoint families. There are no Kurepa trees and thus the Kurepa hypothesis KH fails.*

Proof. AD implies the partition relation $\omega_1 \to (\omega_1)^{\omega_1}_2$. This partition relation implies the club filter W^1_1 on ω_1 is a countably complete normal ultrafilter. In particular, ω_1 is a regular cardinal of uncountable cofinality and thus $\mathscr{P}_{\omega_1}(\omega_1) = \mathscr{B}(\omega_1)$. Using Kunen trees and normality, [2] Fact 5.8 and 5.13 show that ${}^{\omega_1}\omega_1/W^1_1$ is a wellordering without using $\mathsf{DC}_\mathbb{R}$. $\omega_2 = (\omega_1)^+$ is measurable under AD and therefore boldface GCH holds at ω_1 by Fact 3.20.

Let $\mathcal{A}$ be a $\mathscr{P}_{\omega_1}(\omega_1) = \mathscr{B}(\omega_1)$ almost disjoint family. By Proposition 3.17, $\mathcal{A}$ is wellorderable. Therefore by boldface GCH at ω_1, $|\mathcal{A}| \leq \omega_1$. If $|\mathcal{A}| = \omega_1$, then $\mathcal{A}$ cannot be maximal by Fact 2.5. Thus there are no uncountable maximal $\mathscr{P}_{\omega_1}(\omega_1) = \mathscr{B}(\omega_1)$ almost disjoint families.

Let $\mathcal{T}$ be a tree on ω_1. $\mathcal{B}_\mathcal{T}$ is $\mathscr{P}_{\omega_1}(\omega_1)$ almost disjoint family and thus by the above $|\mathcal{B}_\mathcal{T}| \leq \omega_1$. $\mathcal{T}$ is not a Kurepa tree. The Kurepa hypothesis fails. □

Proposition 3.24. *Suppose κ is a cardinal with $\text{cof}(\kappa) > \omega$ and boldface GCH holds at κ. Suppose one of the following two hypotheses holds:*

(1) There is a countably complete ultrafilter μ on κ so that for all $A \subseteq \kappa$ with $|A| < \kappa$, $A \notin \mu$ and ${}^\kappa\kappa/\mu$ is a wellordering.

(2) All sets of reals have the Baire property and the perfect set dichotomy holds for all subsets of $\mathscr{P}(\kappa)$.

Then there are no κ-Kurepa trees (and hence κ-Kurepa hypothesis fails), and if κ is regular, then every maximal $\mathscr{P}_\kappa(\kappa) = \mathscr{B}(\kappa)$ almost disjoint family is wellorderable of size strictly less than κ.

Proof. Let $\mathcal{A}$ be a $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family. $\mathcal{A}$ must be wellorderable by Proposition 3.17 under the first assumption and Theorem 3.12 under the second assumption. By the boldface GCH, $|\mathcal{A}| \leq \kappa$. If κ

was regular, then by Fact 2.5, there are no maximal $\mathscr{P}_\kappa(\kappa) = \mathscr{B}(\kappa)$ almost disjoint family of size $\text{cof}(\kappa) = \kappa$. Thus every maximal almost disjoint family $\mathcal{A}$ on κ must be wellorderable of size strictly less than κ.

If $\mathcal{T}$ is a tree on κ, then $\mathcal{B}_\mathcal{T}$ is a $\mathscr{P}_\kappa(\kappa)$ almost disjoint family. Thus by the previous observation, $|\mathcal{B}_\mathcal{T}| \leq \kappa$. So $\mathcal{T}$ is not a κ-Kurepa tree. The κ-Kurepa hypothesis fails. $\square$

Inner model theoretic arguments of Steel and Woodin show boldface GCH below $\Theta^{L(\mathbb{R})}$ and Θ under AD^+. More combinatorial argument can be used to establish the boldface GCH below $\omega_{\omega+1}$ (and even below the supremum of the projective ordinals).

Fact 3.25. *([20] Steel) Assume* AD. *Boldface* GCH *holds below* $\Theta^{L(\mathbb{R})}$.

(Woodin) Assume AD^+. *Boldface* GCH *holds below* Θ.

Proof. Steel's result is usually stated as a result within $L(\mathbb{R}) \models \mathsf{AD}$. The following are some brief remarks on deriving the first statement from Steel's result: Work in an arbitrary model V of AD. Suppose $\kappa < \Theta^{L(\mathbb{R})}$. There is a surjection $\pi : \mathbb{R} \to \kappa$ with $\pi \in L(\mathbb{R})$. Let $\preceq$ be the prewellordering on $\mathbb{R}$ associated to π, namely $x \preceq y$ if and only $\pi(x) \leq \pi(y)$. Let Γ be a boldface nonselfdual pointclass in $L(\mathbb{R})$ closed under $\exists^\mathbb{R}$, $\wedge$, $\vee$, containing all closed subsets, and containing $\preceq$. Let $U \subseteq \mathbb{R}^3$ be a set in Γ universal for subsets of $\mathbb{R}^2$ in Γ. Since V and $L(\mathbb{R})$ contain the same reals, Γ is also a boldface pointclass of V. For each $A \subseteq \kappa$, let $Z_A \subseteq \mathbb{R}^2$ be defined by $Z_A(w, x)$ if and only if $\pi(w) \in A$ and $x = \bar{1}$ (the constant 1 function) or $\pi(w) \notin A$ and $x = \bar{0}$ (the constant 0 function). By the Moschovakis coding lemma (Fact 3.14) applied in V, there a real e so that $U_e \subseteq Z_A$ and for all $\alpha < \kappa$, there is a $w \in \mathbb{R}$ and an $x \in \mathbb{R}$ so that $\pi(w) = \alpha$ and $U_e(w, x)$ holds. Then $\alpha \in A$ if and only if there exists a $w \in \mathbb{R}$ so that $U_e(w, \bar{1})$ holds. Since $L(\mathbb{R})$ and V have the same reals, $e \in L(\mathbb{R})$. Applying the same definition using e, π, and U in $L(\mathbb{R})$ shows that $A \in L(\mathbb{R})$. Thus $\mathscr{P}(\kappa) \cap V = \mathscr{P}(\kappa) \cap L(\mathbb{R})$. This implies that $\kappa^+ = (\kappa^+)^{L(\mathbb{R})}$. By a similar application of the coding lemma, $\mathscr{P}(\kappa^+) \cap L(\mathbb{R}) = \mathscr{P}(\kappa^+) \cap V$. Since every function $\Phi : \kappa^+ \to \mathscr{P}(\kappa)$ can be coded as a subset of κ^+, this implies that $L(\mathbb{R})$ and V have the same functions from κ^+ into $\mathscr{P}(\kappa)$. Steel ([20]) showed that if $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R})$ believes boldface GCH below its own Θ, namely $\Theta^{L(\mathbb{R})}$. Thus $L(\mathbb{R})$ has no injection of κ^+ into $\mathscr{P}(\kappa)$. Hence V has no injections of κ^+ into $\mathscr{P}(\kappa)$. $\square$

Theorem 3.26. (AD) *If* $\kappa < \Theta^{L(\mathbb{R})}$ *and* $\text{cof}(\kappa) > \omega$, *then the* κ*-Kurepa hypothesis fails, and if* κ *is regular, then there are no maximal* $\mathscr{P}_\kappa(\kappa) = \mathscr{B}(\kappa)$ *almost disjoint families.*

(AD^+). *If* $\kappa < \Theta$ *and* $\text{cof}(\kappa) > \omega$ *, then the* κ*-Kurepa hypothesis fails, and if* κ *is regular, then there are no maximal* $\mathscr{P}_\kappa(\kappa) = \mathscr{B}(\kappa)$ *almost disjoint families.*

Proof. Let $\mathcal{F}$ be the filter consisting of those $A \subseteq \kappa$ so that $|\kappa \setminus A| < \kappa$ which is a countably complete filter on κ since $\text{cof}(\kappa) > \omega$. By Fact 3.16, let μ be a countably complete ultrafilter extending $\mathcal{F}$. Now suppose $\kappa < \Theta^{L(\mathbb{R})}$. Kechris [8] showed that if AD holds, then $L(\mathbb{R}) \models \mathsf{DC}_\mathbb{R}$. By an application of the Moschovakis coding lemma (Fact 3.14) as in the proof of Fact 3.25, ${}^\kappa\kappa \cap L(\mathbb{R}) = {}^\kappa\kappa \cap V$. Thus ${}^\kappa\kappa/\mu$ is a wellordering in $L(\mathbb{R})$ and hence in V. In the AD^+ case, if $\kappa < \Theta$, then ${}^\kappa\kappa/\mu$ is a wellordering since $\mathsf{DC}_\mathbb{R}$ is a part of the theory AD^+. Alternatively, one can also apply Fact 3.11. Boldface GCH holds at κ by Fact 3.25. Therefore Proposition 3.24 implies every maximal almost disjoint family on κ must be wellorderable and have size strictly less than κ. By Fact 3.22, there are no κ-Kurepa trees. $\square$

In the above, the assumption that $\text{cof}(\kappa) > \omega$ is necessary. The following shows that if $\text{cof}(\kappa) = \omega$, then there are non-wellorderable $\mathscr{P}_\kappa(\kappa)$ and $\mathscr{B}(\kappa)$ almost disjoint families on κ.

Fact 3.27. *There is a* $\mathscr{P}_\omega(\omega) = \mathscr{B}(\omega)$ *almost disjoint family which is in bijection with* $\mathbb{R}$.

Proof. For each $f \in {}^\omega 2$, let $A_f = \{f \upharpoonright n : n \in \omega\}$. $\mathcal{A} = \{A_f : f \in {}^\omega 2\}$ is a classical almost disjoint family on ${}^{<\omega}2$. The map $\Phi : \mathbb{R} \to \mathcal{A}$ defined by $\Phi(f) = A_f$ is a bijection. Since $|{}^{<\omega}2| = \omega$, there exists a $\mathscr{P}_\omega(\omega) = \mathscr{B}(\omega)$ almost disjoint family which is in bijection with $\mathbb{R}$. $\square$

Fact 3.28. *Let* κ *be a cardinal. If* $\mathcal{A}$ *is a* $\mathscr{P}_{\text{cof}(\kappa)}(\text{cof}(\kappa)) = \mathscr{B}(\text{cof}(\kappa))$ *almost disjoint family, then there is a* $\mathcal{B}$ *which is both a* $\mathscr{P}_\kappa(\kappa)$ *and* $\mathscr{B}(\kappa)$ *almost disjoint family with* $|\mathcal{A}| = |\mathcal{B}|$.

Proof. Let $\rho : \text{cof}(\kappa) \to \kappa$ be an increasing cofinal map. For each $\alpha < \text{cof}(\kappa)$, let $I_\alpha = \{\gamma : \rho(\alpha) \leq \gamma < \rho(\alpha + 1)$. For each $A \in \mathcal{A}$, let $B_A = \bigcup\{I_\alpha : \alpha \in A\}$. Let $\mathcal{B} = \{B_A : A \in \mathcal{A}\}$. Note that $\mathcal{B} \subseteq \mathscr{P}_\kappa(\kappa)^+ \subseteq \mathscr{B}(\kappa)^+$. Suppose $A_0, A_1 \in \mathcal{A}$ and $A_0 \neq A_1$. Thus $\gamma = \sup(A_0 \cap A_1) < \text{cof}(\kappa)$. Thus $\sup(B_{A_0} \cap B_{A_1}) < \rho(\gamma)$. $\mathcal{B}$ is a $\mathscr{B}(\kappa)$ and $\mathscr{P}_\kappa(\kappa)$ almost disjoint family. The map $\Phi : \mathcal{A} \to \mathcal{B}$ defined by $\Phi(A) = B_A$ is a bijection. $\square$

Fact 3.29. *Assume $\mathbb{R}$ is not wellorderable. Then for every cardinal κ with $\mathrm{cof}(\kappa) = \omega$, there is a $\mathscr{P}_\kappa(\kappa)$ and $\mathscr{B}(\kappa)$ almost disjoint family which is not wellorderable.*

Under AD*, for every cardinal κ with $\mathrm{cof}(\kappa) = \omega$, there is a $\mathscr{P}_\kappa(\kappa)$ and $\mathscr{B}(\kappa)$ almost disjoint family which is not wellorderable.*

Proof. This follows from Fact 3.27 and Fact 3.28. □

Fact 3.30. *Assume $\mathbb{R}$ is not wellorderable. Then for every cardinal κ with $\mathrm{cof}(\kappa) = \omega$, there is a κ-Kurepa tree.*

Under AD*, for every cardinal κ with $\mathrm{cof}(\kappa) = \omega$, there is a κ-Kurepa tree.*

Proof. Suppose κ is a cardinal with $\mathrm{cof}(\kappa) = \omega$ and $\rho : \omega \to \kappa$ is an increasing cofinal sequence. Let $\langle A_\ell : \ell \in {}^{<\omega}2\rangle$ be a sequence in $\mathscr{P}(\kappa)$ with the following properties.

(1) $\bigcup\{A_\ell : \ell \in {}^{<\omega}2\} = \kappa$
(2) For all $\iota, \ell \in {}^{<\omega}2$, if $\iota \neq \ell$, then $A_\iota \cap A_\ell = \emptyset$
(3) For all $\ell \in {}^{<\omega}2$, $\mathrm{ot}(A_\ell) = \rho(|\ell|)$.

Let $\mathcal{T} = (T, \prec)$ be the tree on κ defined by $\alpha \prec \beta$ if and only if the disjunction of the following holds.

(1) There exists an $\iota \in {}^{<\omega}2$ so that $\alpha, \beta \in A_\iota$ and $\alpha < \beta$.
(2) There exists $\iota, \ell \in {}^{<\omega}2$ so that $\alpha \in A_\iota$, $\beta \in B_\iota$, and $\iota \subsetneq \ell$.

For each $f \in {}^\omega 2$, $\bigcup_{n\in\omega} A_{f\restriction n}$ is a branch of $\mathcal{T}$. Thus ${}^\omega 2$ injects into $\mathcal{B}_\mathcal{T}$, the set of branches of $\mathcal{T}$. Thus $\mathcal{B}_\mathcal{T}$ does not inject into κ. □

4. Wellorderable Almost Disjoint Families

Theorem 4.1. *Suppose κ is a cardinal and boldface* GCH *holds at $\mathrm{cof}(\kappa)$. If $\mathcal{A}$ is a wellorderable $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family of cardinality greater than $\mathrm{cof}(\kappa)$, then $\mathcal{A}$ is not maximal.*

Proof. Let $\mathcal{A}$ be a wellorderable $\mathscr{B}(\kappa)$ almost disjoint family of cardinality $\lambda > \mathrm{cof}(\kappa)$. Let $\rho : \mathrm{cof}(\kappa) \to \kappa$ be an increasing cofinal map through κ. Let $\Phi : \lambda \to \mathcal{A}$ be a bijection. For each $\alpha < \mathrm{cof}(\kappa)$, let $A_\alpha = \Phi(\alpha)$. For each $\beta < \lambda$, let $B_\beta = \Phi(\mathrm{cof}(\kappa) + \beta)$. For each $\beta < \lambda$, let $f_\beta : \mathrm{cof}(\kappa) \to \mathrm{cof}(\kappa)$ be defined by $f_\beta(\alpha)$ is the least $\gamma < \mathrm{cof}(\kappa)$ so that $\sup(A_\alpha \cap B_\beta) < \rho(\gamma)$. $\langle f_\beta : \beta < \lambda\rangle$ is essentially a sequence in $\mathscr{P}(\mathrm{cof}(\kappa))$. Since there is no injection of $\mathrm{cof}(\kappa)^+$ into $\mathscr{P}(\mathrm{cof}(\kappa))$ by the boldface GCH at $\mathrm{cof}(\kappa)$, $\{f_\beta : \beta < \lambda\}$ must have wellorderable cardinality less than or equal to $\mathrm{cof}(\kappa)$. Let $\langle g_\alpha : \alpha < \mathrm{cof}(\kappa)\rangle$ be a surjection of $\mathrm{cof}(\kappa)$ onto the set $\{f_\beta : \beta < \lambda\}$. Let $g : \mathrm{cof}(\kappa) \to \mathrm{cof}(\kappa)$ be defined by $g(\alpha) = \sup\{g_\gamma(\alpha) : \gamma < \alpha\}$. For each $\alpha < \mathrm{cof}(\kappa)$, let η_α be the least element of A_α strictly greater than $\rho(\alpha)$, $\sup\{\sup(A_\alpha \cap A_\gamma) : \gamma < \alpha\}$, and $g(\alpha)$. Let $C = \{\eta_\alpha : \alpha < \mathrm{cof}(\kappa)\}$ which belongs to $\mathscr{B}(\kappa)^+$ since it is unbounded. For each $\gamma < \delta < \mathrm{cof}(\kappa)$, note that $\eta_\delta \notin A_\gamma$ since $\eta_\delta > \sup\{\sup(A_\delta \cap A_\gamma) : \gamma < \delta\}$. Thus for all $\gamma < \mathrm{cof}(\kappa)$, $A_\gamma \cap C \subseteq \{\eta_\alpha : \alpha \leq \gamma\}$ which is bounded below κ and hence belongs to $\mathscr{B}(\kappa)$. Let $\beta < \lambda$. There is an $\gamma < \mathrm{cof}(\kappa)$ so that $f_\beta = g_\gamma$. For any $\alpha > \gamma$, $\eta_\alpha \notin B_\beta$ since $\sup(A_\alpha \cap B_\beta) < f_\beta(\alpha) = g_\gamma(\alpha) \leq g(\alpha) < \eta_\alpha$ and $\eta_\alpha \in A_\alpha$. Thus $B_\beta \cap C \subseteq \{\eta_\alpha : \alpha \leq \gamma\}$ which is a bounded set and hence belongs to $\mathscr{B}(\kappa)^+$. $\mathcal{A}$ is not a maximal $\mathscr{B}(\kappa)$ almost disjoint family since $\mathcal{A} \cup \{C\}$ is a strictly larger $\mathscr{B}(\kappa)$ almost disjoint family.

Let $\mathcal{A}$ be a wellorderable $\mathscr{P}_\kappa(\kappa)$ almost disjoint family of cardinality $\lambda > \mathrm{cof}(\kappa)$. Let $\Phi : \lambda \to \mathcal{A}$ be a bijection. For each $\alpha < \mathrm{cof}(\kappa)$, let $A_\alpha = \Phi(\alpha)$. For each $\beta < \lambda$, let $B_\beta = \Phi(\mathrm{cof}(\kappa) + \beta)$. For each $\beta < \lambda$, let $f_\beta : \mathrm{cof}(\kappa) \to \mathrm{cof}(\kappa)$ be defined by $f_\beta(\alpha)$ is the least $\gamma < \mathrm{cof}(\kappa)$ so that $\mathrm{ot}(A_\alpha \cap B_\beta) < \rho(\gamma)$. By the boldface GCH at $\mathrm{cof}(\kappa)$, there is a surjection $\langle g_\alpha : \alpha < \mathrm{cof}(\kappa)\rangle$ of $\mathrm{cof}(\kappa)$ onto $\{f_\beta : \beta < \lambda\}$. For each $\alpha < \mathrm{cof}(\kappa)$, let $E_\alpha = \{A_\alpha \cap B_\beta : \beta < \rho(\alpha) \wedge \mathrm{ot}(A_\alpha \cap B_\beta) < \rho(g(\alpha))\} \cup \{A_\alpha \cap A_\gamma : \gamma < \alpha\}$. Since $|E_\alpha| < \kappa$ and $A_\alpha \in \mathscr{P}_\kappa(\kappa)^+$, $|A_\alpha \setminus E_\alpha| = \kappa$. Let F_α be the next $\rho(\alpha)$ many elements of $A_\alpha \setminus E_\alpha$. Let $C = \bigcup_{\alpha<\mathrm{cof}(\kappa)} F_\alpha$ which has cardinality κ and thus belongs to $\mathscr{P}_\kappa(\kappa)^+$. If $\gamma < \delta < \mathrm{cof}(\kappa)$, then $A_\delta \cap A_\gamma \subseteq E_\alpha$. Since $F_\delta \subseteq A_\delta$, $F_\delta \cap A_\gamma = \emptyset$. Hence $A_\gamma \cap C \subseteq \bigcup_{\alpha\leq\gamma} F_\alpha$ which has cardinality less than κ (and even bounded below κ). Thus $A_\gamma \cap C$ belongs to $\mathscr{P}_\kappa(\kappa)$ for all $\gamma < \mathrm{cof}(\kappa)$. Let $\beta < \lambda$. There is $\gamma < \mathrm{cof}(\kappa)$ so that $f_\beta = g_\gamma$. For any $\alpha > \gamma$, $\mathrm{ot}(A_\alpha \cap B_\beta) < \rho(f_\beta(\alpha)) = \rho(g_\gamma(\alpha)) < \rho(g(\alpha))$. Thus $A_\alpha \cap B_\beta \subseteq E_\alpha$. Since $F_\alpha \subseteq A_\alpha$, $F_\alpha \cap B_\beta = \emptyset$. Thus $C \cap B_\beta \subseteq \bigcup_{\alpha\leq\gamma} F_\alpha$ which has cardinality less than κ. Thus $C \cap B_\beta \in \mathscr{P}_\kappa(\kappa)$. $\mathcal{A}$ is not a maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family since $\mathcal{A} \cup \{C\}$ is a strictly larger $\mathscr{P}_\kappa(\kappa)$ almost disjoint family. □

The following consequence for cardinals of cofinality ω is worth isolating. It states that if κ has cofinality ω, then any potential maximal $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint families must be nonwellorderable.

Proposition 4.2. *Assume there are no uncountable set of reals (i.e. boldface* GCH *at* ω*). If* κ *is a cardinal with* $\text{cof}(\kappa) = \omega$*, then every infinite wellorderable* $\mathscr{P}_\kappa(\kappa)$ *or* $\mathscr{B}(\kappa)$ *almost disjoint family is not maximal.*

As a consequence, under AD*, if* κ *is a cardinal with* $\text{cof}(\kappa) = \omega$*, then every infinite wellorderable* $\mathscr{P}_\kappa(\kappa)$ *or* $\mathscr{B}(\kappa)$ *almost disjoint family is not maximal.*

Proof. Suppose $\mathcal{A}$ is an infinite wellorderable $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family. Let $\lambda = |\mathcal{A}|$. If $\lambda = \omega$, then Fact 2.5 asserts that $\mathcal{A}$ is not maximal. If $\lambda > \omega$, then Theorem 4.1 implies $\mathcal{A}$ is not maximal.

Under AD, ω_1 is measurable (for instance, by the partition relation $\omega_1 \to_* (\omega_1)^{\omega_1}_2$). Fact 3.20 implies boldface GCH holds at ω. This also follows from Fact 3.3. The result under AD follows from the previous statement. ☐

Theorem 4.3. *Assume* AD^+*. If* $\kappa < \Theta$ *is a cardinal and* $\text{cof}(\kappa) > \omega$*, then there are no maximal* $\mathscr{P}_\kappa(\kappa)$ *or* $\mathscr{B}(\kappa)$ *almost disjoint families* $\mathcal{A}$ *so that* $\neg(|\mathcal{A}| < \text{cof}(\kappa))$*.*

Assume $\mathsf{AD}^+ + V = L(\mathscr{P}(\mathbb{R}))$*. If* κ *is a cardinal so that* $\omega < \text{cof}(\kappa) < \Theta$*, then there are no maximal* $\mathscr{P}_\kappa(\kappa)$ *or* $\mathscr{B}(\kappa)$ *almost disjoint families* $\mathcal{A}$ *so that* $\neg(|\mathcal{A}| < \text{cof}(\kappa))$*.*

Proof. For the first statement, Theorem 3.13 or Proposition 3.17 implies all $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint families must be wellorderable if $\kappa < \Theta$ and $\text{cof}(\kappa) > \omega$. Let $\mathcal{A}$ be a $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family. Let $\lambda = |\mathcal{A}|$. If $\lambda = \text{cof}(\kappa)$, then $\mathcal{A}$ cannot be wellorderable by Fact 2.5. If $\lambda > \text{cof}(\kappa)$, then $\mathcal{A}$ is not maximal by Theorem 4.1 since Fact 3.25 implies boldface GCH holds at $\text{cof}(\kappa)$.

For the second statement, the argument is similar after observing that for any κ with $\text{cof}(\kappa) > \omega$, a $\mathscr{P}_\kappa(\kappa)$ or $\mathscr{B}(\kappa)$ almost disjoint family must be wellorderable by Theorem 3.13. ☐

5. Nonwellorderable Almost Disjoint Families on Countable Cofinality

This section will adapt the argument of Schrittesser and Törnquist [16] which shows there are no infinite maximal $\mathscr{P}_\omega(\omega)$-almost disjoint families to show for all cardinals κ with $\text{cof}(\kappa) = \omega$, there are no infinite maximal $\mathscr{B}(\kappa)$ almost disjoint family under AD^+.

Definition 5.1. If $A \subseteq \omega$, then let $[A]^\omega$ be the collection of increasing functions $f : \omega \to A$. If $f \in [\omega]^\omega$ and $n \in \omega$, let $\mathsf{drop}(f, n) \in [\omega]^\omega$ be defined by $\mathsf{drop}(f, n)(k) = f(n + k)$.

For $N \in \omega + 1$, let $\omega \to (\omega)^N_2$ be the statement that for all $P : [\omega]^N \to 2$, there exists an infinite $A \subseteq \omega$ and an $i \in 2$ so that for all $f \in [A]^N$, $P(f) = i$. For all $n \in \omega$, $\omega \to (\omega)^n_2$ holds in ZF by Ramsey theorem. If $\omega \to (\omega)^\omega_2$ holds, then one says that ω satisfies the strong partition property or that all subsets of $[\omega]^\omega$ have the Ramsey property.

Definition 5.2. If $s \in [\omega]^{<\omega}$ and $g \in [\omega]^\omega$ with $\sup(s) < f(0)$, then $s\hat{\ }g \in [\omega]^\omega$ is the concatenation of s followed by g. Suppose $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$ so that $\max(s) < \min A$. Let $N_{s,A} = \{s\hat{\ }g : g \in [A]^\omega\}$.

$X \subseteq [\omega]^\omega$ is Ramsey null if and only for all $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$ (with $\max(s) < \min A$), there exists an infinite $B \subseteq A$ so that $N_{s,B} \cap X = \emptyset$. $X \subseteq [\omega]^\omega$ is Ramsey conull if and only if $[\omega]^\omega \setminus X$ is Ramsey null.

Fact 5.3. *(Mathias) Assume* $\mathsf{DC}_\mathbb{R}$ *and* $\omega \to (\omega)^\omega_2$*. For every function* $\Phi : [\omega]^\omega \to {}^\omega 2$*, there exists an infinite* $B \subseteq \omega$ *so that* $\Phi \restriction [B]^\omega$ *is continuous: for all* $f \in [B]^\omega$*, for all* $m \in \omega$*, there exists an* $n \in \omega$ *so that for all* $g \in [B]^\omega$ *such that* $f \restriction n = g \restriction n$*,* $\Phi(f) \restriction m = \Phi(g) \restriction m$*.*

Proof. Let $B_0 = \emptyset$ and $C_0 = \omega$. Let $t^0_0 = \emptyset$ and $\ell^0_0 = \emptyset$. Let $p^0_0 = (t^0_0, \ell^0_0) = (\emptyset, \emptyset)$. Define $P^{p^0_0} : [C_0]^\omega \to 2$ by $P^{p^0_0}(g) = 0$ if and only if $\Phi(t^0_0\hat{\ }g) \restriction 0 = \ell^0_0$. By $\omega \to (\omega)^\omega_2$, pick an infinite $C^{p^0_0} \subseteq C_0$ which is homogeneous for $P^{p^0_0}$ which means that $P^{p^0_0}$ is constant on $[C^{p^0_0}]^\omega$. (Note that this first step is trivial. One may even let $C^{p^0_0} = C_0$ in which case $P^{p^0_0}$ is necessarily homogeneous taking value 0.) Let $B_1 = B_0 \cup \{\min C^{p^0_0}\}$ and $C_1 = C^{p^0_0} \setminus B_1$.

Suppose B_n and C_n have been defined. Let $p^n_0, ..., p^n_{K_n}$, for some $K_n \in \omega$, enumerate $[B_n]^{<\omega} \times {}^n 2$. For each $k \leq K_n$, let $t^n_k \in [B_n]^{<\omega}$ and $\ell^n_k \in {}^n 2$ be such that $p^n_k = (t^n_k, \ell^n_k)$. Let $P^{p^n_0} : [C_n]^\omega \to 2$ be defined by $P^{p^n_0}(g) = 0$ if and only if $\Phi(t^n_0\hat{\ }g) \restriction n = \ell^n_0$. By $\omega \to (\omega)^\omega_2$, pick an infinite $C^{p^n_0} \subseteq C_n$ which is homogeneous for $P^{p^n_0}$. Suppose for $k < K_n$, $C^{p^n_k}$ has been defined. Define $P^{p^n_{k+1}} : [C^{p^n_k}]^\omega \to 2$ by $P^{p^n_{k+1}}(g) = 0$ if and only if $\Phi(t^n_{k+1}\hat{\ }g) \restriction n = \ell^n_{k+1}$. By $\omega \to (\omega)^\omega_2$, let $C^{p^n_{k+1}} \subset C^{p^n_k}$ be infinite and homogeneous for $P^{p^n_{k+1}}$. Let $B_{n+1} = B_n \cup \{\min C^{p^n_{K_n}}\}$. Let $C_{n+1} = C^{p^n_{K_n}} \setminus B_{n+1}$.

By $\mathsf{DC}_{\mathbb{R}}$, there are sequences $\langle B_n : n \in \omega\rangle$, $\langle C_n : n \in \omega\rangle$, and $\langle C^{p^n_k} : n \in \omega \wedge k \leq K_n\rangle$ compatible with the above construction. Let $B = \bigcup_{n\in\omega} B_n$. The claim is that $\Phi \restriction [B]^\omega$ is continuous: Let $f \in [B]^\omega$ and $m \in \omega$. Let $k \leq K_m$ be such that $\Phi(f) \restriction m = \ell^m_k$ and t^m_k is such that $f = t^m_k\hat{\ }g$ for some $g \in [C_{m+1}]^\omega$. Since $\ell^m_k = \Phi(f) \restriction m = \Phi(t^m_k\hat{\ }g) \restriction m$ and $g \in [C_{m+1}]^\omega$, one has that $C_{m+1} \subseteq C^{p^m_k}$ must be homogeneous for $P^{p^m_k}$ taking value 0. Let $n' = |t^m_k|$. Let $n = n' + 1$. Suppose $h \in [B]^\omega$ is such that $h \restriction n = f \restriction n$. Since $h(n') = f(n') \in C_{m+1}$, $\mathsf{drop}(h, n') \in [C_{m+1}]^\omega$. Thus $P^{p^m_k}(\mathsf{drop}(h, n')) = 0$ which implies that $\Phi(h) \restriction m = \Phi(t^m_k\hat{\ }\mathsf{drop}(h, n')) \restriction m = \ell^m_k = \Phi(f) \restriction m$. This establishes that $\Phi \restriction [B]^\omega$ is continuous. ☐

Fact 5.4. *Assume* $\mathsf{DC}_{\mathbb{R}}$. *Suppose* $\langle X_n : n \in \omega\rangle$ *is such that each* $X_n \subseteq [\omega]^\omega$ *is Ramsey null. Then* $X = \bigcup_{n\in\omega} X_n$ *is Ramsey null.*

Proof. Fix a sequence $\langle X_n : n \in \omega\rangle$ so that each X_n is Ramsey null. Also fix $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$ such that $\max(s) < \min A$.

Let $B_0 = \emptyset$ and $C_0 = A$. Let $t^0_0 = \emptyset$. Since X_0 is Ramsey null, there exists an infinite $C^{t^0_0} \subseteq C_0$ so that $N_{s\hat{\ }t^0_0, C^{t^0_0}} \cap X_0 = \emptyset$. Let $B_1 = B_0 \cup \{\min C^{t^0_0}\}$ and $C_1 = C^{t^0_0} \setminus B_1$.

Now suppose that the finite set B_n and infinite set C_n have been defined. Let $t^n_0, ..., t^n_{K_n}$ for some $K_n \in \omega$ enumerate $[B_n]^{<\omega}$. Since X_n is Ramsey null, there exists an infinite $C^{t^n_0} \subseteq C_n$ so that $N_{s\hat{\ }t^n_0, C^{t^n_0}} \cap X_n = \emptyset$. Suppose for $k < K_n$, $C^{t^n_k}$ have been defined. Again since X_n is Ramsey null, there exists an infinite $C^{t^n_{k+1}} \subseteq C^{t^n_k}$ so that $N_{s\hat{\ }t^n_{k+1}, C^{t^n_{k+1}}} \cap X_n = \emptyset$. Let $B_{n+1} = B_n \cup \{\min C^{t^n_{K_n}}\}$ and $C_{n+1} = C^{t^n_{K_n}} \setminus B_{n+1}$.

Using $\mathsf{DC}_{\mathbb{R}}$, one obtains a sequence of finite sets $\langle B_n : n \in \omega\rangle$ and two sequences of infinite sets $\langle C_n : n \in \omega\rangle$ and $\langle C^{t^n_k} : n \in \omega \wedge k \leq K_n\rangle$ compatible with the above construction. Let $B = \bigcup_{n\in\omega} B_n$. Suppose $N_{s,B} \cap X \neq \emptyset$. Then there is an $n \in \omega$ so that $N_{s,B} \cap X_n \neq \emptyset$. Let $f \in N_{s,B} \cap X_n$. Then there is a (unique) $t^n_k \in [B_n]^{<\omega}$ so that $f = s\hat{\ }t^n_k\hat{\ }g$ where $g \in [C_{n+1}]^\omega$. However by construction, $N_{s\hat{\ }t^n_k, C_{n+1}} \cap X_n = \emptyset$ and $f \in N_{s\hat{\ }t^n_k, C_{n+1}}$. Contradiction. It has been shown that $B \subseteq A$ is infinite and $N_{s,B} \cap X = \emptyset$. Since s and A were arbitrary, X is Ramsey null. ☐

Definition 5.5. If $\kappa \in \mathrm{ON}$, let $\mathsf{Ramsey\ Uniformization}$ for κ be the statement that for all relations $R \subseteq [\omega]^\omega \times \mathscr{P}(\kappa)$, there exists an infinite $A \subseteq \omega$ and a function $\Phi : \mathrm{dom}(R) \cap [A]^\omega \to \mathscr{P}(\kappa)$ so that for all $f \in \mathrm{dom}(R) \cap [A]^\omega$, $R(f, \Phi(f))$.

Definition 5.6. A map $\phi : Q \to \mathrm{ON}$ is called a norm (or prewellordering) on Q, where Q is a subset of $\mathbb{R}$. If $\phi : Q \to \kappa$ is a surjective norm onto κ, then κ is the length of the norm ϕ. (By definition, $\kappa < \Theta$.)

Recall if $\delta \in \mathrm{ON}$, a set $A \subseteq {}^\omega\omega$ is δ-Suslin if and only if there is a tree T on $\omega \times \delta$ so that $A = \{x \in {}^\omega\omega : (\exists f \in {}^\omega\delta)((x, f) \in [T])\}$. A surjective norm $\phi : Q \to \kappa$ is a δ-Suslin bounded norm (or prewellordering) if and only if for every δ-Suslin set $A \subseteq Q$, $\sup(\phi[A]) < \kappa$.

Fact 5.7. *(Steel) Assume* $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. *Suppose* $\kappa < \Theta$ *and* $\omega < \mathrm{cof}(\kappa)$. *There is an* ω*-Suslin bounded prewellordering* $\phi : Q \to \kappa$ *of length* κ.

Proof. This argument can be found in [6] Theorem 2.28 or [4] Fact 3.8. An important tool is a result of Steel ([18] Theorem 2.1) which relates intersections of certain Suslin sets with other sets of certain Wadge degree. ☐

By augmenting the assumptions with $\omega \to (\omega)^\omega_2$, $\mathsf{Ramsey\ Uniformization}$ for ω, and AD, Fact 5.4 can be extended to show the ideal of Ramsey null sets has the full wellordered additivity.

Proposition 5.8. *Assume* AD, $\mathsf{DC}_{\mathbb{R}}$, $\omega \to (\omega)^\omega_2$, *and* $\mathsf{Ramsey\ Uniformization}$ *for* ω. *Let* $\kappa \in \mathrm{ON}$ *and* $\langle X_\alpha : \alpha < \kappa\rangle$ *be a sequence of subsets of* $[\omega]^\omega$ *so that each* X_α *is Ramsey null. Then* $X = \bigcup_{\alpha<\kappa} X_\alpha$ *is Ramsey null.*

Proof. Suppose not. Let κ be the least ordinal so that there exists a sequence $\langle X_\alpha : \alpha < \kappa\rangle$ of Ramsey null sets so that $X = \bigcup_{\alpha<\kappa} X_\alpha$ is not Ramsey null. Let $\pi : X \to \mathrm{ON}$ be defined by $\pi(f)$ is the least α so that $f \in X_\alpha$. Note that $\mathrm{ot}(\pi[X]) = \kappa$. Since otherwise there is an ordinal $\delta < \kappa$ with $\mathrm{ot}(\pi[X]) = \delta$. Let $\tau : \delta \to \pi[X]$ be a bijection. Then $X = \bigcup_{\alpha<\delta} X_{\tau(\alpha)}$. The sequence $\langle X_{\tau(\alpha)} : \alpha < \delta\rangle$ and the ordinal $\delta < \kappa$ would contradict the minimality of κ. Since π induces a surjection of $\mathbb{R}$ onto $\kappa = \mathrm{ot}(\pi[X])$, $\kappa < \Theta$.

Next the claim is that $\text{cof}(\kappa) > \omega$. Suppose otherwise that $\text{cof}(\kappa) = \omega$. Let $\rho : \omega \to \kappa$ be a cofinal map through κ. For each $n \in \omega$, let $Y_n = \bigcup_{\alpha<\rho(n)} X_\alpha$. By the minimality of κ, each Y_n is Ramsey null. Then Fact 5.4 implies that $X = \bigcup_{n\in\omega} Y_n$ is Ramsey null. Contradiction.

Since X is not Ramsey null, there exists an $s \in [\omega]^{<\omega}$ and an infinite A with $\max(s) < \min A$ so that for all infinite $B \subseteq A$, $N_{s,B} \cap X \neq \emptyset$. Define a partition $P : [A]^\omega \to 2$ by $P(g) = 0$ if and only if $s\hat{\ }g \in X$. By $\omega \to (\omega)^\omega_2$, there is an infinite set $C_0 \subseteq A$ which is homogeneous for P. Since $N_{s,C_0} \cap X \neq \emptyset$ by the assumption on s and A, C_0 must be homogeneous for P taking value 0. Thus $N_{s,C_0} \subseteq X$.

Since $\text{cof}(\kappa) > \omega$ and $\kappa < \Theta$, by Fact 5.7, there is a set $Q \subseteq \mathbb{R}$ and a surjective map $\phi : Q \to \kappa$ which is ω-Suslin bounded. Define $R \subseteq [C_0]^\omega \times Q$ by $R(g, z)$ if and only if $s\hat{\ }g \in X_{\phi(z)}$. By $\mathsf{Ramsey\ Uniformization}$ for ω, there is an infinite $C_1 \subseteq C_0$ and a function $\Lambda : [C_1]^\omega \to \mathbb{R}$ so that for all $g \in [C_1]^\omega$, $R(g, \Lambda(g))$. By Fact 5.3, there is an infinite $C_2 \subseteq C_1$ so that $\Lambda \restriction [C_2]^\omega$ is a continuous function. Therefore, $\Lambda[[C_2]^\omega]$ is a $\boldsymbol{\Sigma}^1_1$ and hence ω-Suslin subset of Q. By ω-Suslin bounding, there is a $\delta < \kappa$ so that $\phi[\Lambda[[C_2]^\omega]] \subseteq \delta$.

For each $\alpha < \delta$, let $Y_\alpha = \{s\hat{\ }g : g \in [C_2]^\omega \wedge \phi(\Lambda(g)) = \alpha\}$. Note that for each $\alpha < \delta$, $Y_\alpha \subseteq X_\alpha$ so Y_α is Ramsey null. However, $\bigcup_{\alpha<\delta} Y_\alpha = N_{s,C_2}$ which is not Ramsey null. Thus $\langle Y_\alpha : \alpha < \delta\rangle$ has the property that each Y_α is Ramsey null but $\bigcup_{\alpha<\delta} Y_\alpha$ is not Ramsey null. This contradicts the minimality of κ. $\square$

The next theorem is an almost everywhere continuity result for functions $\Phi : [\omega]^\omega \to \kappa$, where κ is an ordinal.

Theorem 5.9. *Assume* AD, $\mathsf{DC}_\mathbb{R}$, $\omega \to (\omega)^\omega_2$, *and* $\mathsf{Ramsey\ Uniformization}$ *for* ω. *Let* κ *be an ordinal and* $\Phi : [\omega]^\omega \to \kappa$. *Then there is an infinite* $B \subseteq \omega$ *so that* $\Phi \restriction [B]^\omega$ *is continuous: for all* $f \in [B]^\omega$, *there exists an* $n \in \omega$ *so that for all* $g \in [B]^\omega$, *if* $g \restriction n = f \restriction n$, *then* $\Phi(f) = \Phi(g)$.

Proof. Consider the partition $P_0 : [\omega]^\omega \to 2$ defined by $P_0(f) = 0$ if and only if there exists an $n \in \omega$ so that for all $g \in [\text{rang}(f) \setminus \sup(f \restriction n)]^\omega$, $\Phi(f) = \Phi(f \restriction n\hat{\ }g)$. By $\omega \to (\omega)^\omega_2$, there is an infinite $C_0 \subseteq \omega$ which is homogeneous for P_0.

Claim 1: C_0 is homogeneous for P_0 taking value 0.

To see this, suppose otherwise. This means that for all $f \in [C_0]^\omega$, for all $n \in \omega$, there exists $g \in [\text{rang}(f) \setminus \sup(f \restriction n)]^\omega$ so that $\Phi(f) \neq \Phi(f \restriction n\hat{\ }g)$. For each $\alpha < \kappa$, let $X_\alpha = \{f \in [C_0]^\omega : \Phi(f) = \alpha\}$. Note that $[C_0]^\omega = \bigcup_{\alpha<\kappa} X_\alpha$. Since $[C_0]^\omega$ is not Ramsey null, Proposition 5.8 (applied relative to C_0 rather than ω) implies that there is some $\beta < \kappa$ so that X_β is not Ramsey null (relative to C_0). This means that there is some $s \in [C_0]^{<\omega}$ and an infinite $D \subseteq C_0$ with $\sup(s) < \min(D)$ so that for all infinite $E \subseteq D$, $N_{s,E} \cap X_\beta \neq \emptyset$. Consider the partition $P_1 : [D]^\omega \to 2$ by $P_1(g) = 1$ if and only if $s\hat{\ }g \notin X_\beta$. By $\omega \to (\omega)^\omega_2$, there is an infinite $E \subseteq D$ which is homogeneous for P_1. By assumption, $N_{s,E} \cap X_\beta \neq \emptyset$. Thus E is homogeneous for P_1 taking value 0. So $N_{s,E} \subseteq X_\beta$. Pick any $h \in [E]^\omega$. Let $f = s\hat{\ }h$. Since $f \in [C_0]^\omega$, $P_0(f) = 1$. This implies that there exists a $g \in [\text{rang}(h)]^\omega$ so that $\beta = \Phi(f) \neq \Phi(s\hat{\ }g)$. However, for all $g \in [\text{rang}(h)]^\omega$, $g \in [E]^\omega$. Thus $s\hat{\ }g \in N_{s,E} \subseteq X_\beta$. So $\Phi(s\hat{\ }g) = \beta = \Phi(f)$. Contradiction. This establishes Claim 1.

Let $F_0 = \emptyset$ and $G_0 = C_0$. Let $t^0_0 = \emptyset$. Let $G^{t^0_0} \subseteq G_0$ be infinite so that for all $g_0, g_1 \in [G^{t^0_0}]^\omega$, $\Phi(t^0_0\hat{\ }g_0) = \Phi(t^0_0\hat{\ }g_1)$, if such a set exists. Otherwise let $G^{t^0_0} = G_0$. Let $F_1 = F_0 \cup \{\min G_{t^0_0}\}$. Let $G_1 = G^{t^0_0} \setminus F_1$.

Now suppose that F_n and G_n have been defined. Enumerate $[F_n]^{<\omega}$ by $t^n_0, ..., t^n_{K_n}$ for some $K_n \in \omega$. Let $G^{t^n_0} \subseteq G_n$ be infinite so that for all $g_0, g_1 \in [G_n]^\omega$, $\Phi(t^n_0\hat{\ }g_0) = \Phi(t^n_0\hat{\ }g_1)$ if such a set exists. Otherwise, let $G^{t^n_0} = G_n$. Suppose for some $k < K_n$, $G^{t^n_k}$ has been defined. Let $G^{t^n_{k+1}} \subseteq G^{t^n_k}$ be infinite so that for all $g_0, g_1 \in G^{t^n_{k+1}}$, $\Phi(t^n_{k+1}\hat{\ }g_0) = \Phi(t^n_{k+1}\hat{\ }g_1)$, if such a set exists. Otherwise, let $G^{t^n_{k+1}} = G^{t^n_k}$. Let $F_{n+1} = F_n \cup \{\min G^{t^n_{K_n}}\}$. Let $G_{n+1} = G^{t^n_{K_n}} \setminus F_{n+1}$.

Using $\mathsf{DC}_\mathbb{R}$, there are sequences $\langle F_n : n \in \omega\rangle$, $\langle G_n : n \in \omega\rangle$, and $\langle G^{t^n_k} : n \in \omega \wedge k \leq K_n\rangle$ compatible with the above construction. Let $H = \bigcup_{n\in\omega} F_n$.

Claim 2: Φ is continuous on $[H]^\omega$.

To see this, suppose $f \in [H]^\omega$. Since $H \subseteq C_0$, one has that $f \in [C_0]^\omega$. Thus $P_0(f) = 0$ and hence there exists an $n \in \omega$ so that for all $g \in [\text{rang}(f) \setminus \sup(f \restriction n)]^\omega$, $\Phi(f) = \Phi(f \restriction n\hat{\ }g)$. Let m be least so that there is some $k \leq K_m$ with $t^m_k = f \restriction n$. Thus there is an infinite $G^* \subseteq G_k$ (if $k = 0$) or $G^* \subseteq G^{t^m_{k-1}}$ (if $k > 0$) so that for all $g_0, g_1 \in G^*$, $\Phi(t^m_k\hat{\ }g_0) = \Phi(t^m_k\hat{\ }g_1)$ (namely $\text{rang}(f) \setminus \sup(f \restriction n)$). Thus at this stage of the construction, $G^{t^m_k}$ was chosen to be a set with such property. By the minimality of m, one has that $\max(t^m_k) = \max(F_m)$ and thus $\mathsf{drop}(f, n) \in [G^{t^m_k}]^\omega$. Therefore for all $g \in [G^{t^m_k}]^\omega$, $\Phi(t^m_k\hat{\ }g) = \Phi(f \restriction n\hat{\ }g) = \Phi(f)$. For any $h \in [H]^\omega$, if $h \upharpoonright n = f \upharpoonright n = t_k^m$, then $\mathsf{drop}(h,n) \in [G^{t_k^m}]^\omega$ since $\max(t_k^m) = \max(F_m)$ and $H \setminus F_m \subseteq G^{t_k^m}$. Thus $\Phi(h) = \Phi(t_k^m \hat{}\, \mathsf{drop}(h,n)) = \Phi(f)$. Claim 2 has been shown which completes the proof. $\square$

The next result is an almost everywhere weak continuity result for functions $\Phi : [\omega]^\omega \to \mathscr{P}(\kappa)$, where κ is an ordinal.

Theorem 5.10. *Assume* AD, $\mathsf{DC}_\mathbb{R}$, $\omega \to (\omega)^\omega_2$, *and* $\mathsf{Ramsey\ Uniformization}$ *for* ω. *Suppose* $\kappa \in \mathrm{ON}$ *and* $\Phi : [\omega]^\omega \to {}^\kappa 2$. *Then there is an infinite* $B \subseteq \omega$ *so that* $\Phi : [B]^\omega \to {}^\kappa 2$ *is continuous in the following sense: for all* $f \in [B]^\omega$, *for all* $\alpha < \kappa$, *there exists an* $n \in \omega$ *so that for all* $g \in [B]^\omega$ *such that* $f \upharpoonright n = g \upharpoonright n$, $\Phi(f)(\alpha) = \Phi(g)(\alpha)$.

Proof. Say that $h \in [\omega]^\omega$ satisfies weak continuity for Φ if and only if for all $\alpha < \kappa$, for all $f \in [h[\omega]]^\omega$, there exists an $n \in \omega$ so that for all $g \in [h[\omega]]^\omega$ such that $f \upharpoonright n = g \upharpoonright n$, $\Phi(f)(\alpha) = \Phi(g)(\alpha)$. The theorem is equivalent to the existence of an $h \in [\omega]^\omega$ which satisfies weak continuity for Φ (by letting $B = h[\omega]$).

Suppose there are no $h \in [\omega]^\omega$ satisfying weak continuity for Φ. This means that for all $h \in [\omega]^\omega$, there exists an $\alpha < \kappa$, there exists an $f \in [h[\omega]]^\omega$ so that for all $n \in \omega$, there exists a $g \in [h[\omega]]^\omega$ so that $f \upharpoonright n = g \upharpoonright n$ and $\Phi(f)(\alpha) \neq \Phi(g)(\alpha)$. Define $\Psi : [\omega]^\omega \to \kappa$ by $\Psi(h)$ is the least α so that there exists an $f \in [h[\omega]]^\omega$ so that for all $n \in \omega$, there exists a $g \in [h[\omega]]^\omega$ with $f \upharpoonright n = g \upharpoonright n$ and $\Phi(f)(\alpha) \neq \Phi(g)(\alpha)$. If $h \in [\omega]^\omega$ and $f \in [h[\omega]]^\omega$, then f is said to be a witness that h fails weak continuity for Φ if and only if for all $n \in \omega$, there exists a $g \in [h[\omega]]^\omega$ with $f \upharpoonright n = g \upharpoonright n$ and $\Phi(f)(\Psi(h)) \neq \Phi(g)(\Psi(h))$. The assumption implies that every $h \in [\omega]^\omega$ has a witness to the failure of weak continuity for Φ.

By Theorem 5.9, there is a $C \subseteq \omega$ infinite so that $\Psi \upharpoonright [C]^\omega$ is continuous. Fix $h^* \in [C]^\omega$. Thus there is an $m \in \omega$ so that for all $h \in [C]^\omega$, if $h \upharpoonright m = h^* \upharpoonright m$, then $\Psi(h) = \Psi(h^*)$. Let $\alpha^* = \Psi(h^*)$. Let $F = \{h^*(0), ..., h^*(m-1)\}$ and enumerate $[F]^{<\omega}$ by $t_0, ..., t_K$ for some $K \in \omega$. Let $Q_0 : [C \setminus (\sup(h^* \upharpoonright m) + 1)]^\omega \to 2$ be defined by $Q_0(g) = \Phi(t_0 \hat{}\, g)(\alpha^*)$. By $\omega \to (\omega)^\omega_2$, let $D_0 \subseteq C \setminus (\sup(h \upharpoonright m) + 1)$ be an infinite homogeneous set for Q_0. Suppose Q_i and D_i for $i < K$ have been defined. Let $Q_{i+1} : [D_i]^\omega \to 2$ be defined by $Q_{i+1}(g) = \Phi(t_i \hat{}\, g)(\alpha^*)$. By $\omega \to (\omega)^\omega_2$, let $D_{i+1} \subseteq D_i$ be an infinite homogeneous set for Q_{i+1}. Let $D = F \cup D_K$.

Let $h^\star$ be such that $h^\star[\omega] = D$. Since $h^* \in [C]^\omega$ and $h^\star \upharpoonright m = h^* \upharpoonright m$ (as both enumerate F), $\Psi(h^\star) = \Psi(h^*) = \alpha^*$ by the weak continuity. Let $f \in [h^\star[\omega]]^\omega$. Let $i \leq K$ be such that $f = t_i \hat{}\, p$ for some $p \in [D_K]^\omega$. Let $n = |t_i|$. For all $g \in [D]^\omega = [h^\star[\omega]]^\omega$ so that $f \upharpoonright n+1 = g \upharpoonright n+1$, $\mathsf{drop}(g,n) \in [D_i]^\omega$ since $g(n) = p(0) \in D_K \subseteq D_i$. Since $\mathsf{drop}(g,n), \mathsf{drop}(f,n) \in [D_i]^\omega$ and D_i is homogeneous for Q_i, $\Phi(f)(\Psi(h^\star)) = \Phi(f)(\alpha^*) = \Phi(t_i \hat{}\, \mathsf{drop}(f,n))(\alpha^*) = \Phi(t_i \hat{}\, \mathsf{drop}(g,n))(\alpha^*) = \Phi(g)(\alpha^*) = \Phi(g)(\Psi(h^\star))$. Since $g \in [h^\star[\omega]]^\omega$ was arbitrary so that $g \upharpoonright n+1 = f \upharpoonright n+1$, f is not a witness that $h^\star$ fails weak continuity for Φ. Since f was arbitrary in $[h^\star[\omega]]^\omega$, $h^\star$ has no witnesses to the failure of weak continuity for Φ. This contradicts the assumption that every $h \in [\omega]^\omega$ has a witness to the failure of weak continuity for Φ. $\square$

The following is the analog for Proposition 3.5 for the Ramsey null ideal.

Proposition 5.11. *Assume* AD, $\mathsf{DC}_\mathbb{R}$, $\omega \to (\omega)^\omega_2$, *and* $\mathsf{Ramsey\ Uniformization}$ *for* ω. *Suppose* $\Phi : [\omega]^\omega \to \mathscr{P}(\kappa)$. *There is an infinite* $B \subseteq \omega$ *and a countable set* $\mathcal{E} \subseteq \mathscr{P}(\kappa)$ *so that* $\mathcal{E}$ *consists of pairwise disjoint subsets of* κ, $\bigcup \mathcal{E} = \kappa$, *and for all* $f \in [B]^\omega$, *there exists a countable* $\mathcal{F} \subseteq \mathcal{E}$ *so that* $\Phi(f) = \bigcup \mathcal{F}$.

Proof. By Theorem 5.10, there is a $D \subseteq \omega$ so that Φ is weakly continuous on $[D]^\omega$. For each $\alpha < \kappa$, let $B_\alpha = \{f \in [D]^\omega : \alpha \in \Phi(f)\}$ and $U_\alpha = \{s \in [D]^{<\omega} : N_{s, D \setminus s} \subseteq B_\alpha\}$. Observe that by the continuity of Φ on $[D]^\omega$, $B_\alpha = \bigcup_{s \in U_\alpha} N_{s, D \setminus s}$.

Since U_α is a subset of $[D]^{<\omega}$, $\{U_\alpha : \alpha < \kappa\}$ may be regarded as a wellorderable set of reals and thus countable since AD implies there are no uncountable wellorderable set of reals. Let $\langle \hat{U}_n : n \in \omega \rangle$ be a surjection of ω onto $\{U_\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$, let $\mathfrak{n}(\alpha)$ be the least $n \in \omega$ so that $U_\alpha = \hat{U}_n$. Let $E_n = \{\alpha < \kappa : \mathfrak{n}(\alpha) = n\}$. Note that if $m \neq n$, $E_m \cap E_n = \emptyset$ and $\kappa = \bigcup_{n \in \omega} E_n$. Now suppose $\alpha, \beta \in E_n$ and $f \in [D]^\omega$. Observe that $\alpha \in \Phi(f)$ if and only if $f \in B_\alpha$ if and only if $(\exists s \in U_\alpha)(s \subseteq f)$ if and only if $(\exists s \in \hat{U}_n)(s \subseteq f)$ if and only if $(\exists s \in U_\beta)(s \subseteq f)$ if and only if $f \in B_\beta$ if and only if $\beta \in \Phi(\alpha)$. Therefore for all $n \in \omega$ and $f \in [D]^\omega$, $E_n \subseteq \Phi(f)$ or $E_n \cap \Phi(f) = \emptyset$. Let $\mathcal{E} = \{E_n : n \in \omega\}$. If $\mathcal{F} = \{E_n : E_n \subseteq \Phi(f)\}$, then $\Phi(f) = \bigcup \mathcal{F}$. $\square$

Fact 5.12. *(Schrittesser-Törnquist [16]) Assume* $\mathsf{DC}_\mathbb{R}$, $\omega \to (\omega)^\omega_2$, *and* $\mathsf{Ramsey\ Uniformization}$ *for* ω. *There are no maximal* $\mathscr{P}_\omega(\omega) = \mathscr{B}(\omega)$ *almost disjoint families.*

The argument in [16] will be adapted to prove the following result. (A careful inspect of the argument below using Fact 5.3 rather than Theorem 5.10 shows that AD and the other additional assumptions are not needed to prove Fact 5.12.)

Theorem 5.13. *Assume* AD, $\mathsf{DC}_{\mathbb{R}}$, *and* $\omega \to (\omega)^\omega_2$. *Suppose* κ *is an infinite cardinal,* $\text{cof}(\kappa) = \omega$, Ramsey Uniformization *for* κ *holds, and* ω *injects into every infinite subset of* $\mathscr{P}(\kappa)$. *Then there are no infinite maximal* $\mathscr{B}(\kappa)$ *almost disjoint families.*

Proof. Let $\rho : \omega \to \kappa$ be an increasing cofinal sequence through κ. Suppose $\mathcal{A}$ is an infinite maximal $\mathscr{B}(\kappa)$ almost disjoint family. By the assumptions, there is an injection $\langle A_n : n \in \omega\rangle$ of ω into $\mathcal{A}$. For each $i < j < \omega$, let $\eta_{i,j}$ be the least element $\eta \in A_i$ so that $\eta > \rho(j)$ and $\eta \notin \bigcup_{m<i} A_m$. For each $f \in [\omega]^\omega$, let $B_f = \{\eta_{f(n),f(n+1)} : n \in \omega\}$. Note that B_f is unbounded and hence $B_f \in \mathscr{B}(\kappa)^+$. Define $R \subseteq [\omega]^\omega \times \mathscr{P}(\kappa)$ by $R(f, A)$ if and only if $A \in \mathcal{A}$ and $A \cap B_f \in \mathscr{B}(\kappa)^+$ (is unbounded). Since $\mathcal{A}$ is maximal, $\text{dom}(R) = [\omega]^\omega$. By Ramsey Uniformization for κ, let $C_0 \subseteq \omega$ be infinite and $\Phi : [C_0]^\omega \to \mathcal{A}$ so that $R(f, \Phi(f))$, that is, $\Phi(f) \in \mathcal{A}$ and $\Phi(f) \cap B_f$ is unbounded. Let $C_1 \subseteq C_0$ be infinite so that $\Phi \restriction [C_1]^\omega$ is weakly continuous in the sense of Theorem 5.10.

Φ is not constant on $[C_1]^\omega$. To see this, suppose there is an $\hat{A} \in \mathcal{A}$ so that for all $f \in [C_1]^\omega$, $\Phi(f) = \hat{A}$. A function $h \in [C_1]^\omega$ will be constructed by recursion as follows. Let $h(0) = \min(C_1)$. Suppose $h(n)$ has been defined. If $\hat{A} = A_{h(n)}$, then let $h(n + 1)$ be the next element of C_1 greater than $h(n)$. If $\hat{A} \neq A_{h(n)}$, then $\hat{A} \cap A_{h(n)}$ is bounded. Let $h(n+1)$ be the least element of C_1 so that $\eta_{h(n),h(n+1)} \notin \hat{A}$. This completes the construction of h. Suppose there is a $k \in \omega$ so that $\hat{A} = A_k$. Since $\langle A_n : n \in \omega\rangle$ is an injection, there can only be one such k. Thus $\hat{A} \cap B_h = \{\eta_{h(k),h(k+1)}\}$. If there are no k so that $A_k = \hat{A}$, then $\hat{A} \cap B_h = \emptyset$. Since $\Phi(h) = \hat{A}$, $\Phi(h) \cap B_h = \hat{A} \cap B_h$ which is bounded. However $\Phi(h) \cap B_h$ is unbounded by definition of Φ. Contradiction.

Since Φ is not constant on $[C_1]^\omega$, let $f, g \in [C_1]^\omega$ be such that $\Phi(f) \neq \Phi(g)$. Let $\alpha^* < \kappa$ be the least α so that $\Phi(f)(\alpha) \neq \Phi(g)(\alpha)$. Since Φ is weakly continuous on $[C_1]^\omega$, let $s_0 \subseteq f$ and $t_0 \subseteq g$ be such that for all $f' \in [C_1]^\omega$ and $g' \in [C_1]^\omega$, if $s_0 \subseteq f'$ and $t_0 \subseteq g'$, then $\Phi(f')(\alpha^*) = \Phi(f)(\alpha^*)$, $\Phi(g')(\alpha^*) = \Phi(g)(\alpha^*)$, and therefore $\Phi(f')(\alpha^*) \neq \Phi(g')(\alpha^*)$.

Suppose $s_n, t_n \in [C_1]^{<\omega}$ have been defined for some $n \in \omega$. Define $P : [C_1]^2 \to 2$ by $P(i, j) = 1$ if and only if there exists an $f \in [C_1]^\omega$ with $\sup(s_n) < \min(f)$ so that $\eta_{i,j} \in \Phi(s_n\hat{\ }f)$. By the Ramsey theorem $\omega \to (\omega)^2_2$, there is a $D \subseteq C_1$ homogeneous for P. Pick a $f \in [D]^\omega$ so that $\sup(s_n) < f$ and let $h = s_n\hat{\ }f$. By definition of Φ, $\Phi(h) \cap B_h$ is unbounded. Pick an n so that $h(n) > \sup(s_n)$ and $\eta_{h(n),h(n+1)} \in \Phi(h)$. Then $(h(n), h(n+1)) \in [D]^2$ and $\eta_{h(n),h(n+1)} \in \Phi(h) = \Phi(s\hat{\ }f)$. $P(h(n), h(n+1)) = 1$. So D must be homogeneous for P taking value 1. Pick a $g \in [D]^\omega$ with $\sup(t_n) < \min(g)$. Let $q = t_n\hat{\ }g$. $\Phi(q) \cap B_q$ is unbounded. Pick an n so that $q(n) > \sup(t_n)$ and $\eta_{q(n),q(n+1)} \in \Phi(q)$. Since $q(n) > \sup(t_n)$, $(q(n), q(n + 1)) \in [D]^2$. $P(q(n), q(n + 1)) = 1$ implies that there is an $f_0 \in [C_1]^\omega$ with $\sup(f_0) > \sup(s_n)$ so that $\eta_{q(n),q(n+1)} \in \Phi(s_n\hat{\ }f_0)$. Let $p = s_n\hat{\ }f_0$. Observe that $\eta_{q(n),q(n+1)} \in \Phi(p) \cap \Phi(q)$. By the weak continuity of Φ, there are s_{n+1}, t_{n+1} so that $s_n \subsetneq s_{n+1} \subseteq p$, $t_n \subsetneq t_{n+1} \subseteq q$, and for all $p', q' \in [C_1]^\omega$, if $s_{n+1} \subseteq p'$ and $t_{n+1} \subseteq q'$, then $\eta_{q(n),q(n+1)} \in \Phi(p) \cap \Phi(q)$. Note that since $q(n) > \sup(t_n)$, $\eta_{q(n),q(n+1)} > \rho(q(n)) > \rho(\sup(t_n))$. Let $\delta_n = \eta_{q(n),q(n+1)}$.

This construction yields sequences $\langle s_n : n \in \omega\rangle$, $\langle t_n : n \in \omega\rangle$, and $\langle \delta_n : n \in \omega\rangle$ so that the following holds.

1. For all $n \in \omega$, $s_n, t_n \in [C_1]^\omega$, $s_n \subsetneq s_{n+1}$, $t_n \subsetneq t_{n+1}$.
2. For all $n \in \omega$, $\delta_n > \rho(\sup(t_n))$. Since $\sup\{\sup(t_n) : n \in \omega\} = \omega$, $\{\delta_n : n \in \omega\}$ is unbounded in κ.
3. For all $f', g' \in [C_1]^\omega$ with $s_0 \subseteq f'$ and $t_0 \subseteq g'$, $\Phi(f') \neq \Phi(g')$ since $\Phi(f')(\alpha^*) \neq \Phi(g')(\alpha^*)$.
4. For all $n \in \omega$ and $f', g' \in [C_1]^\omega$ with $s_{n+1} \subseteq f'$ and $t_{n+1} \subseteq g'$, $\delta_n \in \Phi(f') \cap \Phi(g')$.

Let $f = \bigcup_{n\in\omega} s_n$ and $g = \bigcup_{n\in\omega} t_n$. By (3), $\Phi(f) \neq \Phi(g)$. By (4), $\{\delta_n : n \in \omega\} \subseteq \Phi(f) \cap \Phi(g)$. Thus by (2), $\Phi(f) \cap \Phi(g)$ is unbounded. Hence $\Phi(f)$ and $\Phi(g)$ are two distinct elements of the $\mathscr{B}(\kappa)$ almost disjoint family $\mathcal{A}$ with $\Phi(f) \cap \Phi(g) \in \mathscr{B}(\kappa)^+$. Contradiction. ☐

Question 5.14. Under similar assumptions, are there no maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint families whenever $\text{cof}(\kappa) = \omega$?

By the result of Schrittesser and Törnquist [16], one needs to analyze the case for singular cardinals of countable cofinality.

For the argument concerning maximal $\mathscr{B}(\kappa)$ almost disjoint families in Theorem 5.13, at each stage of the construction, finite initial segments forced one element to belong to intersections of any future extensions through the weak continuity of Theorem 5.10. An analogous attempt to handle $\mathscr{P}_\kappa(\kappa)$ almost disjoint families would require forcing the intersections to be large. Proposition 5.11 seems potentially useful as it gives a deeper understanding of functions $\Phi : [\omega]^\omega \to \mathscr{P}(\kappa)$.

Next, some remarks will be made concerning the hypotheses of this section and Woodin's theory AD^+.

Fact 5.15. *Assume* AD^+. *Then* $\mathsf{DC}_\mathbb{R}$, $\omega \to (\omega)^\omega_2$, *and* Ramsey Uniformization *for all* $\kappa < \Theta$ *hold. Also,* ω *injects into every infinite set which is a surjective image of* $\mathbb{R}$.

Proof. $\mathsf{DC}_\mathbb{R}$ is already a part of AD^+. Let X be an infinite set which is a surjective image of $\mathbb{R}$. Let $\Pi : \mathbb{R} \to X$. Let $T \subseteq {}^{<\omega}\mathbb{R}$ be defined by $s \in T$ if and only if $\Pi \circ s : |s| \to X$ is an injection. T is a tree on $\mathbb{R}$ ordered by string extension. Since X is infinite, T has no endnodes. By $\mathsf{DC}_\mathbb{R}$, there is a branch $f \in [T]$ (that is, for all $n \in \omega$, $f \restriction n \in T$). Thus $\Pi \circ f : \omega \to X$ is an injection. It has been shown that ω injects into any infinite set which is a surjective image of $\mathbb{R}$.

Suppose $P : [\omega]^\omega \to 2$. P can be coded as a set of reals. AD^+ asserts all sets of reals have an ∞-Borel code. Thus P has an ∞-Borel code (S, φ) where S is a set of ordinals, φ is a formula of set theory, and for all $f \in [\omega]^\omega$, $P(f) = i$ if and only if $L[S, f] \models \varphi(S, f, i)$. Thus P is correctly defined in any inner model M containing S by the formula $L[S, f] \models \varphi(S, f, i)$. The usual proof establishing $\omega \to (\omega)^\omega_2$ using Mathias forcing can be applied here. See [13] or [7] Theorem 26.23.

For simplicity, an argument for Ramsey Uniformization for ω will be sketched assuming AD and $V = L(\mathbb{R})$. By results of Kechris and Woodin, $L(\mathbb{R}) \models \mathsf{AD}^+$. By the previous observation, $L(\mathbb{R}) \models \omega \to (\omega)^\omega_2$. Suppose $L(\mathbb{R}) \models \neg$Ramsey Uniformization. By replacement, there is an $\alpha \in \mathrm{ON}$ so that $L_\alpha(\mathbb{R}) \models \neg$Ramsey Uniformization. Thus $L(\mathbb{R}) \models (\exists \alpha)[L_\alpha(\mathbb{R}) \models \neg$Ramsey Uniformization$]$. The inner statement is Σ_1 in a language possessing a constant symbol for $\mathbb{R}$ and each element $r \in \mathbb{R}$. Since $\boldsymbol{\delta}^2_1 = \boldsymbol{\delta}_\emptyset$ and $\boldsymbol{\delta}_\emptyset$ is the least ordinal δ so that $L_\delta(\mathbb{R}) \prec_{\Sigma_1} L(\mathbb{R})$ (see [10] Theorem 2.28 and Remark 2.29), $L_{\boldsymbol{\delta}^2_1}(\mathbb{R}) \models (\exists \alpha)[L_\alpha(\mathbb{R}) \models \neg$Ramsey Uniformization$]$. Thus there is an $\alpha < \boldsymbol{\delta}^2_1$ so that $L_\alpha(\mathbb{R}) \models \neg$Ramsey Uniformization. Since $\alpha < \boldsymbol{\delta}^2_1$, there is a surjection $\pi : \mathbb{R} \to L_\alpha(\mathbb{R})$ so that the corresponding coding of $L_\alpha(\mathbb{R})$, $\{(x, y) : \pi(x) \in \pi(y)\}$, is $\boldsymbol{\Delta}^2_1$ (see [10] Lemma 2.26). $L_\alpha(\mathbb{R})$ has a relation $R \subseteq [\omega]^\omega \times \mathbb{R}$ which has no almost everywhere uniformization. This relation R is then $\boldsymbol{\Delta}^2_1$. Martin and Steel ([12] and [19]) showed that all $\boldsymbol{\Delta}^2_1$ relation in $L(\mathbb{R})$ (and hence R) has a scale or Suslin representation in $L(\mathbb{R})$. In $L(\mathbb{R})$, R has a uniformization function $\Phi : \mathrm{dom}(R) \to \mathbb{R}$. By Fact 5.3, there is an infinite $C \subseteq \omega$ so that $\Phi \restriction [C]^\omega$ is a continuous function. Such a continuous function can be coded by a real r^* in a simple manner. Since $\mathbb{R} \subseteq L_\alpha(\mathbb{R})$, $r^* \in L_\alpha(\mathbb{R})$ and hence $\Phi \restriction [C]^\omega \in L_\alpha(\mathbb{R})$. By absoluteness, $L_\alpha(\mathbb{R})$ believes $\Phi \restriction [C_0]^\omega$ is an almost everywhere uniformization for R. However $R \in L_\alpha(\mathbb{R})$ was chosen so that $L_\alpha(\mathbb{R}) \models R$ witnesses the failure of Ramsey Uniformization. Contradiction.

In the general setting of AD^+, Ramsey Uniformization for ω holds by a very similar argument after using a result of Woodin under AD^+ which asserts Σ_1 reflection into Suslin co-Suslin sets (see [17]).

Let $R \subseteq [\omega]^\omega \times \mathscr{P}(\kappa)$. By the Moschovakis coding lemma (Fact 3.15), if $\kappa < \Theta$, then there is a surjection $\pi : \mathbb{R} \to \mathscr{P}(\kappa)$. Define $S \subseteq [\omega]^\omega \times \mathbb{R}$ by $S(f, r) \Leftrightarrow R(f, \pi(r))$. Note that $\mathrm{dom}(S) = \mathrm{dom}(R)$. Applying Ramsey Uniformization for ω to S, there is an infinite $A \subseteq \omega$ and a function $\Psi : \mathrm{dom}(S) \cap [A]^\omega \to \mathbb{R}$ such that for all $f \in [A]^\omega$, $R(f, \Psi(f))$. Define $\Phi : [A]^\omega \cap \mathrm{dom}(R) \to \mathscr{P}(\kappa)$ by $\Phi(f) = \pi(\Psi(f))$. Then for all $f \in [A]^\omega \cap \mathrm{dom}(R)$, $R(f, \Phi(f))$. Φ is a uniformization for R on $[A]^\omega \cap \mathrm{dom}(R)$. This establishes Ramsey Uniformization for κ whenever $\kappa < \Theta$. ☐

The following was originally established by Neeman and Norwood using a form of generic absoluteness for proper forcings.

Fact 5.16. *([15]Neeman-Norwood) Under* AD^+, *there are no maximal* $\mathscr{P}_\omega(\omega) = \mathscr{B}(\omega)$ *almost disjoint families.*

Theorem 5.17. *Under* AD^+, *there are no infinite maximal* $\mathscr{B}(\kappa)$ *almost disjoint families for* $\kappa < \Theta$ *with* $\mathrm{cof}(\kappa) = \omega$.

Proof. This follows from Theorem 5.13 and Fact 5.15. ☐

6. Almost Disjoint Families in $L(\mathbb{R})$

Fact 6.1. *Assume* AD *and* $V = L(\mathbb{R})$. *For any ordinal* κ, $\mathsf{Ramsey\ Uniformization}$ *for* κ *holds.*

More generally, if there is a set of ordinals J *so that* $V = L(J,\mathbb{R})$ *and* $L(J,\mathbb{R}) \models \mathsf{AD}^+$, *then for any ordinal* κ, $\mathsf{Ramsey\ Uniformization}$ *for* κ *holds.*

Proof. In $L(\mathbb{R})$, every set is ordinal definable from a real. Thus there is a definable surjection $\Phi : \mathbb{R} \times \mathrm{ON} \to L(\mathbb{R})$. Suppose $R \subseteq [\omega]^\omega \times \mathscr{P}(\kappa)$. For each $f \in [\omega]^\omega$, let α_f be the least α so that there exists an $r \in \mathbb{R}$ so that $R(f, \Phi(r, \alpha))$. Let A be the collection of $x \in \mathscr{P}(\kappa)$ so that there exists an $f \in [\omega]^\omega$ and a $r \in \mathbb{R}$ so that $x = \Phi(r, \alpha_f)$. Note that $f \in \mathrm{dom}(R)$ if and only if there exists some $x \in A$ with $R(f, x)$. Since $\mathbb{R}$ surjects onto A, fix a surjection $\Psi : \mathbb{R} \to A$. Define $S \subseteq [\omega]^\omega \times \mathbb{R}$ by $S(f, r)$ if and only if $R(f, \Psi(r))$. Note $\mathrm{dom}(R) = \mathrm{dom}(S)$. Since $L(\mathbb{R}) \models \mathsf{AD}$ implies $L(\mathbb{R}) \models \mathsf{AD}^+$, $L(\mathbb{R}) \models \mathsf{Ramsey\ Uniformization}$. Thus there is an infinite $B \subseteq \omega$ and a function $\Gamma : \mathrm{dom}(S) \cap [B]^\omega \to \mathbb{R}$ so that for all $f \in \mathrm{dom}(S) \cap [B]^\omega$, $R(f, \Gamma(f))$. Let $\Upsilon : \mathrm{dom}(R) \cap [B]^\omega \to \mathscr{P}(\kappa)$ be defined by $\Upsilon(f) = \Psi(\Gamma(f))$. Then for all $f \in \mathrm{dom}(R) \cap [B]^\omega$, $R(f, \Upsilon(f))$.

The argument for $L(J,\mathbb{R}) \models \mathsf{AD}^+$ is similar using the fact every set is ordinal definable from J and a real. $\square$

Theorem 6.2. *Assume* AD *and* $V = L(\mathbb{R})$. *For any* κ *with* $\mathrm{cof}(\kappa) = \omega$, *there are no infinite* $\mathscr{B}(\kappa)$ *almost disjoint families.*

Proof. Since $L(\mathbb{R}) \models \mathsf{AD}$ implies $L(\mathbb{R}) \models \mathsf{AD}^+$, the result follows from Fact 5.15, Fact 6.1, and Theorem 5.13. $\square$

Theorem 6.3. *Assume* AD *and* $V = L(\mathbb{R})$. *For any cardinal* κ *with* $\mathrm{cof}(\kappa) \geq \Theta$, *there exists a wellorderable maximal* $\mathscr{P}_\kappa(\kappa)$ *almost disjoint family and a wellorderable maximal* $\mathscr{B}(\kappa)$ *almost disjoint family which does not inject into* $\mathrm{cof}(\kappa)$ *and in fact has cardinality* κ^+.

Proof. This argument will use the Vopěnka forcing and Woodin's result that $L(\mathbb{R})$ is a symmetric collapse extension of $\mathrm{HOD}^{L(\mathbb{R})}$. See [2] Section 7 and 8 for an account of these results. For each $1 \leq n < \omega$, let ${}_n\mathbb{A}$ be the forcing of nonempty subsets of $\mathbb{R}^n$ which have OD-infinity Borel codes ordered by $p \leq_{{}_n\mathbb{A}} q$ if and only if $p \subseteq q$. (Woodin showed that ${}_n\mathbb{A}$ is actually the usual Vopěnka forcing of nonempty OD subsets of $\mathbb{R}^n$; see [2] Corollary 7.22.) The projection maps $\pi_{n,m} : \mathbb{R}^n \to \mathbb{R}^m$ for $m \leq n$ induce forcing projection maps $\pi_{n,m} : {}_n\mathbb{A} \to {}_m\mathbb{A}$ (see [2] Fact 7.14). Let ${}_\omega\mathbb{A}$ be the finite support direct limit of $\langle {}_n\mathbb{A} : 1 \leq n < \omega \rangle$. Observe that for all $1 \leq n < \omega$, ${}_n\mathbb{A}$ and ${}_\omega\mathbb{A}$ belong to HOD and are essentially subsets of Θ. $\mathrm{HOD} = L[{}_\omega\mathbb{A}]$ by [2] Corollary 7.21. So by the usual condensation arguments, for all $\kappa \geq \Theta$, HOD satisfies GCH at κ and in fact satisfies GCH below Θ by results of Steel [20]. For each $r \in \mathbb{R}$, $G^1_r = \{p \in {}_1\mathbb{A} : r \in p\}$ is a ${}_1\mathbb{A}$-generic filter over HOD that naturally adds r. For each $r \in \mathbb{R}$, $\mathrm{HOD}[G^1_r] = \mathrm{HOD}[r]$ by [2] Fact 7.6. Also for each $r \in \mathbb{R}$, $\mathrm{HOD}_r = \mathrm{HOD}[G^1_r]$ by [2] Theorem 7.19. By [2] Theorem 7.16, for any $g \subseteq \mathrm{Coll}(\omega, \mathbb{R})$ which is $\mathrm{Coll}(\omega,\mathbb{R})$-generic over $L(\mathbb{R})$, there is, inside $L(\mathbb{R})[g]$, an induced ${}_\omega\mathbb{A}$-generic filter G over HOD so that $L(\mathbb{R}) = L(\dot{\mathbb{R}}_{\mathrm{sym}})^{\mathrm{HOD}[G]}$. ${}_\omega\mathbb{A}$ preserves cardinals above Θ since ${}_\omega\mathbb{A}$ is a forcing of size Θ in HOD. Therefore for all $\kappa \geq \Theta$, $\kappa^+ \leq (\kappa^+)^{\mathrm{HOD}[G]} = (\kappa^+)^{\mathrm{HOD}} \leq \kappa^+$ where the first inequality comes from $L(\mathbb{R}) = L(\dot{\mathbb{R}}_{\mathrm{sym}})^{\mathrm{HOD}[G]}$ being an inner model of $\mathrm{HOD}[G]$. This shows that $\kappa^+ = (\kappa^+)^{\mathrm{HOD}} = (\mathscr{P}(\kappa))^{\mathrm{HOD}}$ where the latter equality comes from HOD satisfying GCH at κ. (Therefore boldface GCH fails for $\kappa \geq \Theta$ in $L(\mathbb{R})$.)

Now fix a κ with $\mathrm{cof}(\kappa) \geq \Theta$. Since $\mathrm{HOD} \models \mathsf{GCH}$, $\mathrm{HOD} \models 2^{<\kappa} = \kappa$. Let $\pi : {}^{<\kappa}2 \to \kappa$ be a bijection with the property that for all $\sigma \subsetneq \tau$, $\pi(\sigma) < \pi(\tau)$. Working in HOD, for each $f \in {}^\kappa 2$, let $A_f = \{\pi(f \restriction \alpha) : \alpha < \kappa\}$ and $X = \{A_f : f \in {}^\kappa 2\}$. X is both a $\mathscr{P}_\kappa(\kappa)$ and $\mathscr{B}(\kappa)$ almost disjoint family and $|X| = |\mathscr{P}(2^{<\kappa})| = |\mathscr{P}(\kappa)| = 2^\kappa = \kappa^+$ since HOD satisfies GCH at κ. Since $\mathrm{HOD} \models \mathsf{AC}$, there is a maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family $\mathcal{A}_0$ and a maximal $\mathscr{B}(\kappa)$ almost disjoint family $\mathcal{A}_1$ so that $X \subseteq \mathcal{A}_0$ and $X \subseteq \mathcal{A}_1$. $|\mathcal{A}_0| = |\mathcal{A}_1| = |X| = \kappa^+$ in HOD.

Claim 1: In $L(\mathbb{R})$, $\mathcal{A}_0$ is a maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family and $\mathcal{A}_1$ is a maximal $\mathscr{B}(\kappa)$ almost disjoint family.

To see Claim 1: Suppose $\mathcal{A}_0$ is not maximal in $L(\mathbb{R})$. Then there is a set $B \in \mathscr{P}_\kappa(\kappa)^+$ (i.e. $|B| = \kappa$) so that for all $A \in \mathcal{A}_0$, $A \cap B \in \mathscr{P}_\kappa(\kappa)$ (i.e. $|A \cap B| < \kappa$). In $L(\mathbb{R})$, all sets are ordinal definable from some real. Thus B is OD_{r^*} for some $r^* \in \mathbb{R}$. Thus $B \in \mathrm{HOD}_{r^*} = \mathrm{HOD}[r^*] = \mathrm{HOD}[G^1_{r^*}]$. $B \in \mathrm{HOD}[G^1_{r^*}]$ and $\mathrm{HOD}[G^1_{r^*}] \models$ for all $A \in \mathcal{A}_0$, $|A \cap B| < \kappa$. Thus there is a ${}_1\mathbb{A}$-name $\dot{B} \in \mathrm{HOD}$ and a $p^* \in G^1_r$ so that $\dot{B}[G^1_{r^*}] = B$ and $p^* \Vdash_{{}_1\mathbb{A}}$ for all $A \in \check{\mathcal{A}}_0$, $|A \cap \dot{B}| < \check{\kappa}$. Let $E = \{\alpha < \kappa : \mathrm{HOD} \models (\exists q \leq_{{}_1\mathbb{A}} p^*)(q \Vdash_{{}_1\mathbb{A}} \check{\alpha} \in \dot{B})\}$.

$E \in \mathscr{P}(\kappa)^{\mathrm{HOD}}$. Since $B = \dot{B}[G^1_{r^*}] \in \mathscr{P}_\kappa(\kappa)^+$ and $B \subseteq E$, $E \in \mathscr{P}_\kappa(\kappa)^+$. Since $E \in \mathrm{HOD}$ and $\mathrm{HOD} \models \mathcal{A}_0$ is a $\mathscr{P}_\kappa(\kappa)$ almost disjoint family, there is some $C \in \mathcal{A}_0$ so that $|C \cap E| = \kappa$. For each $r \in p^*$ (each ${}_1\mathbb{A}$ condition is a set of reals in $L(\mathbb{R})$), let $T_r = \{\alpha \in E \cap C : \alpha \in \dot{B}[G^1_r]\}$.

Note that $E \cap C = \bigcup_{r \in p^*} T_r$. To see this: By definition, $\bigcup_{r \in p^*} T_r \subseteq E \cap C$. If $\alpha \in E \cap C$, then there exists a $q \leq_{{}_1\mathbb{A}} p^*$ so that $q \Vdash_{{}_1\mathbb{A}} \check{\alpha} \in \dot{B}$. Pick any $r \in q \subseteq p^*$. Then $\alpha \in \dot{B}[G^1_r]$ and hence $\alpha \in T_r$.

Claim 1.1: There is an $r \in p^*$ so that $|T_r| = \kappa$.

To see Claim 1.1: Suppose not. Then for all $r \in p^*$, $\mathrm{ot}(T_r) < \kappa$. Define a prewellordering $\preceq$ on p^* by $r \preceq s$ if and only if $\mathrm{ot}(T_r) \leq \mathrm{ot}(T_s)$. The length of $\preceq$ is less than Θ since $\preceq$ is a prewellordering on $p^* \subseteq \mathbb{R}$ and the definition of Θ. Since $\mathrm{cof}(\kappa) \geq \Theta$, there must be a $\delta < \kappa$ so that $\mathrm{ot}(T_r) < \delta$ for all $r \in p^*$. Let $\Sigma_r : T_r \to \delta$ be an injection given by the Mostowski collapse of T_r. For each $\gamma < \delta$, let $D_\gamma = \{\Sigma_r^{-1}(\gamma) : \gamma \in \mathrm{rang}(\Sigma_r) \wedge r \in p^*\}$. Since p^* surjects onto D_γ, $\mathrm{ot}(D_\gamma) < \Theta$. Let $\Lambda_\gamma : D_\gamma \to \mathrm{ot}(D_\gamma)$ be the Mostowski collapse of D_γ. Note also that $\bigcup_{\gamma < \delta} D_\gamma = E \cap C$ since $E \cap C = \bigcup_{r \in p^*} T_r$. Let $\zeta = \sup\{\mathrm{ot}(D_\gamma) : \gamma < \delta\}$. Note that $\zeta \leq \Theta$. If $\kappa = \Theta$, then Θ must be regular since $\mathrm{cof}(\kappa) \geq \Theta$. Thus if $\kappa = \Theta$, then $\zeta = \sup\{\mathrm{ot}(D_\gamma) : \gamma < \delta\} < \Theta = \kappa$ since $\delta < \kappa = \Theta$.

Define $\Psi : E \cap C \to \delta \times \zeta$ by $\Psi(\alpha) = (\gamma, \eta)$ so that γ is the least $\gamma' < \delta$ so that $\alpha \in D_{\gamma'}$, and $\Lambda_\gamma(\alpha) = \eta$. Ψ is an injection and therefore, $|E \cap C| \leq |\delta \times \zeta| = \max\{|\delta|, |\zeta|\} < \kappa$ which is clear if $\Theta < \kappa$ and follow from $\zeta < \Theta$ if $\kappa = \Theta$. This contradicts $|E \cap C| = \kappa$. Claim 1.1. has been shown.

Thus there is an $r \in p^*$ so that $|T_r| = \kappa$. $T_r \subseteq \dot{B}[G^1_r] \cap C$. So $|\dot{B}[G^1_r] \cap C| = \kappa$. Since $C \in \mathcal{A}_0$, this violates $p^* \Vdash_{{}_1\mathbb{A}}$ for all $A \in \check{\mathcal{A}}_0$, $|A \cap \dot{B}| < \check{\kappa}$. This contradiction implies $\mathcal{A}_0$ remains a maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family in $L(\mathbb{R})$.

$\mathcal{A}_1$ remains a maximal $\mathscr{B}(\kappa)$ almost disjoint family by a similar argument with the appropriate change to the ideal $\mathscr{B}(\kappa)$. This completes the proof of the Claim 1.

Note that $|\mathcal{A}_0| = |\mathcal{A}_1| = (\kappa^+)^{\mathrm{HOD}} = \kappa^+$. This completes the proof. □

The following is the complete solution to the maximal $\mathscr{B}(\kappa)$ almost disjoint family problem at every cardinal κ in $L(\mathbb{R}) \models \mathsf{AD}$. A solution for maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint families is known when κ is not a singular cardinal of countable cofinality. The next theorem is a special case of Theorem 6.14 which holds in AD^+ and $V = L(\mathscr{P}(\mathbb{R}))$.

Theorem 6.4. *Assume* AD *and* $V = L(\mathbb{R})$. *For every cardinal* κ*, there is a maximal* $\mathscr{B}(\kappa)$ *almost disjoint family* $\mathcal{A}$ *so that* $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$ *if and only if* $\mathrm{cof}(\kappa) \geq \Theta$.

Proof. If κ is a cardinal with $\mathrm{cof}(\kappa) = \omega$, Theorem 6.2 shows there are no infinite maximal $\mathscr{B}(\kappa)$ almost disjoint families. If κ is a cardinal with $\omega < \mathrm{cof}(\kappa) < \Theta$, then Theorem 4.3 implies there are no maximal $\mathscr{B}(\kappa)$ almost disjoint families which do not strictly inject into $\mathrm{cof}(\kappa)$. If κ is a cardinal with $\mathrm{cof}(\kappa) \geq \Theta$, then there is a maximal $\mathscr{B}(\kappa)$ almost disjoint family of cardinality κ^+ by Theorem 6.3. □

Theorem 6.5. *Assume* AD *and* $V = L(\mathbb{R})$. *Suppose* κ *is a cardinal which is not a singular cardinal of countable cofinality. There is a maximal* $\mathscr{P}_\kappa(\kappa)$ *almost disjoint family* $\mathcal{A}$ *such that* $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$ *if and only if* $\mathrm{cof}(\kappa) \geq \Theta$.

Proof. If $\kappa = \omega$, Theorem 6.2 or Fact 5.12 shows there are no infinite maximal $\mathscr{P}_\omega(\omega)$ almost disjoint families. If κ is a cardinal with $\omega < \mathrm{cof}(\kappa) < \Theta$, then Theorem 4.3 implies there are no maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint families. If κ is a cardinal with $\mathrm{cof}(\kappa) \geq \Theta$, then there is a maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family of cardinality κ^+ by Theorem 6.3. □

Question 6.6. What is the answer to the maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family problem in $L(\mathbb{R}) \models \mathsf{AD}$ when κ is a singular cardinal of countable cofinality? This question is closely related to Question 5.14.

The argument for $L(\mathbb{R})$ from Theorem 6.3, Theorem 6.4, and Theorem 6.5 can be adapted to models satisfying AD^+, $V = L(\mathscr{P}(\mathbb{R}))$, and there is a set of ordinals J so that $V = L(J, \mathbb{R})$. ($\mathsf{AD}_\mathbb{R}$ necessarily fails in this setting.) The following provides some of the details.

Theorem 6.7. *Suppose* AD^+, $V = L(\mathscr{P}(\mathbb{R}))$*, and there is a set of ordinals* J *so that* $V = L(J, \mathbb{R})$.

- *For every cardinal* κ*, there is a maximal* $\mathscr{B}(\kappa)$ *almost disjoint family* $\mathcal{A}$ *so that* $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$ *if and only if* $\mathrm{cof}(\kappa) \geq \Theta$.

- *If κ is not a singular cardinal of countable cofinality, then there is a maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family $\mathcal{A}$ so that $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$ if and only if $\mathrm{cof}(\kappa) \geq \Theta$.*

Proof. For $A, B \in \mathscr{P}(\mathbb{R})$, let $A \leq_L B$ if and only if there is a Lipschitz function $\Xi : \mathbb{R} \to \mathbb{R}$ so that $x \in A$ if and only if $\Xi(x) \in B$. For each $A \in \mathscr{P}(\mathbb{R})$, let $\mathrm{rk}_L(A)$ denote the Lipschitz rank of A. If $r \in \mathbb{R}$, let $\Xi_r : \mathbb{R} \to \mathbb{R}$ denote the Lipschitz continuous function coded by r. Note that if $A \leq_L B$, then there is an $r \in \mathbb{R}$ so that $\Xi_r^{-1}[B] = A$.

Claim 1: There is no surjection of $\mathscr{P}(\mathbb{R})$ onto Θ^+.

To see this: Suppose $\Phi : \mathscr{P}(\mathbb{R}) \to \Theta^+$ is a surjection. Let $\Psi : \Theta^+ \to \Theta$ be defined by $\Psi(\alpha)$ is the least β so that there is a set A with $\mathrm{rk}_L(A) = \beta$ and $\Phi(A) = \alpha$. There is a $\beta < \Theta$ so that $|\Psi^{-1}[\{\beta\}]| \geq \Theta$. Fix $B \in \mathscr{P}(\mathbb{R})$ with $\mathrm{rk}_L(B) = \beta$. Let $C = \{r \in \mathbb{R} : \mathrm{rk}_L(\Xi_r^{-1}[B]) = \beta\}$. Define $\Lambda : C \to \Theta^+$ by $\Lambda(r) = \Phi(\Xi_r^{-1}[B])$. Since $C \subseteq \mathbb{R}$, $\Psi^{-1}[\{\beta\}] \subseteq \Lambda[C]$, and $|\Psi^{-1}[\{\beta\}]| \geq \Theta$, Λ induces a surjection of $\mathbb{R}$ onto Θ which is impossible. This establishes Claim 1.

Now assume AD^+, $V = L(\mathscr{P}(\mathbb{R})) = L(J, \mathbb{R})$ for some set of ordinals J.

Claim 2: For all $\kappa \geq \Theta$, κ^+ is regular.

To see Claim 2: For each $n \in \omega$, let ${}_n\mathbb{A}_J$ be the forcing of nonempty subsets of $\mathbb{R}$ with OD_J-infinity Borel codes. Let ${}_\omega\mathbb{A}_J$ be the finite support direct limit of $\langle {}_n\mathbb{A}_J : 1 \leq n < \omega\rangle$. ${}_\omega\mathbb{A}_J \in \mathrm{HOD}_J$ and $\mathrm{HOD}_J \models |{}_\omega\mathbb{A}_J| = \Theta$. For any $g \subseteq \mathrm{Coll}(\omega, \mathbb{R})$ which is $\mathrm{Coll}(\omega,\mathbb{R})$-generic over $L(J,\mathbb{R})$, there is an induced $G \subseteq {}_\omega\mathbb{A}_J$ which is ${}_\omega\mathbb{A}_J$-generic over HOD_J and $L(J, \dot{\mathbb{R}}_{\mathrm{sym}})^{\mathrm{HOD}_J[G]} = L(J,\mathbb{R}) = V$. Since ${}_\omega\mathbb{A}_J$ is a forcing of cardinality Θ in HOD_J, ${}_\omega\mathbb{A}_J$ preserves cardinals above Θ. Thus for all $\kappa \geq \Theta$, $\kappa^+ \leq (\kappa^+)^{\mathrm{HOD}_J[G]} = (\kappa^+)^{\mathrm{HOD}_J} \leq \kappa^+$ since $L(J,\mathbb{R}) = L(J, \dot{\mathbb{R}}_{\mathrm{sym}})^{\mathrm{HOD}_J[G]}$ is an inner model of $\mathrm{HOD}_J[G]$. Thus $\kappa^+ = (\kappa^+)^{\mathrm{HOD}_J} = (\kappa^+)^{\mathrm{HOD}_J[G]}$. Since $\mathrm{HOD}_J[G] \models \mathsf{AC}$, $\mathrm{HOD}_J[G] \models \kappa^+$ is a regular cardinal. Since $L(J,\mathbb{R}) \subseteq \mathrm{HOD}_J[G]$ and $\kappa^+ = (\kappa^+)^{\mathrm{HOD}_J[G]}$, κ^+ is regular in $L(J,\mathbb{R})$.

Claim 3: For each $\kappa \geq \Theta$, there is no surjection of $\mathscr{P}(\mathbb{R}) \times \kappa$ onto κ^+.

To see this: Suppose $\Upsilon : \mathscr{P}(\mathbb{R}) \times \kappa \to \kappa^+$ is a surjection. For each $B \in \mathscr{P}(\mathbb{R})$, let $\nu_B = \sup\{\Upsilon(B,\alpha) : \alpha < \kappa\}$ which is less than κ^+ since κ^+ is regular by Claim 2. Let $E = \{\nu_B : B \in \mathscr{P}(\mathbb{R})\}$. Since Υ is a surjection onto κ^+, $\sup(E) = \kappa^+$. Since κ^+ is regular, $\mathrm{ot}(\sup(E)) = \kappa^+$. There is a surjection of $\mathscr{P}(\mathbb{R})$ onto E and thus there is a surjection of $\mathscr{P}(\mathbb{R})$ onto $\kappa^+ \geq \Theta^+$. This contradicts Claim 1.

Claim 4: For each $\kappa \geq \Theta$, $\mathscr{P}(\kappa) \subseteq L_{\kappa^+}(\mathscr{P}(\mathbb{R}))$.

To see this: Fix $\kappa \geq \Theta$. Suppose $A \subseteq \kappa$. Let X be the collection of $x \in L_{\kappa^+}(\mathscr{P}(\mathbb{R}))$ such that x is definable in $L_{\kappa^+}(\mathscr{P}(\mathbb{R}))$ from parameters in $\kappa \cup \mathscr{P}(\mathbb{R}) \cup \{A\}$. X is an elementary substructure of $L_{\kappa^+}(\mathscr{P}(\mathbb{R}))$ and $\kappa \cup \mathscr{P}(\mathbb{R}) \cup \{A\} \subseteq X$. There is an $\eta_A \leq \kappa^+$ so that the transitive collapse of X is $L_{\eta_A}(\mathscr{P}(\mathbb{R}))$. Note that $\kappa \cup \mathscr{P}(\mathbb{R}) \cup \{A\} \subseteq L_{\eta_A}(\mathscr{P}(\mathbb{R}))$ and there is a surjection $\mathscr{P}(\mathbb{R}) \times \kappa$ onto $L_{\eta_A}(\mathscr{P}(\mathbb{R}))$. In particular, there is a surjection of $\mathscr{P}(\mathbb{R}) \times \kappa$ onto η_A. By Claim 3, $\eta_A < \kappa^+$. Thus $\mathscr{P}(\kappa) \subseteq \bigcup_{A \in \mathscr{P}(\kappa)} L_{\eta_A}(\mathscr{P}(\mathbb{R})) \subseteq L_{\kappa^+}(\mathscr{P}(\mathbb{R}))$.

Claim 5: For all $\kappa \geq \Theta$, there is no injection of κ^{++} into $\mathscr{P}(\kappa)$. Thus HOD_J satisfies GCH at all $\kappa \geq \Theta$.

To see Claim 5: Suppose there is an injection of κ^{++} into $\mathscr{P}(\kappa)$. Then by Claim 4, there is an injection of κ^{++} into $L_{\kappa^+}(\mathscr{P}(\mathbb{R}))$. Since $\mathscr{P}(\kappa) \times \kappa^+$ surjects onto $L_{\kappa^+}(\mathscr{P}(\kappa))$, $\mathscr{P}(\kappa) \times \kappa^+$ surjects onto κ^{++} which contradicts Claim 3.

With knowledge that HOD_J satisfies GCH at all $\kappa \geq \Theta$, the argument from Theorem 6.3 can be applied over HOD_J using the forcing ${}_1\mathbb{A}_J$ (which has size Θ). The results follows as in the case of $L(\mathbb{R})$ from Theorem 6.4 and Theorem 6.5 ☐

Next, some remarks will be made concerning maximal almost disjoint families in $L(\mathscr{P}(\mathbb{R})) \models \mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$. In this setting, Woodin showed there is an analog of the Vopěnka forcing which adds $\mathscr{P}(\mathbb{R})$ symmetrically over HOD. (See [21] Section 4 for some details of this Vopěnka forcing.) This forcing can be used in manner similar to the argument of Theorem 6.3 to show the following. (It is unknown if $\mathsf{AD}_\mathbb{R}$ implies AD^+ but $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$ does prove AD^+.)

Theorem 6.8. *Assume AD^+, $\mathsf{AD}_\mathbb{R}$, $V = L(\mathscr{P}(\mathbb{R}))$. For any cardinal κ with $\mathrm{cof}(\kappa) \geq \Theta$, there exists a wellorderable maximal $\mathscr{P}_\kappa(\kappa)$ almost disjoint family and a $\mathscr{B}(\kappa)$ almost disjoint family which does not inject into $\mathrm{cof}(\kappa)$ and in fact has cardinality κ^+.*

In this setting, the next result gives some partial results concerning $\mathsf{Ramsey\ Uniformization}$ for all κ.

Fact 6.9. *Assume AD^+, $\mathsf{AD}_\mathbb{R}$, and $V = L(\mathscr{P}(\mathbb{R}))$.*

(1) *Assuming* $\mathrm{cof}(\Theta) > \omega$. *For all ordinals* κ, Ramsey Uniformization *holds for all* κ.
(2) *In general,* Ramsey Uniformization *for* κ *holds on a neighborhood: for all ordinals* κ *and* $R \subseteq [\omega]^\omega \times \mathscr{P}(\kappa)$, *there is an* $s \in [\omega]^{<\omega}$, *an infinite* $C \subseteq \omega$, *and a function* $\Lambda : \mathrm{dom}(R) \cap N_{s,C} \to \mathscr{P}(\kappa)$ *so that for all* $f \in \mathrm{dom}(R) \cap N_{s,C}$, $R(f, \Lambda(f))$.

Proof. Woodin showed that under these assumptions, every set is ordinal definable from an element of $\bigcup_{\lambda<\Theta} \mathscr{P}_\omega(\lambda)$. Thus there is a definable surjection $\Phi : \bigcup_{\lambda<\Theta} \mathscr{P}_\omega(\lambda) \times \mathrm{ON} \to V$. Fix $R \subseteq [\omega]^\omega \times \mathscr{P}(\kappa)$. For each $f \in [\omega]^\omega$, let (λ_f, α_f) be the least pair (λ, α) so that there a $\sigma \in \mathscr{P}_{\omega_1}(\lambda)$ with $R(f, \Phi(\sigma, \alpha))$. Let $\Upsilon : [\omega]^\omega \to \Theta$ be defined by $\Upsilon(f) = \lambda_f$. There is an infinite $C \subseteq \omega$ so that $\Upsilon \upharpoonright [C]^\omega$ is continuous in the sense of Theorem 5.9. Say that $s \in [C]^{<\omega}$ is a continuity point of Υ (relative to C) if and only if for all $f, g \in N_{s,C}$, $\Upsilon(f) = \Upsilon(g)$. Let K be the collection of continuity points of Υ (relative to C). For each $s \in K$, let $\zeta_s < \Theta$ be the value of Υ at the continuity point s; that is, ζ_s is the unique ζ so that for all $f \in N_{s,C}$, $\Upsilon(f) = \zeta$.

To see (1): If $\mathrm{cof}(\Theta) > \omega$, then $\lambda = \sup\{\zeta_s : s \in [C]^{<\omega}\} < \Theta$. By the Moschovakis coding lemma (Fact 3.15), let $\pi : \mathbb{R} \to \mathscr{P}_{\omega_1}(\lambda)$ be a surjection. Define $S \subseteq [C]^\omega \times \mathbb{R}$ by $S(f, r)$ if and only if $R(f, \Phi(\pi(r), \alpha_f))$. Note that for all $f \in [\omega]^\omega$, there is some $n \in \omega$ so that $f \upharpoonright n \in K$ and thus there is some $\sigma \in \mathscr{P}_{\omega_1}(\lambda_f) = \mathscr{P}_{\omega_1}(\zeta_{f\upharpoonright n}) \subseteq \mathscr{P}_{\omega_1}(\lambda)$ so that $R(f, \Phi(\sigma, \alpha_f))$. Thus $\mathrm{dom}(R) = \mathrm{dom}(S)$. Applying Ramsey Uniformization for ω (which follows from Fact 5.15), there is an infinite $D \subseteq C$ and a function $\Gamma : \mathrm{dom}(S) \cap [D]^\omega \to \mathbb{R}$ so that for all $f \in \mathrm{dom}(S) \cap [D]^\omega$, $S(f, \Gamma(f))$. Define $\Lambda : [D]^\omega \cap \mathrm{dom}(R) \to \mathscr{P}(\kappa)$ by $\Lambda(f) = \Phi(\pi(\Gamma(f)), \alpha_f)$.

To see (2): Pick any $s \in K$. By the Moschovakis coding lemma (Fact 3.15), let $\pi : \mathbb{R} \to \mathscr{P}_{\omega_1}(\zeta_s)$ be a surjection. Define $S \subseteq [C]^\omega \times \mathbb{R}$ by $S(h, r)$ if and only if $R(s\hat{\ }h, \Phi(r, \alpha_{s\hat{\ }h}))$. Applying Ramsey Uniformization for ω, there is an infinite $D \subseteq C$ and $\Upsilon : \mathrm{dom}(S) \cap [D]^\omega \to \mathbb{R}$ so that for all $h \in \mathrm{dom}(S) \cap [D]^\omega$, $S(s\hat{\ }h, \Upsilon(s\hat{\ }h))$. Let $n^* = |s|$. Define $\Lambda : \mathrm{dom}(R) \cap N_{s,D} \to \mathscr{P}(\kappa)$ by $\Lambda(f) = \Phi(\pi(\Upsilon(\mathsf{drop}(f, n^*))), \alpha_f)$. Then for all $f \in \mathrm{dom}(R) \cap N_{s,D}$, $R(f, \Lambda(f))$. ☐

Question 6.10. Under AD^+, $\mathsf{AD}_\mathbb{R}$, $V = L(\mathscr{P}(\mathbb{R}))$, and $\mathrm{cof}(\Theta) = \omega$, does Ramsey Uniformization hold for all $\kappa \in \mathrm{ON}$?

Regardless of the answer to the question, the proof of Theorem 5.13 can be modified to use Ramsey Uniformization for κ on a neighorhood.

Theorem 6.11. *Assume* AD, $\mathsf{DC}_\mathbb{R}$, *and* $\omega \to (\omega)^\omega_2$. *Suppose* κ *is an infinite cardinal,* $\mathrm{cof}(\kappa) = \omega$, Ramsey Uniformization *for* κ *holds on a neighborhood, and* ω *injects into every infinite subset of* $\mathscr{P}(\kappa)$. *Then there are no infinite maximal* $\mathscr{B}(\kappa)$ *almost disjoint families.*

Proof. In the proof of Theorem 5.13, now use Ramsey Uniformization for κ on a neighborhood to find some $s \in [\omega]^{<\omega}$, an infinite $C \subseteq \omega$, and a function $\Phi : \mathrm{dom}(R) \cap N_{s,C} \to \mathscr{P}(\kappa)$ so that for all $f \in \mathrm{dom}(R) \cap N_{s,C}$, $R(f, \Phi(f))$, where R is the relation from the proof. Now run the rest of the argument and construction with s as an initial segment of all objects. ☐

Theorem 6.12. *Assume* AD^+, $\mathsf{AD}_\mathbb{R}$, *and* $V = L(\mathscr{P}(\mathbb{R}))$. *If* $\kappa \in \mathrm{ON}$ *and* $\mathrm{cof}(\kappa) = \omega$, *then there are no infinite maximal* $\mathscr{B}(\kappa)$ *almost disjoint families.*

Proof. Use Fact 6.9 and Theorem 6.11. ☐

Theorem 6.13. *Assume* AD^+, $\mathsf{AD}_\mathbb{R}$, *and* $V = L(\mathscr{P}(\mathbb{R}))$.

- *There is a maximal* $\mathscr{B}(\kappa)$ *almost disjoint family* $\mathcal{A}$ *such that* $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$ *if and only if* $\mathrm{cof}(\kappa) \geq \Theta$.
- *If* κ *is not a singular cardinal of countable cofinality, then there is a maximal* $\mathscr{P}_\kappa(\kappa)$ *almost disjoint family* $\mathcal{A}$ *such that* $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$ *if and only if* $\mathrm{cof}(\kappa) \geq \Theta$.

Proof. This follows from Theorem 4.3, Theorem 6.8, and Theorem 6.12. ☐

Combining the previous results gives the following main result.

Theorem 6.14. *Assume* AD^+ *and* $V = L(\mathscr{P}(\mathbb{R}))$.

- *There is a maximal* $\mathscr{B}(\kappa)$ *almost disjoint family* $\mathcal{A}$ *so that* $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$ *if and only if* $\mathrm{cof}(\kappa) \geq \Theta$.
- *If* κ *is not a singular cardinal of countable cofinality, then there is a maximal* $\mathscr{P}_\kappa(\kappa)$ *almost disjoint family* $\mathcal{A}$ *so that* $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$ *if and only if* $\mathrm{cof}(\kappa) \geq \Theta$.

Proof. Woodin showed that if AD^+ and $V = L(\mathscr{P}(\mathbb{R}))$ holds, then either there is a set of ordinals J so that $V = L(J, \mathbb{R})$ or $\mathsf{AD}_{\mathbb{R}}$ holds. In the former case, the result follows from Theorem 6.7. In the latter case, the result follows from Theorem 6.13.

□

References

1. Andrés Eduardo Caicedo and Richard Ketchersid, *A trichotomy theorem in natural models of* AD^+, Set theory and its applications, Contemp. Math., vol. 533, Amer. Math. Soc., Providence, RI, 2011, pp. 227–258. MR 2777751
2. William Chan, *An introduction to combinatorics of determinacy*, Trends in Set Theory, Contemp. Math., vol. 752, Amer. Math. Soc., Providence, RI, 2020, pp. 21–75. MR 4132099
3. William Chan and Stephen Jackson, $\mathbf{L}(\mathbb{R})$ *with determinacy satisfies the Suslin hypothesis*, Adv. Math. **346** (2019), 305–328. MR 3910797
4. William Chan, Stephen Jackson, and Nam Trang, *The size of the class of countable sequences of ordinals*, Trans. Amer. Math. Soc. **375** (2022), no. 3, 1725–1743. MR 4378077
5. P. Erdős and S. H. Hechler, *On maximal almost-disjoint families over singular cardinals*, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I, Colloq. Math. Soc. János Bolyai, Vol. 10, North-Holland, Amsterdam, 1975, pp. 597–604. MR 0376354
6. Steve Jackson, *Structural consequences of* AD, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1753–1876. MR 2768700
7. Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513 (2004g:03071)
8. Alexander S. Kechris, *The axiom of determinacy implies dependent choices in* $L(\mathbb{R})$, J. Symbolic Logic **49** (1984), no. 1, 161–173. MR 736611
9. Richard Ketchersid, Paul Larson, and Jindřich Zapletal, *Ramsey ultrafilters and countable-to-one uniformization*, Topology Appl. **213** (2016), 190–198. MR 3563079
10. Peter Koellner and W. Hugh Woodin, *Large cardinals from determinacy*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1951–2119. MR 2768702
11. Menachem Kojman, Wiesław Kubiś, and Saharon Shelah, *On two problems of Erdős and Hechler: new methods in singular madness*, Proc. Amer. Math. Soc. **132** (2004), no. 11, 3357–3365. MR 2073313
12. Donald A. Martin and John R. Steel, *The extent of scales in* $L(\mathbb{R})$, Cabal seminar 79–81, Lecture Notes in Math., vol. 1019, Springer, Berlin, 1983, pp. 86–96. MR 730590
13. A. R. D. Mathias, *Happy families*, Ann. Math. Logic **12** (1977), no. 1, 59–111. MR 491197
14. S. Müller and P. Lücke, $\mathbf{\Sigma_1}$*-definability at higher cardinals: Thin sets, almost disjoint families and long wellorders*, Forum of Mathematics, Sigma, Accepted.
15. Itay Neeman and Zach Norwood, *Happy and mad families in* $L(\mathbb{R})$, J. Symb. Log. **83** (2018), no. 2, 572–597. MR 3835078
16. David Schrittesser and Asger Törnquist, *The Ramsey property implies no mad families*, Proc. Natl. Acad. Sci. USA **116** (2019), no. 38, 18883–18887. MR 4012549
17. John Steel and Nam Trang, AD^+*, derived models, and* Σ_1*-reflection*, `https://math.berkeley.edu/~steel/papers/AD+reflection.pdf`.
18. John R. Steel, *Closure properties of pointclasses*, Cabal Seminar 77–79 (Proc. Caltech-UCLA Logic Sem., 1977–79), Lecture Notes in Math., vol. 839, Springer, Berlin-New York, 1981, pp. 147–163. MR 611171
19. ———, *Scales in* $L(\mathbf{R})$, Cabal seminar 79–81, Lecture Notes in Math., vol. 1019, Springer, Berlin, 1983, pp. 107–156. MR 730592
20. ———, $\mathrm{HOD}^{L(\mathbb{R})}$ *is a core model below* Θ, Bull. Symbolic Logic **1** (1995), no. 1, 75–84. MR 1324625
21. Nam Trang, *Supercompactness can be equiconsistent with measurability*, Notre Dame J. Form. Log. **62** (2021), no. 4, 593–618. MR 4350949

Institute for Discrete Mathematics and Geometry, Vienna University of Technology, Vienna, Austria
Email address: `William.Chan@tuwien.ac.at`

Department of Mathematics, University of North Texas, Denton, TX 76203
Email address: `Stephen.Jackson@unt.edu`

Department of Mathematics, University of North Texas, Denton, TX 76203
Email address: `Nam.Trang@unt.edu`