CARDINALITY OF THE SET OF BOUNDED SUBSETS OF A CARDINAL

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ABSTRACT. If α and β are cardinals, then the cardinal exponentiation ${}^{\alpha}\beta$ is the set of all functions from α into β. Among the very few behaviors that can be inferred from the definition of cardinal exponentiation and basic set construction principles, one can show that if $\alpha \leq \gamma$ and $\beta \leq \delta$ are cardinals, then $|{}^{\alpha}\beta| \leq |{}^{\gamma}\delta|$. The ABCD hypothesis is the aesthetically simplest behavior for cardinal exponentiation which asserts that this is the only relationship between infinite cardinal exponentiations: For any cardinals $\omega \leq \alpha \leq \beta$ and $\omega \leq \gamma \leq \delta$, $|{}^{\alpha}\beta| \leq |{}^{\gamma}\delta|$ if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$.

The axiom of determinacy AD asserts every two-player infinite game on ω has a winning strategy for one of the two players. AD^+ is Woodin's extension of AD. Let Θ be the supremum of the ordinals onto which $\mathbb{R}$ surjects. Determinacy axioms govern an important initial segment of the universe consisting of the sets onto which $\mathbb{R}$ surjects and Θ is the height of this initial segment.

Assume AD^+. The ABCD hypothesis holds below Θ: For any cardinals $\omega \leq \alpha \leq \beta < \Theta$ and $\omega \leq \gamma < \delta < \Theta$, $|{}^{\alpha}\beta| \leq |{}^{\gamma}\delta|$ if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$. This provides a complete classification of the relationship between any pair of infinite cardinal exponentiations below Θ.

If κ is a cardinal, then let $\mathscr{P}_B(\kappa) = \{A \in \mathscr{P}(\kappa) : \sup(A) < \kappa\}$ be the set of bounded subsets of κ. The previous result follows from the eponymous result of the paper: Assume AD^+. For all cardinals $\omega < \kappa < \Theta$ and $\epsilon < \kappa$, $\mathscr{P}_B(\kappa)$ does not inject into ${}^{\epsilon}\mathrm{ON}$, the class of ϵ-sequences of ordinals.

1. Introduction

The set of natural numbers is denoted ω. A distinguished characteristic of ω and each finite natural number is the ability to make definitions by recursion and proofs by induction. This property is equivalent to the usual ordering on ω being a wellordering. Using recursion, one defines the usual arithmetic operations on ω including addition, multiplication, and exponentiation. Of particular significance here is the exponentiation operation. If m and n are natural numbers, let ${}^{m}n$ be the set of all functions from an m-element set into an n-element set. By arithmetic considerations, the usual exponentiation n^m is the size of the set ${}^{m}n$ and in particular the set ${}^{m}n$ is finite and wellorderable. The birth of set theory began with Cantor's investigation of exponentiation for infinite sets.

Cantor formalized the notion of size for arbitrary infinite sets. If X and Y are two sets, then X and Y have the same cardinality if and only if there is a bijection from X to Y. An equivalence class of the bijection equivalence relation is called a cardinality. If X is a set, then the cardinality of X, denoted $|X|$, is the bijection equivalence class of X. Cardinalities are compared by the injection relation. $|X| \leq |Y|$ if and only if there is an injection of X into Y. $|X| < |Y|$ if and only if $|X| \leq |Y|$ and $\neg(|Y| \leq |X|)$.

The infinite analog of the natural numbers are sets which possess a form of transfinite induction and recursion or equivalently sets endowed with wellorderings. Ordinals are transitive sets α so that its own membership relation $\in$ defines a wellordering. Ordinals are sets with a natural intrinsic wellordering and every wellordering is order isomorphic to an ordinal. Each natural number does not inject into any smaller natural number, but this is not true of all ordinals. A cardinal is an ordinal which does not inject into any smaller ordinal. Cardinals are natural infinite analog of sizes which possess transfinite recursion and induction. If a set X is wellorderable, then $|X|$ has a unique member which is a cardinal. (Thus, if the axiom of choice AC holds, then every cardinality has a cardinal. The investigation of cardinality under AC is tantamount to the study of cardinals.)

For any two set X and Y, ${}^{X}Y$ is the set of all functions f from X into Y which can be regarded as the general exponentiation of sets. Cantor proved the first theorem of set theory that X does not surject into ${}^{X}2$ and thus $|X| < |{}^{X}2|$. The behavior of cardinal exponentiation has received substantial investigation by

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set theorists. However, there are very few relationships between infinite cardinal exponentiations which can be inferred solely from the definition of cardinal exponentiation and very basic set construction principles:

(1) For any cardinal κ, ${}^0\kappa = \{\emptyset\} = 1$.
(2) For any cardinal $\kappa > 0$, ${}^\kappa 0 = \emptyset = 0$.
(3) For any cardinal κ, $|{}^\kappa 1| = |1|$.
(4) For any cardinals $2 \leq \delta \leq \kappa$ with κ infinite, $|{}^\kappa\delta| = |\mathscr{P}(\kappa)| = |{}^\kappa 2| = |{}^\kappa\kappa|$.
(5) For any cardinals α, β, γ, and δ, if $\alpha \leq \gamma$ and $\beta \leq \delta$, then $|{}^\alpha\beta| \leq |{}^\gamma\delta|$.

Exponentiations of finite natural numbers have an intrinsic arithmetic reason for being finite and hence wellorderable. There seems to be no natural way to infer the wellorderability of infinite cardinal exponentiation from the definition of cardinal exponentiation and basic set theoretic construction principles. Cohen [22] formalized this by showing that the Zermelo-Frankel set theory axiom (ZF) cannot prove $\mathbb{R}$ or ${}^\omega 2$ is wellorderable. Thus the aesthetically simplest behavior for infinite cardinal exponentiation compatible with the necessary properties which can be logically inferred from the definition of cardinal exponentiation is called the ABCD hypothesis which states the above behavior completely classifies infinite cardinal exponentiation.

- (ABCD hypothesis) For all cardinals $\omega \leq \alpha \leq \beta$ and $\omega \leq \gamma \leq \delta$, $|{}^\alpha\beta| \leq |{}^\gamma\delta|$ if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$.

The ABCD hypothesis does provide a complete classification of infinite cardinal exponentiation in the following sense: given any four infinite cardinals α, β, γ, and δ, the ABCD hypothesis and property (4) can be used to determine the injection relation between ${}^\alpha\beta$ and ${}^\gamma\delta$. (See Remark 6.10.) If $\omega \leq \alpha \leq \beta$ and $\omega \leq \gamma \leq \delta$ are cardinals so that $\alpha < \gamma$ and $\delta < \beta$, then the ABCD hypothesis would imply that $|{}^\alpha\beta|$ and $|{}^\gamma\delta|$ are incomparable cardinalities. Thus the ABCD hypothesis implies the failure of the axiom of choice.

If one assumes the axiom of choice, then for any cardinal κ, ${}^\kappa 2$ is wellorderable and hence is in bijection with a unique cardinal which is frequently denoted 2^κ. The cardinal successor κ^+ is the least cardinal which does not inject into κ. By Cantor's theorem, $\kappa < 2^\kappa$ and thus the smallest possible value for 2^κ is κ^+. Under the axiom of choice, the aesthetically minimal behavior for infinite cardinal exponentiation is the generalized continuum hypothesis (GCH) which asserts that for all infinite cardinals κ, $2^\kappa = \kappa^+$. However, the argument for the aesthetic minimality of the generalized continuum hypothesis assumes the wellorderability of infinite cardinal exponentiation which seems to be an arbitrary judgement as it cannot be naturally inferred from the definition of cardinal exponentiation and basic set construction principles. In contrast, the failure of the axiom of choice from the ABCD hypothesis is a necessary consequence of the aesthetic minimality considerations. The ABCD hypothesis is a more purely minimal behavior for infinite cardinal exponentiation than the generalized continuum hypothesis from an aesthetic view.

The author does not know if the (global) ABCD hypothesis is consistent. If κ is a cardinal, the ABCD hypothesis below κ is the assertion that for all cardinals $\omega \leq \alpha \leq \beta < \kappa$ and $\omega \leq \gamma \leq \delta < \kappa$, $|{}^\alpha\beta| \leq |{}^\gamma\delta|$ if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$. The ABCD hypothesis below ω is vacuous. The ABCD hypothesis below ω_1 is trivial since the only infinite exponentiation below ω_1 is ${}^\omega\omega$. The first nontrivial instance is the ABCD hypothesis below ω_2 which amounts to showing $|{}^\omega\omega_1| < |{}^{\omega_1}\omega_1|$. Let Θ be the supremum of the ordinals onto which $\mathbb{R}$ surjects. The descriptive set theoretic world can be intuitively regarded as an initial segment of the universe consisting of transitive sets which are images of $\mathbb{R}$. Θ is the ordinal height of the descriptive set theoretic world. $\mathbb{R}$ always surjects onto ω_1 and thus ω_2 is the smallest possible value for Θ. An important instance is whether the ABCD hypothesis holds in the descriptive set theoretic world or namely below Θ. Theorem 3.4 will show that $|{}^\omega\omega_1| < |{}^{\omega_1}\omega_1|$ holds in the Solovay model ([54]) where $\Theta = \omega_2$. Thus the consistency of ZFC and the existence of an inaccessible cardinal implies the consistency of the ABCD hypothesis below $\Theta = \omega_2$.

The classification of cardinal exponentiation is merely one of many other interesting questions of infinitary combinatorics. Rather than investigating consistency strength of the ABCD hypothesis, this paper is more concerned with showing the ABCD hypothesis is a consequence of a practically useful theory that can serve as a foundational framework for many other combinatorial questions. Intuitively, one would like to show that the ABCD hypothesis holds in an idealized universe which possesses all the regularity properties. Descriptive set theory provides an opinion on the nature of this ultimate regular universe at least up to the initial segment of sets which are images of $\mathbb{R}$ (the descriptive set theoretic world). Classical regularity properties include the perfect set property, the property of Baire, and Lebesgue measurability. A particularly prominent regularity property is the axiom of determinacy, AD, which states that all infinite two player games of perfect information with moves from ω have a winning strategy for one of the two players. AD^+ is Woodin's extension of the basic axiom of determinacy, AD. It is open whether AD and AD^+ are equivalent. AD^+ has proven to be one of the most important theory in the modern development of set theory (with or without the axiom of choice) and is central to many aspects of the inner model theory program, Woodin's Ultimate-L, forcing axioms, and descriptive set theory.

The inclusion of AD^+ among the laws governing the ultimate regular universe empirically seems to yield some remarkably fascinating structure to the descriptive set theoretic world. Many classically pathological or paradoxical sets can no longer exist including wellorderings of the reals, Lebesgue nonmeasurable sets, Banach-Tarski decompositions, Suslin lines ([9]), and maximal almost disjoint families ([53], [20]). It seems empirically that AD^+ is very successful at resolving natural combinatorial questions and independence phenomenon among these natural problems are quite rare. The method of forcing is the most powerful technique for establishing independence results. By the results of the Chan-Jackson ([12]) and Ikegami-Trang ([26]), any nontrivial forcing which is an image $\mathbb{R}$ over a model of AD must destroy AD in the forcing extension. Intuitively, the descriptive set theoretic world satisfying AD is quite fragile to the direct application of nontrivial forcings which are themselves in the descriptive set theoretic world. This perhaps explains the empirical rarity of independence phenomenon for natural combinatorial questions under determinacy assumptions. The classification of cardinal exponentiation below Θ seems to belong to the class of natural problems that determinacy should be able to resolve.

The origin of the ABCD hypothesis came from neither philosophical nor aesthetic considerations. Prior to the formalization, many incremental cardinality computations had slowly suggested the possibility. Woodin ([65]) had known $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$ and $\neg(|S_1| \leq |{}^\omega\omega_1|)$ at least under $\mathsf{AD}_\mathbb{R}$ and DC. Jackson and the author ([13]) showed that AD, $\mathsf{DC}_\mathbb{R}$, and all subsets of $\mathbb{R}$ have ∞-Borel codes imply S_1 does not inject into ${}^\omega\mathrm{ON}$ and hence $[\omega_1]^{<\omega_1}$ does not inject into ${}^\omega\mathrm{ON}$. The methods of [13] uses heavily ∞-Borel codes for subsets of $[\omega_1]^{<\omega_1}$ (which does follow from all subsets of $\mathbb{R}$ having ∞-Borel codes) and that generic filters exist for countable forcings that belong to an inner model satisfying ZFC. To generalize this proof beyond ω_1, one would need the existence of generic filters over an inner model satisfying ZFC for a forcing which is uncountable in the real world and the existence of ∞-Borel codes for sets of subsets of uncountable cardinals. This first proof known to the author from [13] will be the most suitable for generalization in this paper but at the time, both problems seemed intractable. Generally generic filters for uncountable forcings cannot exist. Woodin showed AD^+ implies there are subsets of $\mathscr{P}(\omega_1)$ which do not have ∞-Borel codes (see [15]). The goal for the next several years was to find forcing-free and ∞-Borel code-free proofs that $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$ which could generalize to cardinals beyond ω_1. Jackson and the author ([11]) investigated almost everywhere continuity properties for functions of the form $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. Since continuity is a well controlled failure of injectivity, this continuity result was used to show that AD implies $|[\omega_1]^{<\omega_1}| < |\mathscr{P}(\omega_1)|$. Motivated by this almost everywhere continuity result, Jackson, Trang, and the author ([19]) established an almost everywhere continuity property for functions of the form $\Phi : [\omega_1]^\epsilon \to \omega_1$ when $\epsilon < \omega_1$. This was used in [19] to give the first known AD proof that $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$. [19] also established an analogous almost everywhere continuity result for functions of the form $\Phi : [\omega_2]^\epsilon \to \omega_2$ when $\epsilon < \omega_2$. This continuity result was used to show $|[\omega_2]^{\omega_1}| < |[\omega_2]^{<\omega_2}|$ in [19] which was the first instance of these computations beyond ω_1. Using the cardinality computation $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$, [18] showed that AD and $\mathsf{DC}_\mathbb{R}$ prove $[\omega_1]^{<\omega_1}$ does not inject into ${}^\omega\mathrm{ON}$, the class of ω-sequences of ordinals. However, the proof in [18] used very specific properties of ω_1 that could not be generalized to ω_2. [21] then established an almost everywhere continuity property for function of the form $\Phi : [\kappa]^\epsilon \to \mathrm{ON}$ when $\epsilon < \kappa$, $\mathrm{cof}(\epsilon) = \omega$, and κ satisfies the partition relation $\kappa \to_* (\kappa)^{\epsilon\cdot\epsilon}_2$. Using this result, [18] showed that if κ satisfies the weak partition property $\kappa \to_* (\kappa)^{<\kappa}_2$, then for all $\epsilon < \kappa$, $[\kappa]^{<\kappa}$ does not inject into ${}^\epsilon\mathrm{ON}$, the class of ϵ-sequences of ordinals. With this result, these cardinality computations had now been extended to all the familiar weak and strong partition cardinal of determinacy including ω_1, ω_2, $\omega_{\omega+1}$, $\omega_{\omega+2}$, the projective ordinals $\boldsymbol{\delta}^1_n$ (for $n \in \omega$), and $\boldsymbol{\delta}^2_1$. Jackson, Trang, and the author had enough evidence to formulate the seemingly plausible ABCD conjecture which would completely classify cardinal exponentiation below Θ (with the name of the conjecture due to Jackson and Trang):

- (ABCD Conjecture) AD^+ proves that for all cardinals $\omega \leq \alpha \leq \beta < \Theta$ and $\omega \leq \gamma \leq \delta < \Theta$, $|{}^\alpha\beta| \leq |{}^\gamma\delta|$ if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$.

The statement of the ABCD Conjecture is optimal. AD^+ cannot prove the ABCD hypothesis beyond Θ. If AD holds and $V = L(\mathbb{R})$, then for any $\kappa \geq \Theta$, $\kappa^+ = (\kappa^+)^{\mathrm{HOD}}$ (see discussion in the proof of [20] Theorem 6.3) and thus there is even an injection of Θ^+ into $\mathscr{P}(\Theta)$.

The key to solving the ABCD conjecture is to generalize the phenomenon from [21] that $[\kappa]^{<\kappa}$ does not inject into ${}^{\epsilon}\mathrm{ON}$ when $\epsilon < \kappa$ and κ is a weak partition cardinal. At a singular cardinal, it was not clear if this is the correct formulation. Let $\mathscr{P}_B(\kappa) = \{A \in \mathscr{P}(\kappa) : \sup(A) < \kappa\}$ be the set of bounded subsets of κ. If κ is regular, then $|[\kappa]^{<\kappa}| = |\mathscr{P}_B(\kappa)|$. However, if κ is singular, then $|\mathscr{P}_B(\kappa)| < |[\kappa]^{<\kappa}|$ (which follows from Theorem 2.23). The following conjecture is the key insight:

- (Bounded Power Set Conjecture) AD^+ proves that for all cardinal κ with $\omega < \kappa < \Theta$ and $\epsilon < \kappa$, $\mathscr{P}_B(\kappa)$ does not inject into ${}^{\epsilon}\mathrm{ON}$.

The following is the eponymous main theorem of the paper:

- (Theorem 6.7) The Bounded Power Set Conjecture is true.

From the Bounded Power Set Conjecture, one obtains the following:

- (Theorem 6.9) The ABCD Conjecture is true.

The proof of the Bounded Power Set Conjecture will require both the Steel-Woodin inner model theoretic analysis of HOD-type models as a direct system of certain iterable mice ([58], [64]) and Woodin's ∞-Borel code analysis of HOD-type models. The Bounded Power Set Conjecture is proved first in AD and $V = L(\mathbb{R})$ in Theorem 6.2 where both the inner model theoretic and ∞-Borel code analysis of the relevant HOD-type model are much simpler. This should provide most of the main ideas and be accessible to those familiar with the inner model theoretic analysis of $\mathrm{HOD}^{L(\mathbb{R})}$ from [60]. The proof of the Bounded Power Set Conjecture under AD^+ will require an analysis of HOD-type models obtained from Suslin-coSuslin reflection, coarse Woodin mice ([62]), their iteration strategies, and directed system of hybrid structures ([48], [49]). The ∞-Borel code analysis of these HOD-type structures will also be more challenging.

The Nairian models constructed by Blue, Larson, and Sargsyan ([2], [3]) have already been shown to possess on a global scale some of the determinacy features which are used as ingredients of the proof in this paper. There seems to be great promise that the consistency of the (global) ABCD hypothesis could be established by adapting the framework here globally within the Nairian models.

The ABCD conjecture and the Bounded Power Set conjecture merely distinguishes the cardinality of two sets. The existence and nonexistence of injections is the most primitive question of cardinality or infinitary combinatorics. There are more sophisticated combinatorial properties of cardinalities and sets can be endowed with additional mathematical structure. The class of ω-sequences of ordinals corresponds to the ω-exponent cardinal exponentiation and is the simplest nonwellorderable cardinal exponentiation. Many additional combinatorial properties have been established for set of ω-sequences of ordinals under determinacy assumptions.

If κ satisfies $\kappa \to_* (\kappa)^{\omega+\omega}_2$, then ${}^{\omega}\kappa$ does not inject into $\mathscr{P}(\delta) \times \mathrm{ON}$ for any $\delta < \kappa$. The conjecture has been that this should be true of all cardinals $\kappa < \Theta$ under AD^+. For any cardinal κ, let $B(\omega, \kappa)$ be the set of functions $f : \omega \to \kappa$ which are bounded below κ. Note that $B(\omega, \kappa) \subseteq {}^{\omega}\kappa$. If $\mathrm{cof}(\kappa) > \omega$, then $B(\omega, \kappa) = {}^{\omega}\kappa$. Fact 6.11 will show that under AD^+, if $\kappa < \Theta$ and $\mathrm{cof}(\kappa) = \omega$, then $|B(\omega, \kappa)| < |{}^{\omega}\kappa|$. The methods developed to study the ABCD conjecture will resolve the above conjecture.

- (Theorem 7.19) Assume AD^+. For all cardinals κ with $2 \leq \kappa < \Theta$ and all $\chi < \kappa$, $B(\omega, \kappa)$ does not inject into $\mathscr{P}(\chi) \times \mathrm{ON}$. In particular, for all cardinals κ with $2 \leq \kappa < \Theta$ and all $\chi < \kappa$, ${}^{\omega}\kappa$ does not inject into $\mathscr{P}(\chi) \times \mathrm{ON}$.

Jackson, Trang, and the author ([17]) developed a notion of regularity and cofinality in the choiceless framework and calibrated the cofinality of some familiar sets under AD. If X is a set and Y is a class, X has Y-regular cardinality if and only if for any function $\Phi : X \to Y$, there is a $y \in Y$ so that $|\Phi^{-1}[\{y\}]| = |X|$. [17] shows that regular cardinals, $\mathbb{R}$, and $\mathbb{R}/E_0$ have globally regular cardinality and provided substantial evidence for the conjecture that $\mathscr{P}(\omega_1)$ is also globally regular. Much is also known about the cofinality of ω-sequences through certain cardinals. If $\mathrm{cof}(\kappa) > \omega$, then ${}^{\omega}\kappa = B(\omega, \kappa)$ does not have $\mathrm{cof}(\kappa)$-regular cardinality under AD^+. If $\kappa \to_* (\kappa)^{\omega+\omega}_2$ holds, then ${}^{\omega}\kappa = B(\omega, \kappa)$ is δ-regular for all $\delta < \kappa = \mathrm{cof}(\kappa)$. Using the strong partition property $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and Martin's ultrapower representation of ω_{n+1}, [17] can show that ${}^{\omega}(\omega_{n+1})$ has ω_1-regular cardinality but not ω_2-regular cardinality when $1 \leq n < \omega$. (Martin showed

$\mathrm{cof}(\omega_{n+1}) = \omega_2$ for all $1 \leq n < \omega$.) Upon some reflection, when $\mathrm{cof}(\kappa) = \omega$, there seems to be no witness to ${}^\omega\kappa$ failing any form of ordinal regularity (in analog to ${}^\omega\omega$ being ON-regular and even globally regular). The conjecture seems to be that under AD^+, if $\kappa < \Theta$ and $\mathrm{cof}(\kappa) = \omega$, then ${}^\omega\kappa$ has ON-regular cardinality and if $\omega < \kappa < \Theta$, then $B(\omega, \kappa)$ does not have $\mathrm{cof}(\kappa)$-regular cardinality but does have δ-regular cardinality for all $\delta < \mathrm{cof}(\kappa)$. For cardinals κ of countable cofinality, this will be fully verified.

- (Theorem 7.29) Assume AD^+. If $\kappa < \Theta$ and $\mathrm{cof}(\kappa) = \omega$, then ${}^\omega\kappa$ has ON-regular cardinality. If $\omega < \kappa < \Theta$ and $\mathrm{cof}(\kappa) = \omega$, then $B(\omega, \kappa)$ does not have ω-regular cardinality but does have n-regular cardinality for each $n < \omega$.

Some time after the author had proved Theorem 7.29, Jackson, Trang, and the author ([17]) showed that ${}^\omega(\omega_\omega)$ has ON-regular cardinality under AD and $\mathsf{DC}_\mathbb{R}$ using combinatorial and descriptive set theoretic arguments.

To address the full conjecture concerning ordinal regularity for ω-sequences of ordinals, it remains to consider $B(\omega, \kappa) = {}^\omega\kappa$ when $\mathrm{cof}(\kappa) > \omega$. Steel ([58], [60]) showed that if AD and $V = L(\mathbb{R})$ hold, then for any $\delta < \Theta$, $\mathrm{cof}^{L(\mathbb{R})}((\delta^+)^{\mathrm{HOD}}) = \omega$. In the arguments, one will roughly need ω-cofinal sequences through the successor and double successor in HOD of $\mathrm{cof}(\kappa)$-many ordinals below κ. The Uniform Cofinality of HOD Successor and Double Successor Hypothesis (see Definition 7.30) roughly states the desired sequence of ω-cofinal maps can be found. The hypothesis seems quite plausibly to be a consequence of AD^+. When $\mathrm{cof}(\kappa) = \omega$, the hypothesis follows from the appropriate form of countable choice available under AD. The author can verify a substantial instance of the hypothesis for κ with $\mathrm{cof}(\kappa) = \omega_1$ using a simple linear iteration argument and the direct system analysis of $\mathrm{HOD}^{L(\mathbb{R})}$ presented in [60]. In discussion with Schlutzenberg, it seems promising that Schlutzenberg's method of normalization can establish the hypothesis in full generality in $L(\mathbb{R})$. Assuming an appropriate form of the hypothesis, one can fully resolve the extent of ordinal regularity for the exponent ω.

- (Theorem 7.33) Assume AD^+ and the uniform cofinality of HOD successor and double successor hypothesis. Let $\omega \leq \kappa < \Theta$ be a cardinal. Then $B(\omega, \kappa)$ has δ-regular cardinality for all $\delta < \mathrm{cof}(\kappa)$.

The methods developed here should fully resolve many other combinatorial question surrounding the ω-exponent cardinal exponentiation under AD^+: Assuming $\kappa \rightarrow_* (\kappa)^{\omega+\omega}_2$, Jackson, Trang, and the author ([16]) can show that ${}^\omega\kappa$ is prime (see Definition 7.1). Under AD, [16] also shows that for all $n < \omega$, ${}^\omega(\omega_n)$ is prime and ${}^\omega(\omega_\omega)$ is prime. Jackson, Trang, and the author [19] showed that ${}^\omega\kappa$ is Jónsson if $\kappa \rightarrow_* (\kappa)^{\omega\cdot\omega}_2$. The author ([5]) showed there is a four-element basis for the linear orderings on ${}^\omega\kappa$ under order embeddings if $\kappa \rightarrow_* (\kappa)^{\omega+\omega}_2$. For $1 \leq m < \omega$, let $p(m)$ be the set of prelinear orderings on m (which is also known as the Fubini or ordered Bell number of m). Let $c(1) = 1$ and $c(m) = p(m-1)$ for all $2 \leq m < \omega$. The author [7] showed that $c(m)$ is the optimal coloring number for partitions of m-element subsets of ${}^\omega\kappa$ when $\kappa \rightarrow_* (\kappa)^{\omega\cdot\omega}_2$: For every $1 \leq r < \omega$ and $P : \mathscr{P}^m({}^\omega\kappa) \rightarrow r$, there is an $A \subseteq {}^\omega\kappa$ with $|A| = |{}^\omega\kappa|$ so that $|P[\mathscr{P}^m(A)]| \leq c(m)$ and there is a $Q : \mathscr{P}^m({}^\omega\kappa) \rightarrow c(m)$ so that for all $A \subseteq {}^\omega\kappa$, if $|A| = |{}^\omega\kappa|$, then $|Q[\mathscr{P}^m(A)]| = c(m)$. The extensions of these questions to all cardinals below Θ will be addressed in forthcoming work.

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2. Background

Let $\dot{\in}$ be a binary relation symbol. The language of set theory is $\mathscr{L}_{\mathrm{set}} = \{\dot{\in}\}$. ZF is the $\mathscr{L}_{\mathrm{set}}$-theory of Zermelo-Frankel set theory (without the axiom of choice). ZFC will denote ZF with the axiom of choice, AC. The base theory of this paper will be ZF (and the axiom of choice will never be assumed in the real world). All other set theoretical assumptions beyond ZF will be made explicit. However, the most important tools will involve inner models (of the choiceless real world) satisfying ZFC and their forcing extensions.

Definition 2.1. If X is a set and $A \subseteq {}^\omega X$, consider the game G^X_A in which two players alternate taking turns playing elements x_0, x_1, x_2, x_3, x_4, ... from X which jointly produces an element $f \in {}^\omega X$ after ω-many moves. Player 1 is said to win this run of G^X_A if and only if $f \in A$. If one of the two players has a winning strategy then the game G^X_A is said to be determined. AD_X is the assertion that for all $A \subseteq {}^\omega X$, G^X_A is determined.

$$G_A^X \quad \begin{array}{lllllll} \text{I} & x_0 & & x_2 & & x_4 & & \cdots \\ \text{II} & & x_1 & & x_3 & & x_5 & \cdots \end{array} \quad f$$

AD will abbreviate AD_ω which is called the axiom of determinacy. ${}^\omega\omega$ will often be refer to as $\mathbb{R}$. $\mathsf{AD}_\mathbb{R} = \mathsf{AD}_{{}^\omega\omega}$ is the axiom of real determinacy.

Capturing sets by inner models will be a very important notion throughout this paper. If A is a set and M is an inner model, M captures A if and only if $A \cap M \in A$. Here, one will be most interested in capturing functions; however, capturing merely the graph of a function will not be sufficient. If $\Phi : X \to Y$ is a function, then let $\mathrm{G}_\Phi = \{(x,y) \in X \times Y : \Phi(x) = y\}$ be the graph of Φ. M is closed under Φ if and only if for all $x \in X \cap M$, $\Phi(x) \in M$. M function-captures Φ if and only if M is closed under Φ and M captures G_Φ. If M function-captures Φ, then $\Phi \restriction X \cap M = \{(x, \Phi(x)) : x \in X \cap M\} \in M$. If $\Phi : X \to \mathscr{P}(\delta)$ where $\delta \in \mathrm{ON}$, then define the strong graph of Φ to be $\mathrm{Gr}_\Phi = \{(x, \alpha) \in X \times \delta : \alpha \in \Phi(x)\}$. If M captures Gr_Φ, then M will function-capture Φ.

The following result is the basic template with the most elementary form of the notion of function-capturing.[1] In this paper, one will frequently consider inner models of ZFC to prove theorems from certain hypothesis. To help distinguish between these inner models and the background universe, the terms "true universe" or "real world" will often refer to the background universe (which typically do not satisfy choice) in which the hypothesis of the theorem being proved holds.

If κ is a cardinal, $\mathscr{P}_B(\kappa) = \{A \in \mathscr{P}(\kappa) : \sup(A) < \kappa\}$.

Theorem 2.2. *Assume κ is a cardinal such that for all $\epsilon < \kappa$, $\neg(|\kappa| \leq |\mathscr{P}(\epsilon)|)$. Then $\neg(|[\kappa]^{\mathrm{cof}(\kappa)}| \leq |\mathscr{P}_B(\kappa)|)$ and therefore for all $\gamma \leq \kappa$ with $\mathrm{cof}(\gamma) = \mathrm{cof}(\kappa)$, $\neg(|[\kappa]^\gamma| \leq |\mathscr{P}_B(\kappa)|)$ and $|\mathscr{P}_B(\kappa)| < |\mathscr{P}(\kappa)|$. If κ is a regular cardinal, then $|[\kappa]^{<\kappa}| < |\mathscr{P}(\kappa)|$.*

Proof. Let $\rho : \mathrm{cof}(\kappa) \to \kappa$ be an increasing cofinal function. Suppose there is an injection $\Phi : [\kappa]^{\mathrm{cof}(\kappa)} \to \mathscr{P}_B(\kappa)$. Let $\mathrm{Gr}_\Phi = \{(f, \alpha) : \alpha \in \Phi(f)\}$. The relative Gödel constructible universe $L[\rho, \mathrm{Gr}_\Phi]$ is an inner model of ZFC and $L[\rho, \mathrm{Gr}_\Phi] \models \mathrm{cof}(\kappa) = \mathrm{cof}(\kappa)^V$ as witnessed by ρ. For all $\epsilon < \kappa$, $L[\rho, \mathrm{Gr}_\Phi] \models 2^\epsilon < \kappa$. To see this, suppose otherwise that $L[\rho, \mathrm{Gr}_\Phi] \models \kappa \leq 2^\epsilon$. Then there would be an $h \in L[\rho, \mathrm{Gr}_\Phi]$ such that $L[\rho, \mathrm{Gr}_\Phi] \models$ "$h : \kappa \to \mathscr{P}(\epsilon)$ is an injection" and h would still be an injection in the true universe by absoluteness in violation of the hypothesis. Thus $L[\rho, \mathrm{Gr}_\Phi] \models |\mathscr{P}_B(\kappa)| = 2^{<\kappa} = \kappa$. $L[\rho, \mathrm{Gr}_\Phi] \models |[\kappa]^{\mathrm{cof}(\kappa)}| = \kappa^{\mathrm{cof}(\kappa)} > \kappa$ by a basic result of cardinal arithmetic in ZFC ([31] Corollary 5.14). Using comprehension within $L[\rho, \mathrm{Gr}_\Phi]$, let $\tilde{\Phi}$ be defined by $\tilde{\Phi}(f) = \{\alpha \in \kappa : \mathrm{Gr}_\Phi(f, \alpha)\}$ for each $f \in [\kappa]^{\mathrm{cof}(\kappa)}$. Note that $\tilde{\Phi} = \Phi \restriction ([\kappa]^{\mathrm{cof}(\kappa)})^{L[\rho, \mathrm{Gr}_\Phi]}$ and by absoluteness, $L[\rho, \mathrm{Gr}_\Phi] \models \tilde{\Phi} : [\kappa]^{\mathrm{cof}(\kappa)} \to \mathscr{P}_B(\kappa)$ is an injection. This is a contradiction since $L[\rho, \mathrm{Gr}_\Phi] \models |[\kappa]^{\mathrm{cof}(\kappa)}| > \kappa = |\mathscr{P}_B(\kappa)|$. □

Next, one will describe some important instances in which the hypothesis of Theorem 2.2 holds. A cardinal κ is a measurable cardinal if and only if there is a κ-complete nonprincipal ultrafilter on κ. The next fact shows that the hypothesis of Theorem 2.2 holds at measurable cardinals.

Fact 2.3. *If κ is a measurable cardinal, then κ does not inject into $\mathscr{P}(\epsilon)$ for any $\epsilon < \kappa$.*

Proof. Let μ be a κ-complete nonprincipal ultrafilter on κ. Suppose $\epsilon < \kappa$ and $\Phi : \kappa \to {}^\epsilon 2$. For each $\gamma < \epsilon$, there is an $i_\gamma \in 2$ so that $A_\gamma = \{\alpha < \kappa : \Phi(\alpha)(\gamma) = i_\gamma\} \in \mu$. By κ-completeness, $A = \bigcap_{\gamma < \epsilon} A_\gamma \in \mu$. Since μ is nonprincipal, there exist $\alpha_0 < \alpha_1$ with $\alpha_0, \alpha_1 \in A$. Then $\Phi(\alpha_0)(\gamma) = i_\gamma = \Phi(\alpha_1)(\gamma)$ for all $\gamma < \epsilon$. Thus $\Phi(\alpha_0) = \Phi(\alpha_1)$ and hence Φ is not an injection. □

Definition 2.4. If κ is an infinite cardinal, then let boldface GCH at κ be the assertion that there is no injection of κ^+ into $\mathscr{P}(\kappa)$. Say that boldface GCH holds below κ if and only if for all cardinals $\omega \leq \delta < \kappa$, boldface GCH at δ holds.

[1] In [13] Fact 4.6, Jackson and the author first used ∞-Borel codes to obtain the necessary function-capturing to prove $|[\omega_1]^{<\omega_1}| < |\mathscr{P}(\omega_1)|$. In discussion with the author some years ago, Neeman observed that this earlier proof only requires function-capturing in a single model and he suggested a simplification which requires only a simple cardinality hypothesis. This simplification is presented in [13] Theorem Fact 4.1 which is the precursor Theorem 2.2.

The axiom of countable choice for $\mathbb{R}$, $\mathsf{AC}_\omega^\mathbb{R}$, is the assertion that for all $R \subseteq \omega \times \mathbb{R}$, there is a uniformization function $F : \mathrm{dom}(R) \to \mathbb{R}$ for R which means that for all $n \in \mathrm{dom}(R)$, $R(n, F(n))$. A remarkably powerful tool under determinacy is the wellordered additivity of the meager ideal which is precisely stated below. It will be very important in Section 3.

Fact 2.5. *Assuming $\mathsf{AC}_\omega^\mathbb{R}$ and all subsets of $\mathbb{R}$ have the property of Baire. Let $\langle A_\alpha : \alpha < \mathrm{ON}\rangle$ be a sequence such that A_α is a meager subset of $\mathbb{R}$ for all $\alpha \in \mathrm{ON}$. Then $\bigcup_{\alpha\in\mathrm{ON}} A_\alpha$ is a meager subset of $\mathbb{R}$.*

Proof. Suppose $\bigcup_{\alpha\in\mathrm{ON}} A_\alpha$ is a nonmeager subset of ${}^\omega\omega$. By the axiom of replacement, there is a $\xi < \mathrm{ON}$ so that $\bigcup_{\alpha<\xi} A_\alpha = \bigcup_{\alpha\in\mathrm{ON}} A_\alpha$. Let δ be the least ordinals ξ so that $\bigcup_{\alpha<\xi} A_\alpha$ is nonmeager. By the property of Baire, there is an $s \in {}^{<\omega}\omega$ so that $\bigcup_{\alpha<\delta} A_\alpha$ is comeager in $N_s = \{f \in {}^\omega\omega : s \subseteq f\}$. Let $\Phi : N_s \to {}^\omega\omega$ be a homeomorphism and let $B_\alpha = \Phi[A_\alpha]$ for each $\alpha < \delta$. Then for all $\gamma < \delta$, $\bigcup_{\alpha<\gamma} B_\alpha$ is a meager subset of ${}^\omega\omega$ (by the minimality of δ) and $\bigcup_{\alpha<\delta} B_\alpha$ is a comeager subset of ${}^\omega\omega$. Let $\varphi : \bigcup_{\alpha<\delta} B_\alpha \to \delta$ be defined by $\varphi(x)$ is the least α so that $x \in B_\alpha$. Define $S \subseteq {}^\omega\omega \times {}^\omega\omega$ by $S(x, y)$ if and only if $x, y \in \bigcup_{\alpha<\delta} B_\alpha$ and $\varphi(x) < \varphi(y)$. For all x in the comeager set $\bigcup_{\alpha<\delta} B_\alpha$, $S_x = \{y \in {}^\omega\omega : S(x, y)\} = (\bigcup_{\alpha<\delta} B_\alpha) \setminus (\bigcup_{\alpha\leq\varphi(x)} B_\alpha)$ is comeager since $\bigcup_{\alpha\leq\varphi(x)} B_\alpha$ is meager. By the Kuratowski-Ulam theorem, S is comeager. For each y in the comeager set $\bigcup_{\alpha<\delta} B_\alpha$, $S^y = \{x \in {}^\omega\omega : S(x, y)\} = \bigcup_{\alpha<\varphi(y)} B_\alpha$ is meager. Thus S is meager by the Kuratowski-Ulam theorem. This is a contradiction. □

Boldface GCH at ω is a particularly common and useful hypothesis which can equivalently be restated as there are no uncountable wellorderable subsets of $\mathbb{R}$.

Fact 2.6. *Assume $\mathsf{AC}_\omega^\mathbb{R}$ and all sets of reals have the Baire property and the perfect set property. Then boldface GCH holds at ω.*

Proof. Suppose $A \subseteq \mathbb{R}$ is an uncountable wellorderable set. By the perfect set property, there is an injection $\Phi : \mathbb{R} \to A$. Since A is wellorderable, $\mathbb{R}$ is wellorderable and thus 2^ω is the unique cardinal in bijection with $\mathbb{R}$. Let $\langle x_\alpha : \alpha < 2^\omega\rangle$ be a wellordering of $\mathbb{R}$. Since singletons are meager, $\mathbb{R} = \bigcup_{\alpha<2^\omega}\{x_\alpha\}$ is a wellordered union of meager sets. $\mathbb{R}$ is meager by Fact 2.5 which is a contradiction. □

Actually, the perfect set property for subsets of $\mathbb{R}$ already implies the boldface GCH at ω. To see this, as in the proof of Fact 2.6, the failure of boldface GCH at ω and the perfect set property implies $\mathbb{R}$ is wellorderable. A wellordering on $\mathbb{R}$ can be used to create a subset of $\mathbb{R}$ with the perfect set property.

Boldface GCH at ω implies the following cardinality distinctions.

Fact 2.7. *Assume boldface GCH at ω. Then $\neg(|\mathbb{R}| \leq |\omega_1|)$, $\neg(|\omega_1| \leq |\mathbb{R}|)$, $|\mathbb{R}| < |\mathbb{R} \cup \omega_1|$, $|\omega_1| < |\mathbb{R} \cup \omega_1|$, and $|\mathbb{R} \cup \omega_1| < |\mathbb{R} \times \omega_1|$.*

Proof. All except the last are immediate consequences of boldface GCH at ω. Suppose there is an injection $\Phi : \mathbb{R} \times \omega_1 \to \mathbb{R} \cup \omega_1$. Let $\pi_1 : \mathbb{R} \times \omega_1 \to \mathbb{R}$ be the projection onto the first coordinate. $\pi_1[\Phi^{-1}[\omega_1]]$ is a wellorderable subset of $\mathbb{R}$ and thus is countable by boldface GCH at ω. Let $r \in \mathbb{R} \setminus \pi_1[\Phi^{-1}[\omega_1]]$. Then $\Phi \restriction \{r\} \times \omega_1 : \{r\} \times \omega_1 \to \mathbb{R}$ is an injection. This is essentially an injection of ω_1 into $\mathbb{R}$ which is impossible by boldface GCH at ω. □

Let $\mathsf{AC}_\mathbb{R}^\mathbb{R}$ (or $\mathsf{Uniformization}$)[2] be the assertion that for every relation $R \subseteq \mathbb{R} \times \mathbb{R}$, there is a function $\Lambda : \mathrm{dom}(R) \to \mathbb{R}$ so that for all $r \in \mathrm{dom}(R)$, $R(r, \Lambda(r))$. $\mathsf{AC}_\mathbb{R}^\mathbb{R}$ implies that $|\mathbb{R}|$, $|\omega_1|$, $|\mathbb{R} \cup \omega_1|$, and $|\mathbb{R} \times \omega_1|$ are the only uncountable cardinalities below $|\mathbb{R} \times \omega_1|$ (see [13] Corollary 7.6). [13] Theorem 7.33 produced many intermediate cardinalities below $|\mathbb{R} \times \omega_1|$ in $L(\mathbb{R}) \models \mathsf{AD}$ where $\mathsf{AC}_\mathbb{R}^\mathbb{R}$ fails.

An important source of measurable cardinals under AD comes from partition properties. (See [6], [8], [39], [37], and [30] for more information concerning partition properties under determinacy.)

[2] $\mathsf{AD}_{\frac{1}{2}}$ is the statement asserting one of the two players has a winning strategy in games on $\mathbb{R}$ where one of the two players is restricted to playing moves only from ω. Kechris [36] showed that AD and $\mathsf{AC}_\mathbb{R}^\mathbb{R}$ is equivalent to $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$. $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ follows from $\mathsf{AD}_\mathbb{R}$ but is not known to be equivalent. $\mathsf{AD}_\mathbb{R}$ and $\mathsf{AD}_\mathbb{R} + \mathsf{AD}^+$ are also not known to equivalent. $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$ is strictly stronger than all the aforementioned theories by results of Solovay [55].

Definition 2.8. Let ϵ be an ordinal. A function $f : \epsilon \to \mathrm{ON}$ is said to be discontinuous everywhere if and only if for all $\alpha < \epsilon$, $\sup(f \restriction \alpha) < f(\alpha)$. f is said to have uniform cofinality ω if and only if there is a function $F : \epsilon \times \omega \to \mathrm{ON}$ so that for all $\alpha < \epsilon$ and $n \in \omega$, $F(\alpha, n) < F(\alpha, n+1)$ and $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$. f is said to have the correct type if and only if f is both discontinuous everywhere and has uniform cofinality ω. If $X \subseteq \mathrm{ON}$, then let $[X]^\epsilon_*$ be the collection functions $f : \epsilon \to X$ which have the correct type.

If $\epsilon \leq \kappa$ and $\gamma < \kappa$, let the correct-type partition relation[3] $\kappa \to_* (\kappa)^\epsilon_\gamma$ be the assertion that for all $P : [\kappa]^\epsilon_* \to \gamma$, there is a $\beta < \gamma$ and a club $C \subseteq \kappa$ so that for all $f \in [C]^\epsilon_*$, $P(f) = \beta$. If $\epsilon \leq \kappa$, then $\kappa \to_* (\kappa)^{<\epsilon}_\gamma$ is the assertion that for all $\delta < \epsilon$, $\kappa \to_* (\kappa)^\delta_\gamma$. For $\gamma \leq \kappa$, $\kappa \to_* (\kappa)^\epsilon_{<\gamma}$ is the assertion that for all $\beta < \gamma$, $\kappa \to_* (\kappa)^\epsilon_\beta$. It can be shown that for all $\epsilon < \kappa$, $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$ implies $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$.

For $\epsilon \leq \kappa$, let μ^ϵ_κ be the filter on $[\kappa]^\epsilon_*$ defined by $A \in \mu^\epsilon_\kappa$ if and only if there is a club $C \subseteq \kappa$ so that $[C]^\epsilon_* \subseteq A$. μ^ϵ_κ is called the ϵ-exponent partition filter on κ and note that μ^1_κ is the ω-club filter on κ. If $\kappa \to_* (\kappa)^\epsilon_2$, then μ^ϵ_κ is an ultrafilter. For any $\delta < \kappa$, $\kappa \to_* (\kappa)^\epsilon_\delta$ implies μ^ϵ_κ is a δ^+-complete ultrafilter. Hence $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$ implies μ^ϵ_κ is κ-complete. $\kappa \to_* (\kappa)^2_2$ implies that the ω-club filter μ^1_κ is a normal measure on κ.

κ is said to be a strong partition cardinal if and only if $\kappa \to_* (\kappa)^\kappa_2$. κ is said to be a very strong partition cardinal if and only if $\kappa \to_* (\kappa)^\kappa_{<\kappa}$ (or equivalently μ^κ_κ is κ-complete). κ is said to be a weak partition cardinal if and only if $\kappa \to_* (\kappa)^{<\kappa}_2$.

The exponent two correct type partition relation on an uncountable cardinal already implies the failure of AC by the next result.[4]

Fact 2.9. *Let κ be an uncountable cardinal satisfying $\kappa \to_* (\kappa)^2_2$. Then $[\kappa]^\omega$ is not wellorderable.*

Proof. Suppose otherwise that $[\kappa]^\omega$ has a wellordering $\prec$. For each $\alpha \in [\kappa]^1_*$ (that is, $\mathrm{cof}(\alpha) = \omega$), let f_α be the $\prec$-least $f \in [\kappa]^\omega$ so that $\sup(f) = \alpha$. For each $n \in \omega$, let $\Phi_n : [\kappa]^1_* \to \kappa$ be defined by $\Phi_n(\alpha) = f_\alpha(n)$. For all $\alpha \in [\kappa]^1_*$, $\Phi_n(\alpha) < \alpha$. Since $\kappa \to_* (\kappa)^2_2$ implies μ^1_κ is normal, for each $n \in \omega$, there is a λ_n so that $A_n = \{\alpha \in [\kappa]^1_* : \Phi_n(\alpha) = \lambda_n\} \in \mu^1_\kappa$. Since $\kappa \to_* (\kappa)^2_2$ implies $\kappa \to_* (\kappa)^1_{<\kappa}$ and hence the κ-completeness of μ^1_κ, $A = \bigcap_{n\in\omega} A_n \in \mu^1_\kappa$. Let $g : \omega \to \kappa$ be defined by $g(n) = \lambda_n$. For all $\alpha \in A$, $f_\alpha = g$. Since μ^1_κ is a nonprincipal measure, let $\alpha_0, \alpha_1 \in A$ with $\alpha_0 \neq \alpha_1$. Then $\alpha_0 = \sup(f_{\alpha_0}) = \sup(g) = \sup(f_{\alpha_1}) = \alpha_1$. Contradiction. ☐

Fact 2.10. *Suppose κ is an uncountable cardinal, $\omega \leq \epsilon \leq \kappa$, $\delta < \kappa$, and $\kappa \to_* (\kappa)^\epsilon_\delta$ holds. Then there is no injection of ${}^\epsilon\kappa$ into $\mathscr{P}(\delta) \times \mathrm{ON}$.*

Proof. Suppose $\Phi : {}^\epsilon\kappa \to \mathscr{P}(\delta) \times \mathrm{ON}$ is an injection. Let $\pi_1 : \mathscr{P}(\delta) \times \mathrm{ON} \to \mathscr{P}(\delta)$ be the projection onto the first coordinate. For each $\beta < \delta$, let $P_\beta : [\kappa]^\epsilon_* \to 2$ be $P_\beta(f) = 1$ if and only if $\beta \in \pi_1(\Phi(f))$. By $\kappa \to_* (\kappa)^\epsilon_2$, there is an $i_\beta \in 2$ and a club $C \subseteq \kappa$ so that for all $f \in [C]^\epsilon_*$, $P_\beta(f) = i_\beta$. Thus $E_\beta = \{f \in [\kappa]^\epsilon_* : P_\beta(f) = i_\beta\} \in \mu^\epsilon_\kappa$. Since $\kappa \to_* (\kappa)^\epsilon_\delta$ implies that μ^ϵ_κ is δ^+-complete, $\bigcap_{\beta<\delta} E_\beta \in \mu^\epsilon_\kappa$. There is a club $D \subseteq \kappa$ so that $[D]^\epsilon_* \subseteq \bigcap_{\beta<\delta} E_\beta$. So for all $f \in [D]^\epsilon_*$ and all $\beta < \delta$, $P_\beta(f) = i_\beta$. Let $K = \{\beta \in \delta : i_\beta = 1\}$. Thus for all $f \in [D]^\epsilon_*$, $\pi_1(\Phi(f)) = K$. Hence $\Phi \restriction [D]^\epsilon_* : [D]^\epsilon_* \to \{K\} \times \mathrm{ON}$. Since $|[\kappa]^\omega| \leq |[D]^\epsilon_*|$ because $\omega \leq \epsilon$, one has a wellordering of $[\kappa]^\omega$. This is impossible by Fact 2.9. ☐

If $\epsilon \leq \kappa$, Martin defined a good coding system for ${}^\epsilon\kappa$ ([30] Definition 2.33, [6]) which consists of a pointclass and a method of coding elements of ${}^\epsilon\kappa$ by reals with very strict definability constraints relative to the pointclass. Martin showed that if there is a good coding system for ${}^{\omega\cdot\epsilon}\kappa$, then $\kappa \to_* (\kappa)^\epsilon_2$ holds. Determinacy with the method of good coding systems is the only known set-theoretical setting with strong partition cardinals. If Γ is a pointclass, then let $\check{\Gamma}$ be the dual pointclass of Γ consisting of complements of sets of Γ. Let $\boldsymbol{\delta}(\Gamma)$ is the rank of prewellorderings on $\mathbb{R}$ which belong to $\Delta = \Gamma \cap \check{\Gamma}$. The projective ordinal $\boldsymbol{\delta}^1_n$ is defined as $\boldsymbol{\delta}(\boldsymbol{\Sigma}^1_n)$. $\boldsymbol{\delta}^2_1 = \boldsymbol{\delta}(\boldsymbol{\Sigma}^2_1)$. If Γ has the prewellordering property and is closed under $\forall^\mathbb{R}$, $\wedge$, and $\vee$, then for $\epsilon < \omega_1$, the natural coding of ${}^\epsilon\boldsymbol{\delta}(\Gamma)$ by a real (or rather countably many reals relative to a wellordering of ω of ordertype ϵ) defines a good coding system for ${}^\epsilon\boldsymbol{\delta}(\Gamma)$. Hence for such Γ, $\boldsymbol{\delta}(\Gamma)$ satisfies the countable exponent partition relation. Jackson's description theory ([29], [28], [30]) is used to produce good

[3]The ordinary partition relation $\kappa \to (\kappa)^\epsilon_\gamma$ is the assertion that for all $P : [\kappa]^\epsilon \to \gamma$, there is a $\beta < \gamma$ and an $A \subseteq \kappa$ with $|A| = |\kappa|$ such that for all $f \in [A]^\epsilon$, $P(f) = \beta$.

[4]The ordinary exponent two partition relation at an uncountable cardinal characterizes a weakly compact cardinal which exists under AC assuming the consistency of a weakly compact cardinal which is a relatively mild large cardinal axiom.

coding systems on the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$. Kechris-Kleinberg-Moschovakis-Woodin [37] showed that if Γ is a highly closed pointclass, then $\boldsymbol{\delta}(\Gamma)$ has a good coding system for ${}^{\boldsymbol{\delta}(\Gamma)}\boldsymbol{\delta}(\Gamma)$ and hence $\boldsymbol{\delta}(\Gamma)$ is a strong partition cardinal. This result will show $\boldsymbol{\delta}^2_1$ is a strong partition cardinal. For each $A \in \mathscr{P}(\mathbb{R})$, let $\boldsymbol{\delta}_A$ be the least δ so that $L_\delta(A, \mathbb{R}) \prec_{\Sigma_1} L(A, \mathbb{R})$ in the language with a binary symbol $\dot{\in}$ and unary symbol $\dot{\mathbb{R}}$. [37] also implies that $\boldsymbol{\delta}_A$ is a strong partition cardinal for each $A \in \mathscr{P}(\mathbb{R})$. Besides good coding systems, Martin showed that a combinatorial analysis of the ultrapower of a strong partition cardinal by its measures may sometimes yield partition properties. This is used to show that $\omega_2 = \prod_{\omega_1} \omega_1 / \mu^1_{\omega_1}$ is a weak partition cardinal and more generally the even projective ordinals $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ are weak partition cardinals. The following summarizes the most important partition cardinals of determinacy.

Fact 2.11.

(1) *(Martin; [30] Theorem 2.36) Let Γ be a pointclass with the prewellordering property and closed under $\forall^{\mathbb{R}}$, $\wedge$, and $\vee$. Then $\boldsymbol{\delta}(\Gamma) \to_* (\boldsymbol{\delta}(\Gamma))^{<\omega_1}_2$.*
(2) *(Martin; [34] Theorem 12.2, [30] Corollary 4.16, [8] Section 4; [6]) Assume* AD*. ω_1 is a very strong partition cardinal.*
(3) *(Martin-Paris; [8] Theorem 5.19; [11] Theorem 6.2) Assume* AD*. ω_2 is a weak partition cardinal but is not a strong partition cardinal.*
(4) *(Jackson; [28], [29], [30]) Assume* AD*. For all $n \in \omega$, $\boldsymbol{\delta}^1_{2n+1}$ is a strong partition cardinal. For all $n \in \omega$, $\boldsymbol{\delta}^1_{2n+2}$ is a weak partition cardinal but is not a strong partition cardinal.*
(5) *(Kechris-Kleinberg-Moschovakis-Woodin; [37], [6]) Suppose Γ is a pointclass with the prewellordering property and closed under $\forall^{\mathbb{R}}$, $\wedge$, and $\vee$ and $\Delta = \Gamma \cap \check{\Gamma}$ is closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$. Then $\boldsymbol{\delta}(\Gamma)$ is a very strong partition cardinal.*
(6) *(Kechris-Kleinberg-Moschovakis-Woodin; [37], [6], [42] Theorem 5.8) Assume* AD*. For all $A \in \mathscr{P}(\mathbb{R})$, $\boldsymbol{\delta}_A$ is a very strong partition cardinal.*
(7) *(Kechris-Kleinberg-Moschovakis-Woodin; [37], [42] Theorem 5.5) Assume* AD *and* $\mathsf{DC}_{\mathbb{R}}$*. $\boldsymbol{\delta}^2_1$ is a very strong partition cardinal.*

By Fact 2.3, ω_1 does not inject into $\mathscr{P}(\omega)$ and ω_2 does not inject into $\mathscr{P}(\omega_1)$. This gives another proof that boldface GCH holds at ω and shows boldface GCH holds ω_1. Using Martin's analysis of $\omega_{n+1} = \prod_{[\omega_1]^n} \omega_1 / \mu^n_{\omega_1}$, for each $n \in \omega$, one has that $\mathrm{cof}(\omega_n) = \omega_2$ for all $2 \leq n < \omega$. (Thus ω_3 is the first singular cardinal of determinacy.) Using this ultrapower analysis, Jackson, Trang, and the author ([14]) can show boldface GCH below ω_ω under AD. These arguments generalize using Jackson's description theory ([30], [29], and [28]) to show boldface GCH below $\sup\{\boldsymbol{\delta}^1_n : n \in \omega\}$. The following is obtained from Theorem 2.2.

Theorem 2.12. *Assume* AD*.*

(1) *$|[\omega_1]^{<\omega_1}| < |\mathscr{P}(\omega_1)|$ and $|[\omega_2]^{<\omega_2}| < |\mathscr{P}(\omega_2)|$. For all $3 \leq n < \omega$, $\neg(|[\omega_n]^{\omega_2}| \leq |\mathscr{P}_B(\omega_n)|)$.*
(2) *For all $\kappa < \sup\{\boldsymbol{\delta}^1_n : n \in \omega\}$, $\neg(|[\kappa]^{\mathrm{cof}(\kappa)}| < |\mathscr{P}_B(\kappa)|)$.*
(3) *For all $A \in \mathscr{P}(\mathbb{R})$, $|[\boldsymbol{\delta}_A]^{<\boldsymbol{\delta}_A}| < |\mathscr{P}(\boldsymbol{\delta}_A)|$.*
(4) *Assume* $\mathsf{DC}_{\mathbb{R}}$*, $|[\boldsymbol{\delta}^2_1]^{<\boldsymbol{\delta}^2_1}| < |\mathscr{P}(\boldsymbol{\delta}^2_1)|$.*

Note that under AD, $\mathrm{cof}(\omega_3) = \omega_2$ and hence Theorem 2.2 and its proof method cannot show that $|[\omega_3]^{\omega_2}| < |\mathscr{P}(\omega_3)|$ or even $|[\omega_3]^{\omega_2}| < |[\omega_3]^{<\omega_3}|$. These cardinality distinctions will be established in Theorem 6.12 as a consequence of the main theorem of the paper.

Since AD implies $\omega_1 \to_* (\omega_1)^\omega_\omega$, Fact 2.10 implies the following. In the next section, this cardinality computation will be shown to follow from classical regularity properties by a proof which is more suitable for generalization (see Section 7).

Fact 2.13. *Assume* AD*. ${}^\omega\omega_1$ does not inject to $\mathbb{R} \times \mathrm{ON}$. Thus $|\mathbb{R} \times \omega_1| < |[\omega_1]^\omega| = |{}^\omega\omega_1|$.*

Forcing is a technique developed by Cohen ([22]) to extend a given model of set theory to larger model of set theory in a constructive and minimal manner. Let $\mathbb{P}$ be a nonempty set. Let $\leq_{\mathbb{P}}$ be a partial ordering on $\mathbb{P}$. Assume $\mathbb{P}$ has a $\leq_{\mathbb{P}}$-largest element denoted by $1_{\mathbb{P}}$. $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}})$ is called a forcing. An element $p \in \mathbb{P}$ is often called a condition of $\mathbb{P}$. A $\mathbb{P}$-filter (in the sense of forcing) is a set $F \subseteq \mathbb{P}$ satisfying the following properties:

(1) $1_{\mathbb{P}} \in F$.

(2) For all $p, q \in \mathbb{P}$, if $p \leq_{\mathbb{P}} q$ and $p \in F$, then $q \in F$.
(3) For all $p_0, p_1 \in \mathbb{P}$, there is a $q \in F$ so that $q \leq_{\mathbb{P}} p_0$ and $q \leq_{\mathbb{P}} p_1$.

If $A \subseteq \mathbb{P}$, then let $\langle A \rangle_{\mathbb{P}} = \{p \in \mathbb{P} : (\exists q \in A)(q \leq_{\mathbb{P}} p)\}$ which is the $\leq_{\mathbb{P}}$-upward closure of A. $D \subseteq \mathbb{P}$ is dense if and only if for all $p \in \mathbb{P}$, there is a $q \in D$ such that $q \leq_{\mathbb{P}} p$. $A \subseteq \mathbb{P}$ is a $\mathbb{P}$-antichain if and only if for all $p_0, p_1 \in A$, if $p_0 \neq p_1$, then p_0 and p_1 are $\mathbb{P}$-incompatible which means there is no $q \in \mathbb{P}$ with $q \leq_{\mathbb{P}} p_0$ and $q \leq_{\mathbb{P}} p_1$. A $\mathbb{P}$-antichain A is maximal if and only if A is an antichain and for all $\mathbb{P}$-antichain B with $B \supseteq A$, $B = A$. Let $\mathcal{D}$ be a collection of dense subsets of $\mathbb{P}$. A $\mathbb{P}$-filter G is $\mathbb{P}$-generic over $\mathcal{D}$ if and only if $D \cap G \neq \emptyset$ for all $D \in \mathcal{D}$. Let M be an inner model of ZFC with $\mathbb{P} \in M$. Let $\mathcal{D}_M$ be the collection of all dense subset of $\mathbb{P}$ in M. Let $\mathcal{E}_M$ be the collection of all maximal antichain of $\mathbb{P}$ in M. A filter G is said to be $\mathbb{P}$-generic over M if and only if G is $\mathbb{P}$-generic over $\mathcal{D}_M$ or equivalently G intersects every maximal antichain in $\mathcal{E}_M$. If G is $\mathbb{P}$-generic over M, then $M[G]$ is the $\mathbb{P}$-generic extension of M by G. $M[G] \models \mathsf{ZFC}$ and is the minimal model of ZFC containing G. See [31] or [41] for more information on the method of forcing, the explicit construction of the generic extension, and details concerning the examples of forcing which will be used below.

An important application of boldface GCH at ω is the existence of generic filters over inner models of the axiom of choice for forcings which are countable in the true universe. The proof of the next result is a well known argument which will be presented in the following manner to motivate Lemma 5.6 later. The use of this fact will be ubiquitous in this paper.

Fact 2.14. *Assume boldface* GCH *at* ω. *Let* M *be an inner model of* ZFC *and* $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}}) \in M$ *be a forcing which is countable in the true universe. Then there is a* $G \subseteq \mathbb{P}$ *which is* $\mathbb{P}$*-generic over* M.

Proof. Let $\sigma : \mathbb{P} \to \omega$ be a bijection. Let $\tau : \mathscr{P}(\mathbb{P})^M \to \mathscr{P}(\omega)$ be defined by $\tau(A) = \sigma[A]$ and observe that τ is an injection. Since $M \models \mathsf{AC}$, there is an injection $\iota : \mathscr{P}(\mathbb{P})^M \to \gamma$ for some ordinal γ with $\iota \in M$. Then $\iota \circ \tau^{-1} : \tau[\mathscr{P}(\mathbb{P})^M] \to \gamma$ is an injection. Thus $\tau[\mathscr{P}(\mathbb{P})^M]$ is a wellorderable subset of $\mathscr{P}(\omega)$ and must be countable in the true universe by boldface GCH at ω. Thus $\mathscr{P}(\mathbb{P})^M$ is countable in the true universe. Let $\mathcal{D}$ be the collection of all dense subsets of $\mathbb{P}$ in M. Since $\mathscr{P}(\mathbb{P})^M$ is countable and $\mathcal{D} \subseteq \mathscr{P}(\mathbb{P})^M$, there is a bijection $\nu : \omega \to \mathcal{D}$. Note the following important observation:

(1) Every forcing is $< \omega$-closed and thus $M \models$ "$\mathbb{P}$ is $< \omega$-closed".
(2) For every $n \in \omega$, $\nu \restriction n \in M$ since it is finite.

Since $M \models \mathsf{AC}$, fix a wellordering $\sqsubset$ for $\mathbb{P}$ which belongs to M. By the two observations, one can work within M to define a sequence $\langle p^n_m : m < n \rangle$ so that for all $m < n$, p^n_m is the $\sqsubset$-least $p \in \mathbb{P}$ so that $p \leq_{\mathbb{P}} p^n_k$ for all $k < m$ and $p \in \nu(m)$. By the uniformity obtained by using $\sqsubset$, for all $m < n_0 \leq n_1$, $p^{n_0}_m = p^{n_1}_m$. For each $m \in \omega$, let $p_m = p^{m+1}_m$. Let $G = \langle \{p_n : n \in \omega\} \rangle_{\mathbb{P}} = \{p \in \mathbb{P} : (\exists n \in \omega)(p_n \leq_{\mathbb{P}} p)\}$. Since ν enumerates all dense subsets of $\mathbb{P}$ in M and $\{p_n : n \in \omega\} \subseteq G$, G is $\mathbb{P}$-generic over M. ☐

Let Θ be the supremum of the ordinals α for which there is a surjection $\pi : \mathbb{R} \to \alpha$. The Moschovakis coding lemma is a powerful tool of determinacy proved using the recursion theorem. It can be used to show that if AD holds, M is an inner model of AD with $\mathbb{R} \subseteq M$, and $\kappa < \Theta^M$, then $\mathscr{P}(\kappa)^M = \mathscr{P}(\kappa)$. However, one of its frequently used coarse consequences is that if $\kappa < \Theta$ (that is, $\mathbb{R}$ surjects onto κ), then $\mathbb{R}$ surjects onto $\mathscr{P}(\kappa)$. The proof of various forms of the coding lemma and background concerning pointclasses can be found in [45] Section 7D, [30] Theorem 2.12, [37] Lemma 1.5, and [40] Theorem 3.2 and Theorem 3.4.

Fact 2.15. *(Moschovakis coding lemma; [44]) Assume* AD. *Let* Γ *be a nonselfdual pointclass closed under* $\exists^{\mathbb{R}}$ *and* $\wedge$. *Let* $P \subseteq \mathbb{R}$, $\kappa \in \mathrm{ON}$, *and* $\varphi : P \to \kappa$ *is a surjection. Define* $\prec$ *on* $\mathbb{R}^2$ *by* $x \prec y$ *if and only if* $x, y \in P$ *and* $\varphi(x) \leq \varphi(y)$. *Assume* $\prec \in \Gamma$. *Let* $R \subseteq P \times \mathbb{R}$. *Then there is an* $S \subseteq R$ *with* $S \in \Gamma$ *so that for all* $\alpha < \kappa$, *if there exists* $x \in P$ *with* $\varphi(x) = \alpha$ *and* $x \in \mathrm{dom}(R)$, *then there exists* $y \in P$ *with* $\varphi(y) = \alpha$ *and* $y \in \mathrm{dom}(S)$.

As a consequence, if $\kappa < \Theta$, *there is a surjection of* $\mathbb{R}$ *onto* $\mathscr{P}(\kappa)$.

The following illustrates one use of the Moschovakis coding lemma.

Fact 2.16. *Assume* AD. *For any* $\kappa < \Theta$, $\mathsf{AC}^{\mathscr{P}(\kappa)}_{\omega}$ *holds: For any* $R \subseteq \omega \times \mathscr{P}(\kappa)$, *there is a function* $\Lambda : \mathrm{dom}(R) \to \mathscr{P}(\kappa)$ *such that for all* $n \in \mathrm{dom}(R)$, $R(n, \Lambda(n))$.

Proof. Since $\kappa < \Theta$, there is a surjection $\pi : \mathbb{R} \to \mathscr{P}(\kappa)$ by the Moschovakis coding lemma (Fact 2.15). Given $R \subseteq \omega \times \mathscr{P}(\kappa)$, let $S \subseteq \omega \times \mathbb{R}$ be defined by $S(n, x)$ if and only if $R(n, \pi(x))$. Since AD implies

$\mathsf{AC}_\omega^{\mathbb{R}}$, there is a $\Sigma : \mathrm{dom}(S) \to \mathbb{R}$ such that for all $n \in \mathrm{dom}(S)$, $S(n, \Sigma(n))$. Since $\mathrm{dom}(R) = \mathrm{dom}(S)$, for all $n \in \mathrm{dom}(R)$, $R(n, \pi(\Sigma(n)))$ holds. Thus $\Lambda : \mathrm{dom}(R) \to \mathscr{P}(\kappa)$ defined by $\Lambda = \pi \circ \Sigma$ is the desired uniformization function. □

Under AC, successor cardinals are regular. This is not true under AD since ω_3 is the least singular cardinal with $\mathrm{cof}(\omega_3) = \omega_2$. However under AD and $\kappa < \Theta$, $\mathsf{AC}_\omega^{\mathscr{P}(\kappa)}$ can recover a fragment of the argument to show $\mathrm{cof}(\kappa^+) > \omega$.

Fact 2.17. *Assume* AD. *If* $\omega \leq \kappa < \Theta$, *then* $\mathrm{cof}(\kappa^+) > \omega$.

The following consequence of boldface GCH will be needed later. If $\kappa \in \mathrm{ON}$ and $\alpha \in \mathrm{ON}$, then $\kappa^{+\alpha}$ will refer to the α^{th} cardinal greater than κ.

Fact 2.18. *Assume boldface* GCH *holds at a cardinal* κ. *Let* $M \models \mathsf{ZFC}$ *be an inner model. The cardinals of* M *below* κ^+ *is unbounded in* κ^+. *Thus for all* $\alpha < \mathrm{cof}(\kappa)$, $(\kappa^{+\alpha})^M < \kappa^+$.

Proof. Suppose not. Let $\lambda < \kappa^+$ be the largest ordinal $\delta < \kappa^+$ so that $M \models$ "δ is a cardinal". This implies $(\lambda^+)^M = \kappa^+$. Since $\kappa \leq \lambda < \kappa^+$, let $\Phi : \lambda \to \kappa$ be a bijection. Define $\Psi : (\mathscr{P}(\lambda))^M \to \mathscr{P}(\kappa)$ by $\Psi(A) = \Phi[A] = \{\Phi(\alpha) : \alpha \in A\}$. Ψ is an injection. By Cantor's theorem in M, $M \models$ "λ does not surject onto $\mathscr{P}(\lambda)$". Since $M \models \mathsf{AC}$, M has an injection $\Upsilon : (\lambda^+)^M \to (\mathscr{P}(\lambda))^M$. Since $\kappa^+ = (\lambda^+)^M$, $\Psi \circ \Upsilon : \kappa^+ \to \mathscr{P}(\kappa)$ is an injection which violates the boldface GCH at κ. □

Fact 2.19. *Assume* AD. *Let* $\omega \leq \kappa < \Theta$ *and boldface* GCH *holds at* κ. *Let* $M \models \mathsf{ZFC}$ *be an inner model. Then for all* $\alpha < \omega_1$, $(\kappa^{+\alpha})^M < \kappa^+$.

Proof. The result follows by Fact 2.17 and Fact 2.18. □

Becker and Kechris showed that AD implies $\mathrm{cof}(\kappa^+) > \omega_1$ for all $\omega_1 \leq \kappa < \Theta$ using a simple version of the Kechris-Woodin generic coding function on ω_1 ([38]). Thus Fact 2.19 can be strengthened to show that $(\kappa^{+\alpha})^M < \kappa^+$ for all $\alpha < \omega_2$ when $\omega_1 \leq \kappa < \Theta$. One can say much more when $M = \mathrm{HOD}^{L(\mathbb{R})}$. Jackson, Ketchersid, Schlutzenburg, and Woodin (Fact 7.16; [27]) showed that for all $\omega_1 \leq \kappa < \Theta^{L(\mathbb{R})}$, there are κ^+ many cardinals (even measurable cardinals) of $\mathrm{HOD}^{L(\mathbb{R})}$ below κ^+.

Definition 2.20. An ∞-Borel code is a pair $p = (S, \varphi)$ where S is a set of ordinals and φ is a first order $\mathscr{L}_{\mathrm{set}}$-formula with two free variables. Let $\mathscr{B}_\infty$ denote the set of all ∞-Borel codes. Let X be a set of sets of ordinals. Let $\mathfrak{B}_p^X = \{x \in X : L[S, x] \models \varphi(S, x)\}$. A set $A \subseteq X$ is ∞-Borel if and only if there is an ∞-Borel code p so that $A = \mathfrak{B}_p^X$.

∞-Borel codes provide the most absolute and uniform way to capture a set: Suppose $A \subseteq \mathbb{R}$ and (S, φ) is an ∞-Borel code for A. Then any inner model M of ZF with $S \in M$ can capture A by using comprehension and the observation that $x \in A$ if and only if $L[S, x] \models \varphi(S, x)$.

The following is Woodin's theory AD^+ which is a very powerful extension of determinacy.

Definition 2.21. (Woodin) AD^+ is the conjunction of the following three statements.

1. $\mathsf{DC}_{\mathbb{R}}$ (dependent choice for $\mathbb{R}$).
2. For all $\kappa < \Theta$, continuous function $\pi : {}^\omega\kappa \to {}^\omega\omega$, and $A \subseteq {}^\omega\omega$, the game $G^\kappa_{\pi^{-1}[A]}$ is determined.
3. For all $A \subseteq {}^\omega\omega$, A is ∞-Borel.

Under AD^+, boldface GCH holds below Θ.

Fact 2.22. *(Steel, Woodin;* [61] *Theorem 2.16) Assume* AD^+. *For all* $\kappa < \Theta$, *boldface* GCH *holds at* κ.

Since boldface GCH below κ implies the hypothesis of Theorem 2.2 at κ, the following theorem is obtained as a corollary.

Theorem 2.23. *Assume* AD^+. *For all cardinals* $\kappa < \Theta$, $\neg(|[\kappa]^{\mathrm{cof}(\kappa)}| \leq |\mathscr{P}_B(\kappa)|)$ *and therefore for all* $\gamma \leq \kappa$ *with* $\mathrm{cof}(\gamma) = \mathrm{cof}(\kappa)$, $\neg(|[\kappa]^\gamma| \leq |\mathscr{P}_B(\kappa)|)$ *and* $|\mathscr{P}_B(\kappa)| < |\mathscr{P}(\kappa)|$. *If* κ *is a regular cardinal, then* $|[\kappa]^{<\kappa}| < |\mathscr{P}(\kappa)|$.

Theorem 2.12 shows that $|[\omega_1]^{<\omega_1}| < |\mathscr{P}(\omega_1)| = |[\omega_1]^{\omega_1}|$. Other proofs of this fact using more substantial amount of determinacy or partition properties can provide more information about the cardinality such as insight into the cofinality of $\mathscr{P}(\omega_1)$. (Cofinality and regularity in the choiceless context is developed in [17].) [11] showed that every function $\Phi : [\omega_1]^{\omega_1} \to \omega_1$ is continuous $\mu^{\omega_1}_{\omega_1}$-almost everywhere in the sense that there is a club $C \subseteq \omega_1$ so that for all $f \in [C]^{\omega_1}_*$, there is an $\alpha < \omega_1$ so that for all $g \in [C]^{\omega_1}_*$, if $f \upharpoonright \alpha = g \upharpoonright \alpha$, then $\Phi(f) = \Phi(g)$. This implies that $\mathscr{P}(\omega_1)$ has ω_1-regular cardinality: If $\Phi : \mathscr{P}(\omega_1) \to \omega_1$, then there is an $\alpha < \omega_1$ so that $|\Phi^{-1}[\{\alpha\}]| = |\mathscr{P}(\omega_1)|$. $\mathscr{P}_B(\omega_1)$ does not have ω_1-regular cardinality since $\mathscr{P}_B(\omega_1) = \bigcup_{\alpha<\omega_1} \mathscr{P}(\alpha)$ and $|\mathscr{P}(\alpha)| = |\mathscr{P}(\omega)|$ whenever $\alpha < \omega_1$ (and note that $|\mathscr{P}(\omega)| \neq |\mathscr{P}_B(\omega_1)|$ by boldface GCH at ω). This gives another proof that $|[\omega_1]^{<\omega_1}| = |\mathscr{P}_B(\omega_1)| < |\mathscr{P}(\omega_1)|$.

The next natural cardinality computation to consider would be the relation between $|{}^\omega\omega_1| = |[\omega_1]^\omega|$ and $|\mathscr{P}_B(\omega_1)| = |[\omega_1]^{<\omega_1}|$. Woodin [65] investigated the cardinality below $|[\omega_1]^{<\omega_1}|$ under $\mathsf{AD}_\mathbb{R}$ and DC (which is known to imply AD^+). Woodin also isolated an important subset of $[\omega_1]^{<\omega_1}$ which he called $S_1 = \{f \in [\omega_1]^{<\omega_1} : \omega_1^{L[f]} = \sup(f)\}$. (See Fact 3.10 below for some basic properties of S_1.)

In [65], Woodin proved two dichotomy results below $[\omega_1]^{<\omega_1}$ under $\mathsf{AD}_\mathbb{R}$ and $\mathsf{DC}_\mathbb{R}$.

(1) If $X \subseteq [\omega_1]^\omega$, then either $|X| \leq |\mathbb{R} \times \omega_1|$ or $|X| = |[\omega_1]^\omega|$.
(2) If $X \subseteq [\omega_1]^{<\omega_1}$, then $|X| \leq |[\omega_1]^\omega|$ or $|S_1| \leq |X|$.

The author can prove the first dichotomy for $|[\omega_1]^\omega|$ under AD and $\mathsf{AC}^\mathbb{R}_\mathbb{R}$ (or equivalently $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$) using classical determinacy techniques for ω_1. The first dichotomy also provides a complete classification of the cardinality below $|[\omega_1]^\omega|$ under $\mathsf{AD}_\mathbb{R}$ and DC (or just $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$): If $X \subseteq [\omega_1]^\omega$ is uncountable, then $|X|$ is $|\omega_1|$, $|\mathbb{R}|$, $|\mathbb{R} \sqcup \omega_1|$, $|\omega_1 \times \mathbb{R}|$, or $|[\omega_1]^\omega|$. It was unknown whether the first dichotomy could fail without $\mathsf{AC}^\mathbb{R}_\mathbb{R}$. As an application of the material developed in Section 4, Theorem 4.23 will produced an intermediate cardinality which is example of this failure.

The second dichotomy of Woodin is only meaningful if $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$ and $|\mathbb{R}| < |S_1|$. Although the proofs of these cardinality distinctions do not seem to be included in [65], Woodin had known these cardinality computations at least under $\mathsf{AD}_\mathbb{R}$ and DC. [13] provides a proof that S_1 does not inject into ${}^\omega\mathrm{ON}$, the class of ω-sequences of ordinals, using just AD, $\mathsf{DC}_\mathbb{R}$, and all sets of reals have ∞-Borel code. To motivate the use of function capturing in generic extensions in the main result of the paper, weak versions of the main result will be shown and a survey of the relevant results from [13] will be provided.

Under ZF alone, $\omega_1 < \Theta$ can be proved with a very explicit surjection of $\mathbb{R}$ onto ω_1. Let $\pi : \omega \times \omega \to \omega$ be a recursive bijection. If $x \in {}^\omega\omega$, let $\mathcal{R}_x \subseteq \omega \times \omega$ be defined by $R_x(a,b) \Leftrightarrow x(\pi(a,b)) = 1$. Let WO be the set of $w \in \mathbb{R}$ such that $\mathrm{dom}(\mathcal{R}_w) = \omega$ and $(\omega, \mathcal{R}_w)$ is a wellordering. If $w \in \mathrm{WO}$, then let $\mathrm{ot}(w)$ be the ordertype of $(\omega, \mathcal{R}_w)$. $\mathrm{ot} : \mathrm{WO} \to \omega_1$ is a surjection. If $n \in \omega$ and $w \in \mathrm{WO}$, then let $\mathrm{ot}(w,n)$ be the rank of n in the wellordering $(\omega, \mathcal{R}_w)$. If $\alpha < \mathrm{ot}(w)$, then let $\mathrm{num}(w,\alpha)$ be the unique $n \in \omega$ so that $\mathrm{ot}(w,n) = \alpha$. If $x \in {}^\omega\omega$ and $m \in \omega$, let $x^{[m]} \in {}^\omega\omega$ be defined by $x^{[m]}(n) = x(\pi(m,n))$. Let $u \in \mathsf{BS}$ if and only if $u^{[0]} \in \mathrm{WO}$ and for all $n \in \omega$, $(u^{[1]})^{[n]} \in \mathrm{WO}$. If $u \in \mathsf{BS}$, then let $\mathsf{seq}(u) : \mathrm{ot}(u^{[0]}) \to \omega_1$ be defined by $\mathsf{seq}(u)(\alpha) = \mathrm{ot}((u^{[1]})^{[\mathrm{num}(w,\alpha)]})$ for each $\alpha < \mathrm{ot}(u^{[0]})$. $\mathsf{seq} : \mathsf{BS} \to {}^{<\omega_1}\omega_1$ is a surjection. Let $\mathsf{iseq} : \mathsf{BS} \to \mathscr{P}_B(\omega_1)$ be defined by $\mathsf{iseq}(u) = \mathsf{seq}(u)[\mathrm{ot}(u^{[0]})]$ and note that iseq is a surjection. From the explicit nature of the coding, if $\ell \in {}^{<\omega_1}\omega_1$ and $A \in \mathscr{P}_B(\omega_1)$, then in any inner model M with $\ell \in M$ and $M \models$ "$\max\{\mathrm{dom}(\ell), \sup(\ell)\}$ is countable", there is a $u \in \mathsf{BS} \cap M$ so that $\mathsf{seq}(u) = \ell$ and in any inner model M with $A \in M$ and $M \models$ "$\sup(A)$ is countable", then there is a $u \in \mathsf{BS} \cap M$ so that $\mathsf{iseq}(u) = A$.

Fact 2.24. *If boldface* GCH *at* ω *holds and all subsets of* $\mathbb{R}$ *have* ∞*-Borel codes, then all subsets of* $\mathscr{P}_B(\omega_1)$ *and all subsets of* ${}^{<\omega_1}\omega_1$ *have* ∞*-Borel codes.*

Proof. Let $X \subseteq \mathscr{P}_B(\omega_1)$. Let $\tilde{X} = \{u \in \mathsf{BS} : \mathsf{iseq}(u) \in X\}$. Since $\tilde{X} \subseteq \mathbb{R}$, the hypothesis implies there is an ∞-Borel code (S,φ) for $\tilde{X}$. If ξ is an ordinal, then let $\mathrm{Coll}(\omega,\xi)$ be the forcing of finite partial functions from ω into ξ ordered by reverse function inclusion. The claim is that for all $A \in \mathscr{P}_B(\omega_1)$, $A \in X$ if and only if

$$L[S,A] \models 1_{\mathrm{Coll}(\omega,\sup(A))} \Vdash_{\mathrm{Coll}(\omega,\sup(A))} (\exists u \in \mathsf{BS})(\mathsf{iseq}(u) = \check{A} \wedge L[\check{S},u] \models \varphi(\check{S},u)).$$

To see this: ($\Rightarrow$) Suppose $A \in X$. Since $L[S,A] \models \mathsf{AC}$ and $\sup(A)$ is countable in the true world, $\mathrm{Coll}(\omega,\sup(A))$ is countable in the true universe. By Fact 2.14, there is a $G \subseteq \mathrm{Coll}(\omega,\sup(A))$ which is $\mathrm{Coll}(\omega,\sup(A))$-generic over $L[S,A]$. Since $L[S,A][G] \models \sup(A)$ is countable, there is a $u \in \mathsf{BS} \cap L[S,A][G]$ such that $L[S,A][G] \models \mathsf{iseq}(u) = A$. Since G belongs to the true universe, by absoluteness, $\mathsf{iseq}(u) = A$.

Since $A \in X$, one has $u \in \tilde{X}$. Since (S, φ) is the ∞-Borel code for $\tilde{X}$, $L[S, u] \models \varphi(S, u)$. By absoluteness, $L[S, A][G] \models$ "$\mathsf{iseq}(u) = A \wedge L[S, u] \models \varphi(S, u)$". Thus

$$L[S, A][G] \models (\exists u \in \mathsf{BS})(\mathsf{iseq}(u) = A \wedge L[S, u] \models \varphi(S, u)).$$

By the forcing theorem, there is a $p \in \mathrm{Coll}(\omega, \sup(A))$ so that

$$L[S, A][G] \models p \Vdash_{\mathrm{Coll}(\omega,\sup(A))} (\exists u \in \mathsf{BS})(\mathsf{iseq}(u) = \check{A} \wedge L[S, u] \models \varphi(\check{S}, u)).$$

By the homogeneity of $\mathrm{Coll}(\omega, \sup(A))$,

$$L[S, A][G] \models 1_{\mathrm{Coll}(\omega,\sup(A))} \Vdash_{\mathrm{Coll}(\omega,\sup(A))} (\exists u \in \mathsf{BS})(\mathsf{iseq}(u) = \check{A} \wedge L[S, u] \models \varphi(\check{S}, u)).$$

($\Leftarrow$) Again by Fact 2.14, let $G \subseteq \mathrm{Coll}(\omega, \sup(A))$ be $\mathrm{Coll}(\omega, \sup(A))$-generic over $L[S, A]$. By the forcing theorem,

$$L[S, A][G] \models (\exists u \in \mathsf{BS})(\mathsf{iseq}(u) = A \wedge L[S, u] \models \varphi(S, u)).$$

Let $u \in \mathsf{BS} \cap L[S, A][G]$ be such that $L[S, A][G] \models \mathsf{iseq}(u) = A \wedge L[S, A] \models \varphi(S, u)$. Since G belongs to the true universe and hence u belongs to the true universe, one has by absoluteness, $\mathsf{iseq}(u) = A$ and $L[S, A] \models \varphi(S, u)$. Since (S, φ) is the ∞-Borel code for $\tilde{X}$, $u \in \tilde{X}$. By definition of $\tilde{X}$, $\mathsf{iseq}(u) = A \in X$. The completes the proof of the claim.

Let $\phi(S, A)$ be the statement

$$1_{\mathrm{Coll}(\omega,\sup(A))} \Vdash_{\mathrm{Coll}(\omega,\sup(A))} (\exists u \in \mathsf{BS})(\mathsf{iseq}(u) = \check{A} \wedge L[S, u] \models \varphi(\check{S}, u)).$$

The claim implies that (S, ϕ) is an ∞-Borel code for X. ☐

The proof of the next result will introduce the forcing and the basic skeleton of the capturing arguments used throughout the paper.

Fact 2.25. *Assume boldface* GCH *at* ω *and all subsets of* $\mathbb{R}$ *have* ∞*-Borel codes. Then* $|{}^{\omega}\omega_1| = |[\omega_1]^{\omega}| < |\mathscr{P}_B(\omega)| = |{}^{<\omega_1}\omega_1| = |[\omega_1]^{<\omega_1}|$.

Proof. Suppose $\Phi : \mathscr{P}_B(\omega_1) \to {}^{\omega}\omega_1$ is an injection. Let $\mathrm{Gr}_\Phi = \{(A, n, \alpha) \in \mathscr{P}_B(\omega_1) \times \omega \times \omega_1 : \Phi(A)(n) = \alpha\}$ and $\mathrm{Gr}_{\Phi^{-1}} = \{(f, \alpha) \in {}^{\omega}\omega_1 \times \omega_1 : f \in \Phi[\mathscr{P}_B(\omega_1)] \wedge \alpha \in \Phi^{-1}(f)\}$. By Fact 2.24, let (S_0, φ_0) and (S_1, φ_1) be ∞-Borel codes for Gr_Φ and $\mathrm{Gr}_{\Phi^{-1}}$, respectively. Let $\mathrm{Coll}(\omega^+, \omega^{++})^{L[S_0,S_1]} = \mathrm{Coll}(\omega_1, \omega_2)^{L[S_0,S_1]}$ be the forcing defined in $L[S_0, S_1]$ consisting of countable partial functions from $\omega_1^{L[S_0,S_1]}$ into $\omega_2^{L[S_0,S_1]}$ ordered by reverse function inclusion. Since this forcing is countable in the real world by boldface GCH at ω and Fact 2.18, Fact 2.14 implies there is a $G \subseteq \mathrm{Coll}(\omega_1, \omega_2)^{L[S_0,S_1]}$ which is generic over $L[S_0, S_1]$. G adds a generic surjection $g : \omega_1^{L[S_0,S_1]} \to \omega_2^{L[S_0,S_1]}$. Let $A = \{(\alpha, \beta) : g(\alpha) = \beta\}$ be the graph of g which may be regarded as an element of $\mathscr{P}_B(\omega_1)$ (after coding pairs using the Gödel coding function) using the fact that $\omega_2^{L[S_0,S_1]} < \omega_1$ by Fact 2.18. Note that for all $n \in \omega$ and $\alpha \in \omega_1$, $\Phi(A)(n) = \alpha$ if and only if $\mathrm{Gr}_\Phi(A, n, \alpha)$ if and only if $L[S_0, A] \models \varphi_0(S_0, A, n, \alpha)$ if and only if $L[S_0, S_1][G] \models L[S_0, A] \models \varphi_0(S_0, A, n, \alpha)$ by the fact that (S_0, φ_0) is the ∞-Borel code for Gr_Φ and absoluteness. Thus by applying comprehension in $L[S_0, S_1][G]$, this shows that $\Phi(A) \in {}^{\omega}\omega_1 \cap L[S_0, S_1][G]$. Since $L[S_0, S_1] \models$ "$\mathrm{Coll}(\omega_1, \omega_2)^{L[S_0,S_1]}$ is $< \omega_1$-closed", no new ω-sequence through $L[S_0, S_1]$ is added by this forcing. Hence $\Phi(A) \in L[S_0, S_1]$. Note that for all $\alpha \in \omega_1$, $\alpha \in A$ if and only if $\alpha \in \Phi^{-1}(\Phi(A))$ if and only if $\mathrm{Gr}_{\Phi^{-1}}(\Phi(A), \alpha)$ if and only if $L[S_1, \Phi(A)] \models \varphi_1(S_1, \Phi(A), \alpha)$ if and only if $L[S_0, S_1] \models L[S_1, \Phi(A)] \models \varphi_1(S_1, \Phi(A), \alpha)$ since (S_1, φ_1) is the ∞-Borel code for $\mathrm{Gr}_{\Phi^{-1}}$ and by absoluteness. By comprehension, $A \in L[S_0, S_1]$. Since A is the graph of g, $g \in L[S_0, S_1]$. Since $L[S_0, S_1][G] \models$ "$g : \omega_1^{L[S_0,S_1]} \to \omega_2^{L[S_0,S_1]}$ is a surjection", $L[S_0, S_1] \models$ "$g : \omega_1^{L[S_0,S_1]} \to \omega_2^{L[S_0,S_1]}$ is a surjection" by absoluteness. This is a contradiction. ☐

Inspecting the proof, one found an inner model M which was used to solve two crucial problems:

(1) (Generic existence problem) There is a generic $G \subseteq \mathrm{Coll}(\omega_1, \omega_2)^M$ which is $\mathrm{Coll}(\omega_1, \omega_2)^M$-generic over M. (This is provided by Fact 2.14 and boldface GCH at ω.)
(2) (Capturing problem)
   - $M[G]$ is closed under Φ: for all $A \in \mathscr{P}_B(\omega_1) \cap M[G]$, $\Phi(A) \in M[G]$.
   - M is closed under the partial function Φ^{-1}: For all $f \in \mathrm{dom}(\Phi^{-1}) \cap M$, $\Phi^{-1}(f) \in M$.

Since Gr_Φ and $\mathrm{Gr}_{\Phi^{-1}}$ have ∞-Borel codes (S_0, φ_0) and (S_1, φ_1), any inner model of ZF containing S_0 and S_1 can capture Φ and Φ^{-1} in the above sense.

Toward the goal of classifying cardinal exponentiation, one needs to prove a strong instance of Fact 2.25; namely, $\mathscr{P}_B(\omega_1)$ does not inject into ${}^\omega\mathrm{ON}$. Suppose $\Phi : \mathscr{P}_B(\omega_1) \to {}^\omega\kappa$ where κ is any cardinal below Θ. The challenge is to find an inner model $M \models \mathsf{ZFC}$ which is closed under the partial function $\Phi^{-1} : {}^\omega\kappa \to \mathscr{P}_B(\kappa)$. $\mathrm{Gr}_{\Phi^{-1}}$ is a subset of ${}^\omega\kappa \times \omega_1$. If $\kappa > \omega_1$, then Fact 2.24 cannot be applied to show that $\mathrm{Gr}_{\Phi^{-1}}$ has an ∞-Borel code. Instead, [13] used the ∞-Borel code forcing (see Section 4) and Fact 4.17 to obtain the necessary model closed under Φ^{-1}. However at the time of [13], only Fact 4.17 for the Woodin's classical ∞-Borel code forcing $\mathbb{B}_n^{\omega;Z}$ which adds reals (for some set Z and $n \in \omega$) was known at least to the authors. The forcing $\mathrm{Coll}(\omega^+, \omega^{++})^M$ (for some suitable inner model M of ZFC) used in the proof of Fact 2.25 does not add any new reals and hence the classical Fact 4.17 for adding reals could not be used. It is for this reason that the more natural forcing $\mathrm{Coll}(\omega^+, \omega^{++})^M$ was not used in [13] but rather the Lévy collapse $\mathrm{Coll}(\omega, <\xi)^M$ where ξ is an inaccessible cardinal of M which is countable in the real world was used since it adds many new reals. One additional benefit of using the Lévy collapse is that it provides a cardinality result even for S_1.

Fact 2.26. *([13] Theorem 5.7) Assume* AD, $\mathsf{DC}_\mathbb{R}$, *and all sets of reals have ∞-Borel codes. S_1 does not inject into ${}^\omega\mathrm{ON}$ and thus $\mathscr{P}_B(\omega_1)$ does not inject into ${}^\omega\mathrm{ON}$.*

Several questions arises from these early results.

- (a) Can AD alone prove $[\omega_1]^{<\omega_1}$ does not inject into ${}^\omega\mathrm{ON}$?
- (b) Can AD alone prove S_1 does not inject into ${}^\omega\mathrm{ON}$?
- (c) (Bounded Power Set Conjecture) From any determinacy hypothesis, can it be shown that for all cardinals κ with $\omega < \kappa < \Theta$ and ordinals $\epsilon < \kappa$, $\mathscr{P}_B(\kappa)$ does not inject into ${}^\epsilon\mathrm{ON}$?

Question (a) has been solved using descriptive set theoretic properties and partition properties. These methods also provide partial solutions to Question (c). A few remarks about these combinatorial methods will be mentioned below. Question (b) will be solved in the next section and the techniques will motivate the methods used to solve Question (c). The solution to Question (c) is the main result of this paper.

Using descriptive set theoretic properties of the pointclass $\mathbf{\Sigma}_1^1$ and Kunen trees ([30]) from the study of measures below ω_ω under AD, the following almost everywhere continuity result with respect to the partition measures $\mu_{\omega_1}^\epsilon$ ($\epsilon < \omega_1$) was established.

Fact 2.27. *([19] Theorem 2.14, [18] Fact 2.5) Assume* AD. *For every $\epsilon < \omega_1$ and function $\Phi : [\omega_1]^\epsilon_* \to \omega_1$, there is a club $C \subseteq \omega_1$ and a $\delta < \epsilon$ so that for all $f, g \in [C]^\epsilon_*$, if $f \restriction \delta = g \restriction \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.*

The continuity property of Fact 2.27 is a very well controlled failure of injectivity. This is used to obtain a proof of Fact 2.25 from AD alone.

Fact 2.28. *([19] Theorem 2.16) Assume* AD. $|{}^\omega\omega_1| < |[\omega_1]^{<\omega_1}| = |\mathscr{P}_B(\omega_1)|$.

Building on Fact 2.27 and Martin's ultrapower representation of $\omega_{n+1} = \prod_{[\omega_1]^n_*} \omega_1/\mu^n_{\omega_1}$ for all $1 \leq n < \omega$, one can show $\mathscr{P}_B(\omega_1)$ does not inject into ${}^\omega(\omega_\omega)$. Furthermore using Fact 2.27 and extensive descriptive set theoretic properties of ω_1 and the pointclasses of $\mathbf{\Sigma}_1^1$ and $\mathbf{\Sigma}_2^1$, one can show that $\mathscr{P}_B(\omega_1)$ does not inject into ${}^\omega\mathrm{ON}$.

Fact 2.29. *([18] Theorem 2.9 and Theorem 4.4)*

- *Assume* AD. $\neg(|\mathscr{P}_B(\omega_1)| \leq |{}^\omega(\omega_\omega)|)$.
- *Assume* AD *and* $\mathsf{DC}_\mathbb{R}$. *$\mathscr{P}_B(\omega_1)$ does not inject into ${}^\omega\mathrm{ON}$.*

The proof of Fact 2.29 uses many descriptive set theoretic property surrounding ω_1 which are not known to generalize to any higher cardinals. Moreover, $\mathsf{DC}_\mathbb{R}$ is used directly in the proof and indirectly in several descriptive set theoretic property (such as the wellfoundedness of the Wadge degree in pointclass arguments). To remove $\mathsf{DC}_\mathbb{R}$ and answer instances of Question (c) for higher cardinals, a strengthening of Fact 2.27 was established using pure combinatorics.

Fact 2.30. *([21] Theorem 3.7) Assume $\epsilon < \kappa$, $\mathrm{cof}(\epsilon) = \omega$, and $\kappa \to_* (\kappa)^{\epsilon\cdot\epsilon}_2$ holds. For any function $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$, there is a $\delta < \epsilon$ and a club $C \subseteq \kappa$ such that for all $f, g \in [C]^\epsilon_*$, if $f \restriction \delta = g \restriction \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.*

This is used to answer Question (a) and Question (c) at weak partition cardinals. The following result gave the most compelling evidence to formulate the Bounded Power Set Conjecture.

Fact 2.31. *([21] Theorem 4.4) Let κ be a weak partition cardinal (that is, $\kappa \to_* (\kappa)^{<\kappa}_2$) and $\epsilon < \kappa$. Then $\mathscr{P}_B(\kappa)$ does not inject into ${}^\epsilon\mathrm{ON}$.*

Thus under AD, $\mathscr{P}_B(\omega_1)$ does not inject into ${}^\omega\mathrm{ON}$ and $\mathscr{P}_B(\omega_2)$ does not inject into ${}^{\omega_1}\mathrm{ON}$. Also $\mathscr{P}_B(\omega_3)$ does not inject into ${}^{\omega_1}\omega_3$. Since ω_3 is not a partition cardinal, the first relevant open cardinality computation is whether $\mathscr{P}_B(\omega_3)$ injects into ${}^{\omega_2}\omega_3$ which will be solved here. This is the history of these questions until this paper.

The rest of the paper will prove the main result. An important step is finding an appropriate form of capturing in certain inner models of ZFC and it generic extension without the given function possessing ∞-Borel codes. In fact, Woodin showed under AD^+ that there is always a subset of $\mathscr{P}(\omega_1)$ which does have an ∞-Borel code. (See [15].) So Question (c) (the Bounded Power Set Conjecture) cannot be solved using ∞-Borel codes for functions of the form $\Phi : \mathscr{P}_B(\kappa) \to {}^\epsilon\mathrm{ON}$. Fact 2.26 says that S_1 does not inject into ${}^\omega\mathrm{ON}$ if all subsets of $\mathbb{R}$ have ∞-Borel codes. Question (b) asks how this can be done without ∞-Borel codes. The next section will solve Question (b) and the technique will motivate how to resolve the Question (c), the eponymous Bounded Power Set Conjecture.

3. Capturing without ∞-Borel Code within the Hereditarily Countable Sets

This section will provide an answer to question (b) above and show that under AD (and in fact, less than AD), S_1 does not inject into ${}^\omega\mathrm{ON}$. Fact 2.26 used the existence of ∞-Borel code. A solution must do without ∞-Borel codes. Capturing will be obtained by using an appropriate HOD-type model. This will motivate some of the techniques of the main theorem of the paper where one will also need to work without the assumption that subsets of the power set of uncountable ordinals have ∞-Borel codes.

Let $\mathbb{P}$ be a countable forcing. Let $\mathrm{Fin}(\mathbb{P}, 2)$ be the set of all finite partial functions from $\mathbb{P}$ into 2. For each $s \in \mathrm{Fin}(\mathbb{P}, 2)$, let $N^{\mathbb{P}}_s = \{f \in {}^{\mathbb{P}}2 : s \subseteq f\}$. Give ${}^{\mathbb{P}}2$ the topology generated by $\{N^{\mathbb{P}}_s : s \in \mathrm{Fin}(\mathbb{P}, 2)\}$ as a basis. ${}^{\mathbb{P}}2$ is homeomorphic to ${}^\omega 2$ with its usual topology and thus ${}^{\mathbb{P}}2$ is a Polish space. Let $\mathcal{F}_{\mathbb{P}} \subseteq {}^{\mathbb{P}}2$ be the set of all $\mathbb{P}$-filters (or more accurately the characteristic function of $\mathbb{P}$-filters).

Note that $\mathcal{F}_{\mathbb{P}}$ is a $\mathbf{\Pi}^0_2$ subset of ${}^{\mathbb{P}}2$ (using the fact that basic neighborhoods are clopen). Thus $\mathcal{F}_{\mathbb{P}}$ is a Polish space with the subspace topology inherited from ${}^{\mathbb{P}}2$. One can then define the usual notions of category for this Polish topology. If $\mathsf{AC}^{\mathbb{R}}_\omega$ holds and all subsets $\mathcal{F}_{\mathbb{P}}$ has the property of Baire (defined using it natural Polish topology), then the argument of Fact 2.5 shows that wellordered unions of meager subsets of $\mathcal{F}_{\mathbb{P}}$ are meager subsets of $\mathcal{F}_{\mathbb{P}}$. (Under AD, all subsets of all Polish spaces have the Baire property relative to its Polish topology.)

For each $s \in \mathrm{Fin}(\mathbb{P}, 2)$, let $M^{\mathbb{P}}_s = N^{\mathbb{P}}_s \cap \mathcal{F}_{\mathbb{P}}$. It may be the case that some $s \in \mathrm{Fin}(\mathbb{P}, 2)$, $M^{\mathbb{P}}_s = \emptyset$. Let $\mathrm{sFin}(\mathbb{P}, 2)$ be the collection of $s \in \mathrm{Fin}(\mathbb{P}, 2)$ such that $M^{\mathbb{P}}_s \neq \emptyset$. Say that $s \in \mathrm{Fin}(\mathbb{P}, 2)$ is $\mathbb{P}$-suitable if and only if there is a $\tilde{p} \in \mathbb{P}$ so that for all $p \in \mathrm{dom}(s)$ with $s(p) = 1$, $\tilde{p} \leq_{\mathbb{P}} p$ and for all $p \in \mathrm{dom}(s)$ with $s(p) = 0$, p and $\tilde{p}$ are not compatible. Suppose s is $\mathbb{P}$-suitable. Let $\tilde{p} \in \mathbb{P}$ be such that for all $p \in \mathrm{dom}(s)$ with $s(p) = 1$, $\tilde{p} \leq_{\mathbb{P}} p$ and for all $p \in \mathrm{dom}(s)$ with $s(p) = 0$, p and $\tilde{p}$ are not compatible. Let $f : \mathbb{P} \to 2$ be defined by $f(p) = 1$ if and only if $\tilde{p} \leq_{\mathbb{P}} p$. Then $f \in \mathcal{F}_{\mathbb{P}}$ and $s \subseteq f$. This shows that $M^{\mathbb{P}}_s \neq \emptyset$. Now suppose $s \in \mathrm{Fin}(\mathbb{P}, 2)$ be such that $M^{\mathbb{P}}_s \neq \emptyset$. Let $f \in M^{\mathbb{P}}_s$. Since f codes a filter, there is a $\tilde{p}$ in the filter coded by f so that $\tilde{p} \leq_{\mathbb{P}} p$ for all $p \in \mathrm{dom}(s)$ with $s(p) = 1$ and for all $p \in \mathrm{dom}(s)$ with $s(p) = 0$, p and $\tilde{p}$ are not compatible. This shows that s is $\mathbb{P}$-suitable. Thus $\{M^{\mathbb{P}}_s : s \in \mathrm{Fin}_{\mathbb{P}}\} = \{M^{\mathbb{P}}_s : s \text{ is } \mathbb{P}\text{-suitable}\}$ is a basis for the topology on $\mathcal{F}_{\mathbb{P}}$ and $\mathrm{sFin}(\mathbb{P}, 2)$ is the collection of $\mathbb{P}$-suitable $s \in \mathrm{Fin}(\mathbb{P}, 2)$.

If $D \subseteq \mathbb{P}$ is dense, then let $\mathcal{F}^D_{\mathbb{P}} = \{f \in \mathcal{F}_{\mathbb{P}} : (\exists p \in D)(f(p) = 1)\}$. Suppose $f \in \mathcal{F}^D_{\mathbb{P}}$. Then there is some $p \in D$ so that $f(p) = 1$. Let $s : \{p\} \to 2$ be defined by $s(p) = 1$. Note $f \in M^{\mathbb{P}}_s \subseteq F^{\mathcal{D}}_{\mathbb{P}}$. This shows $\mathcal{F}^D_{\mathbb{P}}$ is open. Suppose $s \in \mathrm{sFin}_{\mathbb{P}}$ or equivalently $\mathbb{P}$-suitable. There is a $\tilde{p} \in \mathbb{P}$ so that for all $p \in \mathrm{dom}(s)$ with $s(p) = 1$, $\tilde{p} \leq_{\mathbb{P}} p$ and for all $p \in \mathrm{dom}(s)$ with $s(p) = 0$, $\tilde{p}$ and p are not compatible. Since D is dense, there is a $\bar{p} \in D$ with $\bar{p} \leq_{\mathbb{P}} \tilde{p}$. Let $s' = s \cup \{(\bar{p}, 1)\}$ and note that s' is still $\mathbb{P}$-suitable and $M^{\mathbb{P}}_{s'} \subseteq M^{\mathbb{P}}_s \cap \mathcal{F}^D_{\mathbb{P}}$. This shows that $\mathcal{F}^D_{\mathbb{P}}$ is dense open.

Suppose $\mathcal{D}$ is a countable collection of dense subsets of $\mathbb{P}$. Let $\mathsf{Gen}^{\mathcal{D}}_{\mathbb{P}} = \bigcap_{D \in \mathcal{D}} \mathcal{F}^D_{\mathbb{P}}$ be the collection of $f \in {}^{\mathbb{P}}2$ which codes $\mathbb{P}$-generic filter over $\mathcal{D}$. Note that $\mathsf{Gen}^{\mathcal{D}}_{\mathbb{P}}$ is comeager since it is a countable intersection of dense open subsets of $\mathcal{F}_{\mathbb{P}}$.

Recall that if Z is a set, then OD_Z and HOD_Z refer to ordinal definability with parameters from Z. If $Z \in \mathrm{OD}_Z$, then $\mathrm{HOD}_Z \models \mathsf{ZF}$. If $Z \in \mathrm{OD}_Z$ and Z has an OD_Z wellordering, then $\mathrm{HOD}_Z \models \mathsf{ZFC}$.

Lemma 3.1. *(Woodin; [40] Theorem 5.42, Claim 2) Assume $\mathsf{AC}^{\mathbb{R}}_\omega$, all subsets of all Polish spaces have the Baire property, and boldface GCH at ω. Let Z be a set such that there is an OD_Z wellordering of Z. Let $\mathbb{P} \in \mathrm{HOD}_Z$ be a forcing which is countable in the real world. Let $\varsigma^{\mathbb{P}}_Z$ be the least ordinal γ greater than ω_1^V such that $Z \cup \{\mathbb{P}\} \in V_\gamma$[5] Let $\mathcal{D}^{\mathbb{P}}_Z$ denote the collection of $\mathbb{P}$-dense subsets of $\mathbb{P}$ in HOD_Z. Let $\mathcal{S}^{\mathbb{P}}_Z$ be the collection of all tuples $(\varphi, \gamma, \vec{z}, \vec{\xi})$ such that $\gamma \in \mathrm{ON}$ with $\varsigma^{\mathbb{P}}_Z \leq \gamma$, $\vec{z}$ is a finite tuple of elements from Z, $\vec{\xi}$ is a finite tuple of ordinals less than γ, and φ is a $\mathscr{L}_{\mathrm{set}}$-formula with $|\vec{z}| + |\vec{\xi}| + 1$ free variables. Uniformly from $\mathbb{P}$ and Z[6], there is a $T^{\mathbb{P}}_Z \subseteq \mathsf{Gen}^{\mathcal{D}^{\mathbb{P}}_Z}_{\mathbb{P}}$ which is comeager in the topology of $\mathcal{F}_{\mathbb{P}}$ and sequence $\langle P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}, N^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}} : (\varphi, \gamma, \vec{z}, \vec{\xi}) \in \mathcal{S}^{\mathbb{P}}_Z\rangle$ with the following properties:*

1. *For any $(\varphi, \gamma, \vec{z}, \vec{\xi}) \in \mathcal{S}^{\mathbb{P}}_Z$, $P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}} \subseteq \mathrm{sFin}(\mathbb{P}, 2)$ and $N^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}} \subseteq \mathrm{sFin}(\mathbb{P}, 2)$.*
2. *For any $G \in T^{\mathbb{P}}_Z$ and $(\varphi, \gamma, \vec{z}, \vec{\xi}) \in \mathcal{S}^{\mathbb{P}}_Z$, $V_\gamma \models \varphi(G, \vec{z}, \vec{\xi})$ if and only if there is an $s \in P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}$ so that $G \in M^{\mathbb{P}}_s$.*
3. *For any $G \in T^{\mathbb{P}}_Z$ and $(\varphi, \gamma, \vec{z}, \vec{\xi}) \in \mathcal{S}^{\mathbb{P}}_Z$, $V_\gamma \models \neg\varphi(G, \vec{z}, \vec{\xi})$ if and only if there is an $s \in N^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}$ so that $G \in M^{\mathbb{P}}_s$.*
4. *For any finite $\vec{z}$ in Z, let $\mathcal{S}^{\mathbb{P}}_{Z,\vec{z}}$ be the collection of $(\varphi, \gamma, \vec{\xi})$ such that $(\varphi, \gamma, \vec{z}, \vec{\xi}) \in \mathcal{S}^{\mathbb{P}}_Z$. Then $\mathcal{S}^{\mathbb{P}}_{Z,\vec{z}} \in \mathrm{HOD}_Z$ and $\langle P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}, N^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}} : (\varphi, \gamma, \vec{\xi}) \in \mathcal{S}^{\mathbb{P}}_{Z,\vec{z}}\rangle \in \mathrm{HOD}_Z$.*

Proof. Let $(\varphi, \gamma, \vec{z}, \vec{\xi}) \in \mathcal{S}^{\mathbb{P}}_Z$. Let $P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}$ be the collection of $s \in \mathrm{sFin}(\mathbb{P}, 2)$ so that $A^s_{\varphi,\gamma,\vec{z},\vec{\xi}} = \{F \in M^{\mathbb{P}}_s \cap \mathsf{Gen}^{\mathcal{D}^{\mathbb{P}}_Z}_{\mathbb{P}} : V_\gamma \models \varphi(F, \vec{z}, \vec{\xi})\}$ is comeager in $M^{\mathbb{P}}_s$. Let $A_{\varphi,\gamma,\vec{z},\vec{\xi}} = \bigcup_{s \in P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}} A^s_{\varphi,\gamma,\vec{z},\vec{\xi}}$. Let $N^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}$ be the collection of $s \in \mathrm{sFin}(\mathbb{P}, 2)$ so that $B^s_{\varphi,\gamma,\vec{z},\vec{\xi}} = \{F \in M^{\mathbb{P}}_s \cap \mathrm{Gen}^{\mathcal{D}^{\mathbb{P}}_Z}_{\mathbb{P}} : V_\gamma \models \neg\varphi(F, \vec{z}, \vec{\xi})\}$ is comeager in $M^{\mathbb{P}}_s$. Let $B_{\varphi,\gamma,\vec{z},\vec{\xi}} = \bigcup_{s \in N^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}} B^s_{\varphi,\gamma,\vec{z},\vec{\xi}}$. Let $C_{\varphi,\gamma,\vec{z},\vec{\xi}} = A_{\varphi,\gamma,\vec{z},\vec{\xi}} \cup B_{\varphi,\gamma,\vec{z},\vec{\xi}}$. Note that $C_{\varphi,\gamma,\vec{z},\vec{\xi}} \subseteq \mathsf{Gen}^{\mathcal{D}^{\mathbb{P}}_Z}_{\mathbb{P}}$ is a comeager subset of $\mathcal{F}_{\mathbb{P}}$. Let $T^{\mathbb{P}}_Z = \bigcap_{(\varphi,\gamma,\vec{z},\vec{\xi}) \in \mathcal{S}^{\mathbb{P}}_Z} C_{\varphi,\gamma,\vec{z},\vec{\xi}}$ which is a comeager subset of $\mathcal{F}_{\mathbb{P}}$ since wellordered intersection of comeager subsets of $\mathcal{F}_{\mathbb{P}}$ are comeager because all subsets of the Polish space $\mathcal{F}_{\mathbb{P}}$ has the property of Baire and $\mathcal{S}^{\mathbb{P}}_Z$ is a wellorderable set (since Z is wellorderable by the hypothesis that there is even an OD_Z wellordering of Z). Note that for all $(\varphi, \gamma, \vec{z}, \vec{\xi}) \in \mathcal{S}^{\mathbb{P}}_Z$, $P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}} \in \mathrm{HOD}_Z$ and $N^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}} \in \mathrm{HOD}_Z$ and for any finite $\vec{z} \in Z$, $\mathcal{S}^{\mathbb{P}}_{Z,\vec{z}} \in \mathrm{HOD}_Z$ and $\langle P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}, N^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}} : (\varphi, \gamma, \vec{\xi}) \in \mathcal{S}^{\mathbb{P}}_{Z,\vec{z}}\rangle \in \mathrm{HOD}_Z$. Let $G \in T^{\mathbb{P}}_Z$ and $(\varphi, \gamma, \vec{z}, \vec{\xi}) \in \mathcal{S}^{\mathbb{P}}_Z$. Then $G \in T^{\mathbb{P}}_Z \subseteq C_{\varphi,\gamma,\vec{z},\vec{\xi}} = A_{\varphi,\gamma,\vec{z},\vec{\xi}} \cup B_{\varphi,\gamma,\vec{z},\vec{\xi}}$. Since $A_{\varphi,\gamma,\vec{z},\vec{\xi}}$ and $B_{\varphi,\gamma,\vec{z},\vec{\xi}}$ are disjoint, G is in exactly one of the two sets. If $G \in A_{\varphi,\gamma,\vec{z},\vec{\xi}} = \bigcup_{s \in P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}} A^s_{\varphi,\gamma,\vec{z},\vec{\xi}}$, then there is some $s \in P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}$ so that $G \in A^s_{\varphi,\gamma,\vec{z},\vec{\xi}} \subseteq M^{\mathbb{P}}_s$ and $V_\gamma \models \varphi(G, \vec{z}, \vec{\xi})$. If $G \in B_{\varphi,\gamma,\vec{z},\vec{\xi}} = \bigcup_{s \in N^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}} B^s_{\varphi,\gamma,\vec{z},\vec{\xi}}$, then there is some $s \in N^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}$ so that $G \in B^s_{\varphi,\gamma,\vec{z},\vec{\xi}} \subseteq M^{\mathbb{P}}_s$ and $V_\gamma \models \neg\varphi(G, \vec{z}, \vec{\xi})$. This completes the argument. ☐

Lemma 3.2. *(Woodin; [40] Theorem 5.42, Claim 2) Assume $\mathsf{AC}^{\mathbb{R}}_\omega$, all subsets of all Polish spaces have the Baire property, and boldface GCH at ω. Let Z be a set such that there is an OD_Z wellordering of Z. Let $\mathbb{P} \in \mathrm{HOD}_Z$ be a forcing which is countable in the real world. Let $T^{\mathbb{P}}_Z \subseteq \mathcal{F}_{\mathbb{P}}$ be the comeager set from Lemma 3.1. For all $G \in T^{\mathbb{P}}_Z$, $\mathrm{HOD}_Z[G] = \mathrm{HOD}_{Z \cup \{G\}}$.*

Proof. Let $G \in T^{\mathbb{P}}_Z$. It is clear that $\mathrm{HOD}_Z[G] \subseteq \mathrm{HOD}_{Z \cup \{G\}}$. Suppose $Y \in \mathrm{HOD}_{Z \cup \{G\}}$. Let U be the transitive closure of $\{Y\}$. Because there is an OD_Z wellordering of Z, $\mathrm{HOD}_{Z \cup \{G\}} \models \mathsf{AC}$. Let $\pi : \delta \to U$ be a bijection where δ is an ordinal and $\pi \in \mathrm{HOD}_{Z \cup \{G\}}$. Within $\mathrm{HOD}_{Z \cup \{G\}}$, define $R \subseteq \delta \times \delta$ by $R(\alpha, \beta)$ if and only if $\pi(\alpha) \in \pi(\beta)$. There is some ordinal $\gamma > \max\{\delta, \varsigma^{\mathbb{P}}_Z\}$, finite $\vec{z}$ in Z, and finitely many ordinals $\vec{\xi} < \gamma$,

[5] V_γ refers to the sets of rank less than γ in the real world.

[6] The comeager set $T^{\mathbb{P}}_Z$ and the sequence $\langle P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}}, N^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}} : (\varphi, \gamma, \vec{z}, \vec{\xi}) \in \mathcal{S}^{\mathbb{P}}_Z\rangle$ depend only on $\mathbb{P}$ and Z such that there is an OD_Z wellordering of Z and $\mathbb{P} \in \mathrm{HOD}_Z$ is countable in the real world and does not depend on any witness that $\mathbb{P}$ is countable in the real world.

and a formula with $|\vec{z}|+|\vec{\xi}|+3$ variables so that for all $\alpha,\beta<\delta$, $R(\alpha,\beta)$ if and only if $V_\gamma \models \varphi(G,\vec{z},\vec{\xi},\alpha,\beta)$. For $\alpha,\beta\in\delta$, let $\vec{\xi}_{\alpha,\beta}=\vec{\xi}\hat{\ }\langle\alpha,\beta\rangle$. By Lemma 3.1, the set $\langle P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}_{\alpha,\beta}} : \alpha,\beta\in\delta\rangle \in \mathrm{HOD}_Z$. For all $\alpha,\beta\in\delta$, $R(\alpha,\beta)$ if and only if there is an $s\in P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}_{\alpha,\beta}}$ with $G\in M^{\mathbb{P}}_s$ if and only if

$$\mathrm{HOD}_Z[G]\models(\exists s\in P^{\mathbb{P},Z}_{\varphi,\gamma,\vec{z},\vec{\xi}_{\alpha,\beta}})(G\in M^{\mathbb{P}}_s).$$

By the axiom of comprehension inside of $\mathrm{HOD}_Z[G]$, $R\in\mathrm{HOD}_Z[G]$. Then $U\in\mathrm{HOD}_Z[G]$ since U is the Mostowski collapse of the wellfounded relation (δ,R). Finally $Y\in\mathrm{HOD}_Z[G]$ since Y is the unique element of highest rank in U. Since $Y\in\mathrm{HOD}_{Z\cup\{G\}}$ was arbitrary, $\mathrm{HOD}_{Z\cup\{G\}}\subseteq\mathrm{HOD}_Z[G]$. Thus $\mathrm{HOD}_Z[G]=\mathrm{HOD}_{Z\cup\{G\}}$. ☐

Theorem 3.3. *Assume* $\mathsf{AC}^{\mathbb{R}}_\omega$, *all subsets of all Polish spaces have the Baire property, and boldface* GCH *at* ω. *Then there is no injection of* $\mathscr{P}_B(\omega_1)$ *into* ${}^\omega\mathrm{ON}$ *and hence* $|{}^\omega\omega_1|=|[\omega_1]^\omega|<|\mathscr{P}_B(\omega_1)|=|{}^{<\omega_1}\omega_1|=|[\omega_1]^{<\omega_1}|$.

Proof. Suppose there is an injection $\Phi:\mathscr{P}_B(\omega_1)\to{}^\omega\mathrm{ON}$. Let $\mathbb{P}=\mathrm{Coll}(\omega^+,\omega^{++})^{\mathrm{HOD}_{\{\Phi\}}}=\mathrm{Coll}(\omega_1,\omega_2)^{\mathrm{HOD}_{\{\Phi\}}}$ which is the forcing of countable partial functions from $\omega_1^{\mathrm{HOD}_{\{\Phi\}}}$ into $\omega_2^{\mathrm{HOD}_{\{\Phi\}}}$ as defined within $\mathrm{HOD}_{\{\Phi\}}$. $\mathbb{P}$ is a countable forcing from the view of the real world. By Lemma 3.2, there is $G\subseteq\mathbb{P}$ which is $\mathbb{P}$-generic over $\mathrm{HOD}_{\{\Phi\}}$ such that $\mathrm{HOD}_{\{\Phi,G\}}=\mathrm{HOD}_{\{\Phi\}}[G]$. $\mathbb{P}$ adds a generic surjection $g:\omega_1^{\mathrm{HOD}_{\{\Phi\}}}\to\omega_2^{\mathrm{HOD}_{\{\Phi\}}}$. Let $\pi:\omega_1\times\omega_1\to\omega_1$ be the Gödel pair function which belongs to $\mathrm{HOD}_{\{\Phi\}}$. Let $A=\pi[g]$ which is a bounded subset of the true ω_1 and thus $A\in\mathscr{P}_B(\omega_1)\cap\mathrm{HOD}_{\{\Phi\}}[G]$. Since A is ordinal definable from G, $\Phi(A)$ is ordinal definable from Φ and G. Thus $\Phi(A)\in\mathrm{HOD}_{\{\Phi,G\}}=\mathrm{HOD}_{\{\Phi\}}[G]$. So $\Phi(A)\in\mathrm{HOD}_{\{\Phi\}}[G]\cap{}^\omega\mathrm{ON}$. Since $\mathrm{HOD}_{\{\Phi\}}\models$ "$\mathbb{P}=\mathrm{Coll}(\omega_1,\omega_2)$ is $<\omega_1$-closed", no new ω-sequences through $\mathrm{HOD}_{\{\Phi\}}$ are added by forcing $\mathbb{P}$ over $\mathrm{HOD}_{\{\Phi\}}$. Thus $\Phi(A)\in\mathrm{HOD}_{\{\Phi\}}$. Since $\Phi(A)$ is ordinal definable from Φ, $A=\Phi^{-1}(\Phi(A))$ is ordinal definable from Φ. Thus $A\in\mathrm{HOD}_{\{\Phi\}}$. Then $g=\pi^{-1}[A]$ is ordinal definable from Φ and so $g\in\mathrm{HOD}_{\{\Phi\}}$. By absoluteness, $\mathrm{HOD}_{\{\Phi\}}\models$ "$g:\omega_1\to\omega_2$ is a surjection". Contradiction. ☐

The hypothesis of Theorem 3.3 follows from AD but moreover it follows from $\mathsf{AC}^{\mathbb{R}}_\omega$ and all subsets of all Polish spaces have the Baire property and the perfect set property by Fact 2.6. Thus $\mathscr{P}_B(\omega_1)$ does not inject into ${}^\omega\mathrm{ON}$ is a consequence of classical regularity properties.

Theorem 3.4. *The consistency of "*ZFC *and there exists an inaccessible cardinal" implies the consistency of "the Bounded Power Set Conjecture and the ABCD Conjecture below* $\Theta=\omega_2$*".*

Proof. The Solovay model ([54]) constructed from the Lévy collapse of an inaccessible satisfies the hypothesis of Theorem 3.3 and $\Theta=\omega_2$. Theorem 3.3 shows that the Solovay model satisfies the Bounded Powerset Conjecture below ω_2: $\mathscr{P}_B(\omega_1)$ does not inject into ${}^\omega\mathrm{ON}$. Since $\Theta=\omega_2$, the only nontrivial instance of the ABCD conjecture below Θ is that $|{}^\omega\omega_1|<|{}^{\omega_1}\omega_1|$ which follows from $\mathscr{P}_B(\omega_1)$ not injecting into ${}^\omega\mathrm{ON}$ since $|\mathscr{P}_B(\omega_1)|\leq|{}^{\omega_1}\omega_1|$. ☐

Theorem 3.5. *Assume* $\mathsf{AC}^{\mathbb{R}}_\omega$, *all subsets of all Polish spaces have the Baire property, and boldface* GCH *at* ω. ${}^\omega\omega_1$ *does not inject into* $\mathscr{P}(\omega)\times\mathrm{ON}$. *Thus* $|\mathbb{R}\times\omega_1|<|{}^\omega\omega_1|=|[\omega_1]^\omega|$.

Proof. Suppose otherwise that there is an injection $\Phi:{}^\omega\omega_1\to\mathscr{P}(\omega)\times\mathrm{ON}$. Let $\mathrm{Coll}(\omega_1,2^\omega)^{\mathrm{HOD}_{\{\Phi\}}}$ be the forcing defined in $\mathrm{HOD}_{\{\Phi\}}$ consisting of countable partial functions from $\omega_1^{\mathrm{HOD}_{\{\Phi\}}}$ into the $\mathbb{R}^{\mathrm{HOD}_{\{\Phi\}}}$ ordered by reverse function extension. Let $\dot{\mathbb{N}}\in\mathrm{HOD}_{\{\Phi\}}$ be such that $\mathrm{HOD}_{\{\Phi\}}\models$ "$\dot{\mathbb{N}}$ is the $\mathrm{Coll}(\omega_1,\mathbb{R})$-name for Namba forcing". (Namba forcing $\mathbb{N}$ consists of $p\subseteq{}^{<\omega}\omega_2$ which are nonempty trees on ω_2 so that each node of p has ω_2 many extensions. Let $\leq_{\mathbb{N}}$ be the subset relation. See [31] Theorem 28.10 for more information on Namba forcing.) Let $\mathbb{P}$ be the two step iterated forcing $\mathrm{Coll}(\omega_1,2^\omega)*\dot{\mathbb{N}}$. $\mathbb{P}$ is a countable forcing in the real world.[7] Thus by Theorem 3.2, there is a $\mathbb{P}$-generic filter over $\mathrm{HOD}_{\{\Phi\}}$ with the property that $\mathrm{HOD}_{\{\Phi\}}[G]=\mathrm{HOD}_{\{\Phi,G\}}$. There are $H\subseteq\mathrm{Coll}(\omega_1,2^\omega)^{\mathrm{HOD}_{\{\Phi\}}}$ which is $\mathrm{Coll}(\omega_1,2^\omega)^{\mathrm{HOD}_{\{\Phi\}}}$-generic over $\mathrm{HOD}_{\{\Phi\}}$ and $K\subseteq\dot{\mathbb{N}}[H]$ which is $\dot{\mathbb{N}}[H]$-generic over $\mathrm{HOD}_{\{\Phi\}}[H]$ so that $G=H*K$. $\mathrm{Coll}(\omega_1,2^\omega)^{\mathrm{HOD}_{\{\Phi\}}}$ adds no new reals or even ω-sequences through $\mathrm{HOD}_{\{\Phi\}}$ since $\mathrm{HOD}_{\{\Phi\}}\models$ "$\mathrm{Coll}(\omega_1,2^\omega)$ is $<\omega_1$-closed" and

[7] Here, one assumes that some reasonable presentation of the iterated forcing $\mathrm{Coll}(\omega,2^\omega)*\dot{\mathbb{N}}$ is used to limit the $\mathrm{Coll}(\omega_1,2^\omega)$-names for elements of $\dot{\mathbb{N}}$

$\text{HOD}_{\{\Phi\}}[H] \models \mathsf{CH}$ (which is the primary purpose of this first forcing). Since $\text{HOD}_{\{\Phi\}}[H] \models \mathsf{CH}$, the $\dot{\mathbb{N}}[H]$-generic filter K adds a generic cofinal function $k : \omega \to \omega_2^{\text{HOD}_{\{\Phi\}}}$ and adds no new reals by [31] Theorem 28.10. Therefore $\mathscr{P}(\omega) \cap \text{HOD}_{\{\Phi\}} = \mathscr{P}(\omega) \cap \text{HOD}_{\{\Phi\}}[G] = \mathscr{P}(\omega) \cap \text{HOD}_{\{\Phi\}}[H * K]$. Since boldface GCH at ω holds, $\omega_2^{\text{HOD}_{\{\Phi\}}}$ is less than the true ω_1 and thus $k \in [\omega_1]^\omega$. Since $\Phi(k) \in \text{OD}_{\{\Phi,K\}} \subseteq \text{OD}_{\{\Phi,G\}}$, $\Phi(k) \in \text{HOD}_{\{\Phi,G\}} = \text{HOD}_{\{\Phi\}}[G]$. Since $\Phi(k) \in \mathscr{P}(\omega) \times \text{ON}$ and $\mathscr{P}(\omega) \cap \text{HOD}_{\{\Phi\}}[G] = \mathscr{P}(\omega) \cap \text{HOD}_{\{\Phi\}}$, $\Phi(k) \in \text{HOD}_{\{\Phi\}}$. Thus $k = \Phi^{-1}(\Phi(k))$ is $\text{OD}_{\{\Phi\}}$ and thus $k \in \text{HOD}_{\{\Phi\}}$. By absoluteness, $\text{HOD}_{\{\Phi\}} \models$ "$k : \omega \to \omega_2$ is cofinal" which is impossible since successor cardinals are regular in $\text{HOD}_{\{\Phi\}} \models \mathsf{AC}$. ☐

At the moment, the only known proof of the cardinality computation, $|\mathbb{R} \times \omega_1| < |{}^\omega\omega_1|$, under classical regularity property involves HOD-type models and Namba forcing.[8] However, this proof involving Namba forcing is the only proof of this result suitable for generalization to higher cardinals. Section 7 will use a form of Namba forcing to prove the analog of this result for all cardinals below Θ in AD^+.

Next one will resolve Question (b) from this same hypothesis.

Fact 3.6. *Let $\mathbb{P} = (\mathbb{P}, \leq_\mathbb{P}, 1_\mathbb{P})$ and $\mathbb{Q} = (\mathbb{Q}, \leq_\mathbb{Q}, 1_\mathbb{Q})$ be two forcings. Let $\pi : \mathbb{Q} \to \mathbb{P}$ be a map such that for all $q_0, q_1 \in \mathbb{Q}$, $q_0 \leq_\mathbb{P} q_1$ implies $\pi(q_0) \leq_\mathbb{P} \pi(q_1)$ and $\pi(1_\mathbb{Q}) = 1_\mathbb{P}$. The map $\Lambda_\pi : \mathcal{F}_\mathbb{Q} \to \mathcal{F}_\mathbb{P}$ defined by $\Lambda_\pi(F) = \langle \pi[F] \rangle_\mathbb{P} = \{p \in \mathbb{P} : (\exists q \in F)(\pi(q) \leq_\mathbb{P} p)\}$ is a well defined continuous map (with respect to the topology on $\mathcal{F}_\mathbb{Q}$ and the topology on $\mathcal{F}_\mathbb{P}$).*

Proof. Suppose $F \in \mathcal{F}_\mathbb{Q}$. Then $\langle \pi[F] \rangle_\mathbb{P} \in \mathcal{F}_\mathbb{P}$. To see this, suppose $p_0, p_1 \in \langle \pi[F] \rangle_\mathbb{P}$. Let $q_0, q_1 \in \mathbb{Q}$ be such that $\pi(q_0) \leq_\mathbb{P} p_0$ and $\pi(q_1) \leq_\mathbb{P} p_1$. Since F is a $\mathbb{Q}$-filter, there is a $q_2 \in F$ such that $q_2 \leq_\mathbb{Q} q_0$ and $q_2 \leq_\mathbb{Q} q_1$. Thus $\pi(q_2) \in \langle \pi[F] \rangle_\mathbb{P}$, $\pi(q_2) \leq_\mathbb{P} \pi(q_0) \leq_\mathbb{P} p_0$, and $\pi(q_2) \leq_\mathbb{P} \pi(q_1) \leq_\mathbb{P} p_1$. $\langle \pi[F] \rangle_\mathbb{P}$ is clearly $\leq_\mathbb{P}$-upward closed.

Let s be $\mathbb{P}$-suitable. Suppose $F \in \Lambda_\pi^{-1}[M_s^\mathbb{P}]$. Since $\Lambda_\pi(F) = \langle \pi[F] \rangle_\mathbb{P} \in M_s^\mathbb{P}$ is a filter, there is a $\tilde{p} \in \Lambda_\pi(F)$ so that $\tilde{p} \leq_\mathbb{P} p$ for all $p \in \text{dom}(s)$ with $s(p) = 1$ and $\tilde{p}$ is incompatible with all $p \in \text{dom}(s)$ with $s(p) = 0$. By definition of $\tilde{p} \in \langle \pi[F] \rangle_\mathbb{P}$, there is a $\tilde{q} \in F$ so that $\pi(\tilde{q}) \leq \tilde{p}$. Let $t : \{\tilde{q}\} \to 2$ be define by $t(\tilde{q}) = 1$. It is clear that $F \in M_t^\mathbb{Q}$. Now suppose that $H \in M_t^\mathbb{Q}$. Since $\pi(\tilde{q}) \leq_\mathbb{P} \tilde{p}$, $\tilde{p} \in \langle \pi[H] \rangle_\mathbb{P} = \Lambda_\pi(H)$. Since $\tilde{p} \leq_\mathbb{P} p$ for all $p \in \text{dom}(s)$ with $s(p) = 1$ and $\tilde{p}$ is not compatible with any $p \in \text{dom}(s)$ with $s(p) = 0$, one has that $p \in \Lambda_\pi(H)$ for all $p \in \text{dom}(s)$ with $s(p) = 1$ and $p \notin \Lambda_\pi(H)$ for all $p \in \text{dom}(s)$ with $s(p) = 0$. Thus $\Lambda_\pi(H) \in M_s^\mathbb{Q}$. It has been shown that $F \in M_t^\mathbb{Q} \subseteq \Lambda_\pi^{-1}[M_s^\mathbb{P}]$. This shows that Λ_π is continuous. ☐

Some basic properties of forcing projection will be presented next. (Forcing projection of the ∞-Borel code forcings will be very important in Section 4.)

Definition 3.7. Let $\mathbb{P} = (\mathbb{P}, \leq_\mathbb{P}, 1_\mathbb{P})$ and $\mathbb{Q} = (\mathbb{Q}, \leq_\mathbb{Q}, 1_\mathbb{Q})$ be two forcings. A function $\pi : \mathbb{Q} \to \mathbb{P}$ is a forcing projection if and only if the following holds:

(1) For all $q_0, q_1 \in \mathbb{Q}$, $q_0 \leq_\mathbb{Q} q_1$ implies $\pi(q_0) \leq_\mathbb{P} \pi(q_1)$ and $\pi(1_\mathbb{Q}) = 1_\mathbb{P}$.
(2) For all $q \in \mathbb{Q}$ and $p' \in \mathbb{P}$ such that $p' \leq_\mathbb{P} \pi(q)$, there is a $q' \in \mathbb{Q}$ so that $q' \leq_\mathbb{Q} q$ and $\pi(q') = p'$.

Suppose M is an inner model of ZF so that $\mathbb{P}, \mathbb{Q}, \pi \in M$ and $M \models$ "$\pi : \mathbb{Q} \to \mathbb{P}$ is a forcing projection". If $H \subseteq \mathbb{Q}$ be $\mathbb{Q}$-generic over M, then $\Lambda_\pi(H) = \langle \pi[H] \rangle_\mathbb{P} = \{p \in \mathbb{P} : (\exists q \in H)(\pi(q) \leq_\mathbb{P} p)\}$ is a $\mathbb{P}$-generic filter over M. To see this, suppose $D \in M$ is $\mathbb{P}$-dense. Let $\tilde{D} = \pi^{-1}[D]$ which belongs to M. Let $q \in \mathbb{Q}$. Since D is dense there is a $p' \in D$ so that $p' \leq_\mathbb{P} \pi(q)$. By the properties of a forcing projection, there is a $q' \leq_\mathbb{Q} q$ so that $\pi(q') = p'$ and hence $q' \in \tilde{D}$. This shows that $\tilde{D}$ is dense in $\mathbb{Q}$. Since H is $\mathbb{Q}$-generic over M, $H \cap \tilde{D} \neq \emptyset$. Let $q \in H \cap \tilde{D}$. Then $\pi(q) \in \langle \pi[H] \rangle_\mathbb{P} \cap D$.

Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over M. Let $\mathbb{Q}/_\pi G = \{q \in \mathbb{Q} : \pi(q) \in G\}$, $\leq_{\mathbb{Q}/_\pi Q} = \leq_\mathbb{Q} \restriction \mathbb{Q}/_\pi G$, and $1_{\mathbb{Q}/_\pi G} = 1_\mathbb{Q}$. $\mathbb{Q}/_\pi G = (\mathbb{Q}/_\pi G, \leq_{\mathbb{Q}/_\pi G}, 1_{\mathbb{Q}/_\pi G})$ is a forcing in $M[G]$ and is called the remainder forcing of $\mathbb{Q}$ by G. Let $\dot{G}$ denote the canonical $\mathbb{P}$-name for its generic filter. Let $\mathbb{Q}/_\pi \dot{G}$ be the canonical $\mathbb{P}$-name such that $1_\mathbb{P} \Vdash_\mathbb{P} \mathbb{Q}/_\pi \dot{G} = \{q \in \check{\mathbb{Q}} : \check{\pi}(q) \in \dot{G}\}$. Note that if G is $\mathbb{P}$-generic over M, then $(\mathbb{Q}/_\pi \dot{G})[G] = \mathbb{Q}/_\pi G$. $\mathbb{Q}$ is forcing equivalent to the two step iterated forcing $\mathbb{P} * (\mathbb{Q}/\dot{G})$. If $H \subseteq \mathbb{Q}$ is $\mathbb{Q}$-generic over M and $G = \langle \pi[H] \rangle_\mathbb{P}$

[8]The important feature of Namba forcing is that it adds a new ω-sequence below the true ω_1 without adding any new reals. Prikry forcing at a measurable cardinal adds a new ω-sequence without adding any bounded subsets below the measurable. Under the hypothesis of Theorem 3.5, there may be no inner models with a measurable. In fact, even in $L(\mathbb{R}) \models \mathsf{AD}$, no ordinals which is countable in the real world can be measurable in $\text{HOD}^{L(\mathbb{R})}$. Steel's HOD-analysis in $L(\mathbb{R})$ ([60] Lemma 8.25) shows that the real world never has a Prikry generic sequence over $\text{HOD}^{L(\mathbb{R})}$ for any measurable cardinal of $\text{HOD}^{L(\mathbb{R})}$.

which is $\mathbb{P}$-generic over M, then H is $\mathbb{Q}/G$-generic over $M[G]$, $G * H$ is $\mathbb{P} * (\mathbb{Q}/_\pi \dot{G})$-generic over M, and $M[H] = M[G][H] = M[G * H]$. (See the introduction of [1] for more information.)

Fact 3.8. *Suppose $\pi : \mathbb{Q} \to \mathbb{P}$ is a forcing projection. If E is dense open in the topology on $\mathcal{F}_\mathbb{P}$, then $\Lambda_\pi^{-1}[E]$ is dense open in the topology of $\mathcal{F}_\mathbb{Q}$. Thus for any $C \subseteq \mathcal{F}_\mathbb{P}$ which is comeager in the topology of $\mathcal{F}_\mathbb{P}$, $\Lambda_\pi^{-1}[C] \subseteq \mathcal{F}_\mathbb{Q}$ is comeager in the topology of $\mathcal{F}_\mathbb{Q}$.*

Proof. Since Λ_π is continuous by Fact 3.6, $\Lambda_\pi^{-1}[E]$ is open. Suppose s is $\mathbb{Q}$-suitable. There is a $\tilde{q}$ so that $\tilde{q} \leq_\mathbb{Q} q$ for all $q \in \text{dom}(s)$ with $s(q) = 1$ and $\tilde{q}$ is not compatible with any $q \in \text{dom}(s)$ with $s(q) = 0$. Let $u = \{(\pi(\tilde{q}), 1)\}$ which is $\mathbb{P}$-suitable. Since E is dense open, there is a $\mathbb{P}$-suitable t so that $M_t^\mathbb{P} \subseteq M_u^\mathbb{P} \cap E$, $\pi(\tilde{q}) \in \text{dom}(t)$, and $t(\pi(\tilde{q})) = 1$. Since t is $\mathbb{P}$-suitable, there is a $p' \in \mathbb{P}$ so that for all $p \in \text{dom}(t)$ with $t(p) = 1$, $p' \leq_\mathbb{P} p$ and p' is not compatible with any $p \in \text{dom}(t)$ with $t(p) = 0$. Since $\pi(\tilde{q}) \in \text{dom}(t)$, one has $p' \leq_\mathbb{P} \pi(\tilde{q})$. By the property of projections, there is a $q' \leq_\mathbb{Q} \tilde{q}$ so that $\pi(q') = p'$. Let $v = s \cup \{(q', 1)\}$ which is $\mathbb{Q}$-suitable. Note that $\Lambda_\pi[M_v^\mathbb{Q}] \subseteq M_t^\mathbb{P} \subseteq E$. Thus $M_v^\mathbb{Q} \subseteq \Lambda_h^{-1}[E] \cap M_s^\mathbb{Q}$. This shows $\Lambda_\pi^{-1}[E]$ is dense open.

Suppose $C \subseteq \mathcal{F}_\mathbb{P}$ is comeager. There is a countable sequence $\langle E_n : n \in \omega \rangle$ of dense open subsets of $\mathcal{F}_\mathbb{P}$ such that $\bigcap_{n \in \omega} E_n \subseteq C$. Then $\bigcap_{n\in\omega} \Lambda_\pi^{-1}[E_n] = \pi^{-1}\big(\bigcap_{n\in\omega} E_n\big) \subseteq \Lambda_\pi^{-1}[C]$. Since $\bigcap_{n\in\omega} \Lambda_\pi^{-1}[E_n]$ is a countable intersection of dense open subsets of $\mathcal{F}_\mathbb{Q}$, $\Lambda_\pi^{-1}[C]$ is comeager. ☐

Lemma 3.9. *Assume $\mathsf{AC}_\omega^\mathbb{R}$, all subsets of all Polish spaces have the Baire property, and boldface GCH at ω. Let Z be a set such that there is an OD_Z wellordering of Z. Let $\mathbb{Q} \in \text{HOD}_Z$ be a forcing which is countable in the real world. There is a $U_Z^\mathbb{Q} \subseteq \mathsf{Gen}_\mathbb{Q}^{\mathcal{D}_Z^\mathbb{Q}}$ comeager in the topology of $\mathcal{F}_\mathbb{Q}$ and obtained uniformly from $\mathbb{Q}$ and Z with the property that for all $G \in U_Z^\mathbb{Q}$ and forcing projection $\pi : \mathbb{Q} \to \mathbb{P}$ such that $\pi \in \text{HOD}_Z$ and $\mathbb{P} \in \text{HOD}_Z$ is a countable forcing in the real world, $\text{HOD}_Z[\langle \pi[G] \rangle_\mathbb{P}] = \text{HOD}_{Z \cup \{\langle \pi[G] \rangle_\mathbb{P}\}}$.*

Proof. Let $\langle \pi_\alpha : \alpha \in \text{ON} \rangle$ be an enumeration of all forcing projections in HOD_Z of $\mathbb{Q}$ into some $\mathbb{P}_\alpha$ where $\mathbb{P}_\alpha$ is countable in the real world according to the canonical global wellordering of HOD_Z. (The enumeration does not need to belong to HOD_Z.) Let $T_Z^{\mathbb{P}_\alpha}$ be the comeager set from Lemma 3.2 for $\mathbb{P}_\alpha$ (which is obtained uniformly from $\mathbb{P}_\alpha$ and Z). For each $\alpha \in \text{ON}$, $\Lambda_{\pi_\alpha}^{-1}[T_Z^{\mathbb{P}_\alpha}] \subseteq \mathcal{F}_\mathbb{Q}$ is comeager by Fact 3.8. Then $U_Z^\mathbb{Q} = \mathsf{Gen}_\mathbb{Q}^{\mathcal{D}_Z^\mathbb{Q}} \cap \bigcap_{\alpha\in\text{ON}} \Lambda_{\pi_\alpha}^{-1}[T_Z^{\mathbb{P}_\alpha}]$ is a comeager subset of $\mathsf{Gen}_\mathbb{Q}^{\mathcal{D}_Z^\mathbb{Q}}$ since wellordered intersections of comeager subsets of $\mathcal{F}_\mathbb{Q}$ remain comeager subsets of $\mathcal{F}_\mathbb{Q}$ under the given hypothesis. If $\pi : \mathbb{Q} \to \mathbb{P}$ is such a forcing projection, then there is an $\alpha \in \text{ON}$ so that $\pi = \pi_\alpha$ and $\mathbb{P} = \mathbb{P}_\alpha$. Let $G \in U_Z^\mathbb{Q}$. Then $\langle \pi[G] \rangle_\mathbb{P} = \Lambda_{\pi_\alpha}(G) \in T_Z^{\mathbb{P}_\alpha} = T_Z^\mathbb{P}$. By the property of $T_Z^\mathbb{P}$ coming from Lemma 3.2, $\text{HOD}_Z[\langle \pi[G] \rangle_\mathbb{P}] = \text{HOD}_{Z \cup \{\langle \pi[G] \rangle_\mathbb{P}\}}$. ☐

Fact 3.10. $|\mathbb{R}| \leq |S_1|$. *If boldface* GCH *at ω holds, then $\neg(|\omega_1| \leq |S_1|)$.*

Proof. For $r \in [\omega]^\omega$, let $f_r : \omega_1^{L[r]} \to \omega_1^{L[r]}$ be defined by

$$f_r(\alpha) = \begin{cases} r(\alpha) & \alpha \in \omega \\ \alpha & \omega \leq \alpha < \omega_1^{L[r]} \end{cases}.$$

Note that $f_r \in L[r]$ and that $\omega_1^{L[f_r]} = \omega_1^{L[r]} = \sup(f_r)$. Thus $f_r \in S_1$. The map $\Phi : \mathbb{R} \to S_1$ defined by $\Phi(r) = f_r$ is an injection.

Now assume boldface GCH at ω. Suppose $\Psi : \omega_1 \to S_1$ is an injection. The claim is that $\omega_1 = \sup\{\sup(\Psi(\alpha)) : \alpha < \omega_1\}$. If not, then this supremum is some countable ordinal δ. Then one has that $\Psi : \omega_1 \to [\delta]^{<\delta}$. Since δ is countable, there is a bijection $\Upsilon : [\delta]^{<\delta} \to \mathbb{R}$. Then $\Upsilon \circ \Psi : \omega_1 \to \mathbb{R}$ is an injection which violates boldface GCH at ω. Let $A = \{(\alpha, \beta, \gamma) \in \omega_1 \times \omega_1 \times \omega_1 : \Psi(\alpha)(\beta) = \gamma\}$. $L[A] \models \mathsf{ZFC}$ and thus by boldface GCH at ω, $\omega_1^{L[A]} \leq (2^\omega)^{L[A]} < \omega_1$. By the claim, there is some $\alpha < \omega_1$ so that $\sup(\Psi(\alpha)) > \omega_1^{L[A]}$. Since $\Psi(\alpha) \in L[A]$ and $\Psi(\alpha) \in S_1$, one has that $\omega_1^{L[A]} \geq \omega_1^{L[\Psi(\alpha)]} = \sup(\Psi(\alpha)) > \omega_1^{L[A]}$ which is a contradiction. ☐

Theorem 3.11. *Assume $\mathsf{AC}_\omega^\mathbb{R}$, all subsets of all Polish spaces have the Baire property, and boldface GCH at ω. Then S_1 does not inject into ${}^\omega\text{ON}$.*

Proof. Suppose there is an injection $\Phi : S_1 \to {}^\omega\text{ON}$. For $\alpha < \omega_1^{\text{HOD}_{\{\Phi\}}}$, let $\mathbb{K}_\alpha = \text{Coll}(\omega, \alpha)$ be the forcing of finite partial function p on ω into α. Let $\leq_{\mathbb{K}_\alpha}$ be reverse function extension. Let $1_{\mathbb{K}_\alpha} = \emptyset$. For $\beta \leq \omega_1^{\text{HOD}_{\{\Phi\}}}$, let $\mathbb{Q}_\beta = \mathrm{Coll}(\omega, <\beta)$ be the collection of all functions $\mathfrak{p}$ with domain β so that for all $\alpha < \beta$, $\mathfrak{p}(\alpha) \in \mathbb{K}_\alpha$ and $\mathrm{supp}^\beta(\mathfrak{p}) = \{\alpha < \beta : \mathfrak{p}(\alpha) \neq 1_{\mathbb{K}_\alpha}\}$ is finite. Let $\mathfrak{p} \leq_{\mathbb{Q}_\beta} \mathfrak{q}$ if and only if for all $\alpha < \beta$, $\mathfrak{p}(\alpha) \leq_{\mathbb{K}_\alpha} \mathfrak{q}(\alpha)$. Let $1_{\mathbb{Q}_\beta}$ be defined by $1_{\mathbb{Q}_\beta}(\alpha) = 1_{\mathbb{K}_\alpha}$. $\mathbb{Q}_\beta = (\mathbb{Q}_\beta, \leq_{\mathbb{Q}_\beta}, 1_{\mathbb{Q}_\beta})$ is the Lévy collapse of the ordinals below β to ω. Let $\mathbb{Q} = \mathbb{Q}_{\omega_1^{\mathrm{HOD}_{\{\Phi\}}}} = \mathrm{Coll}(\omega, < \omega_1^{\mathrm{HOD}_{\{\Phi\}}})$.[9] For $\beta < \omega_1^{\mathrm{HOD}_{\{\Phi\}}}$, let $\pi_\beta : \mathbb{Q} \to \mathbb{Q}_\beta$ be defined by $\pi_\beta(\mathfrak{p}) = \mathfrak{p} \upharpoonright \beta$. Suppose $\mathfrak{p} \in \mathbb{Q}_\beta$, $\mathfrak{q} \in \mathbb{Q}$, and $\mathfrak{p} \leq_{\mathbb{Q}_\beta} \mathfrak{q} \upharpoonright \beta = \pi_\beta(\mathfrak{q})$. Let $\mathfrak{q}' \in \mathbb{Q}$ be defined by

$$\mathfrak{q}'(\alpha) = \begin{cases} \mathfrak{p}(\alpha) & \alpha < \beta \\ \mathfrak{p}(\alpha) = \mathfrak{q}(\alpha) & \beta \leq \alpha < \omega_1^{\mathrm{HOD}_{\{\Phi\}}} \end{cases}.$$

Then $\mathfrak{q}' \leq_{\mathbb{Q}} \mathfrak{q}$ and $\pi_\beta(\mathfrak{q}') = \mathfrak{p}$. This shows that π_β is a forcing projection. By boldface GCH at ω, $\mathbb{Q}_\beta$ and $\mathbb{Q}$ are countable for all $\beta < \omega_1^{\mathrm{HOD}_{\{\Phi\}}}$. By Lemma 3.9, let $G \subseteq \mathbb{Q}$ be $\mathbb{Q}$-generic over $\mathrm{HOD}_{\{\Phi\}}$ having the property that $\mathrm{HOD}_{\{\Phi\}}[G] = \mathrm{HOD}_{\{\Phi,G\}}$ and for all $\beta < \omega_1^{\mathrm{HOD}_{\{\Phi\}}}$, $\mathrm{HOD}_{\{\Phi\}}[\langle \pi_\beta[G]\rangle_{\mathbb{Q}_\beta}] = \mathrm{HOD}_{\{\Phi, \langle \pi_\beta[G]\rangle_{\mathbb{Q}_\beta}\}}$. The generic G can be coded naturally as a subset of $\omega_1^{\mathrm{HOD}_{\{\Phi\}}}$ and hence as an element $\tilde{g} \in [\omega_1^{\mathrm{HOD}_{\{\Phi\}}}]^{\omega_1^{\mathrm{HOD}_{\{\Phi\}}}}$. Thus $\sup(\tilde{g}) \leq \omega_1^{\mathrm{HOD}_{\{\Phi\}}}$. If done naturally, one will have $L[G] = L[\tilde{g}]$. $\mathrm{HOD}_{\{\Phi\}} \models$ "$\mathbb{Q}$ has the $\omega_1^{\mathrm{HOD}_{\{\Phi\}}}$-chain condition" (see [50] Lemma 6.44 using the Δ-system lemma). Thus $\omega_1^{\mathrm{HOD}_{\{\Phi\}}[G]} = \omega_1^{\mathrm{HOD}_{\{\Phi\}}}$. Since $\tilde{g} \in \mathrm{HOD}_{\{\Phi\}}[G]$, $\omega_1^{L[\tilde{g}]} \leq \omega_1^{\mathrm{HOD}_{\{\Phi\}}[G]} = \omega_1^{\mathrm{HOD}_{\{\Phi\}}}$. Since $\tilde{g}$ codes a $\mathbb{K}_\beta$-generic filter for each $\beta < \omega_1^{\mathrm{HOD}_{\{\Phi\}}}$, $\omega_1^{L[\tilde{g}]} \geq \omega_1^{\mathrm{HOD}_{\{\Phi\}}}$ and also $\sup(\tilde{g}) \geq \omega_1^{\mathrm{HOD}_{\{\Phi\}}}$. This shows that $\sup(\tilde{g}) = \omega_1^{\mathrm{HOD}_{\{\Phi\}}} = \omega_1^{L[\tilde{g}]}$. So $\tilde{g} \in S_1$. Since $\tilde{g}$ is $\mathrm{OD}_{\{G\}}$, $\Phi(\tilde{g}) \in \mathrm{HOD}_{\{\Phi,G\}} = \mathrm{HOD}_{\{\Phi\}}[G]$. $\Phi(\tilde{g}) \in {}^\omega\mathrm{ON}$ and thus there is a $\mathbb{Q}$-name τ so that $\tau[G] = \Phi(\tilde{g})$. Without loss of generality, $1_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ "τ is a function from $\check{\omega}$ into ON". In $\mathrm{HOD}_{\{\Phi\}}$, for each $n \in \omega$, let A_n be a maximal antichain inside of $\{\mathfrak{q} \in \mathbb{Q} : \mathfrak{q} \Vdash_{\mathbb{Q}} (\exists \xi \in \mathrm{ON})(\tau(\check{n}) = \xi)\}$. Thus $\mathrm{HOD}_{\{\Phi\}} \models$ "A_n is countable" by the $\omega_1^{\mathrm{HOD}_{\{\Phi\}}}$-chain condition. Then $\delta_n = \sup\{\max(\mathrm{supp}(\mathfrak{q})) : \mathfrak{q} \in A_n\} < \omega_1^{\mathrm{HOD}_{\{\Phi\}}}$ since $\omega_1^{\mathrm{HOD}_{\{\Phi\}}}$ is regular in $\mathrm{HOD}_{\{\Phi\}}$. Similarly, $\delta = \sup\{\delta_n : n \in \omega\}$ is less than $\omega_1^{\mathrm{HOD}_{\{\Phi\}}}$. For each $n \in \omega$ and $\mathfrak{q} \in A_n$, let $\alpha^n_{\mathfrak{q}}$ be the unique α so that $\mathrm{HOD}_{\{\Phi\}} \models \mathfrak{q} \Vdash_{\mathbb{Q}} \tau(\check{n}) = \check{\alpha}$. Let $\sigma = \{((n, \alpha^n_{\mathfrak{q}})\check{}, \mathfrak{q} \upharpoonright \delta) : n \in \omega \wedge \mathfrak{q} \in A_n\}$. σ is a $\mathbb{Q}_\delta$-name and $\Phi(\tilde{g}) = \tau[G] = \sigma[\langle \pi_\delta[G]\rangle_{\mathbb{Q}_\delta}]$. This shows that $\Phi(\tilde{g}) \in \mathrm{HOD}_{\{\Phi\}}[\langle \pi_\delta[G]\rangle_{\mathbb{Q}_\delta}] = \mathrm{HOD}_{\{\Phi, \langle \pi_\delta[G]\rangle_{\mathbb{Q}_\delta}\}}$. Since $\Phi(\tilde{g})$ is $\mathrm{OD}_{\{\Phi, \langle \pi_\delta[G]\rangle_{\mathbb{Q}_\delta}\}}$, $\tilde{g} = \Phi^{-1}(\Phi(\tilde{g}))$ is also $\mathrm{OD}_{\{\Phi, \langle \pi_\delta[G]\rangle_{\mathbb{Q}_\delta}\}}$. Thus $\tilde{g} \in \mathrm{HOD}_{\{\Phi, \langle \pi_\delta[G]\rangle_{\mathbb{Q}_\delta}\}} = \mathrm{HOD}_{\{\Phi\}}[\langle \pi_\delta[G]\rangle_{\mathbb{Q}_\delta}]$. Since $L[\tilde{g}] = L[G]$, one has that $G \in \mathrm{HOD}_{\{\Phi\}}[\langle \pi_\delta[G]\rangle_{\mathbb{Q}_\delta}]$ which is impossible since G is $(\mathbb{Q}/_{\pi_\delta} \dot{G})$-generic over $\mathrm{HOD}_{\{\Phi\}}[\langle \pi_\delta[G]\rangle_{\mathbb{Q}_\delta}]$ and $\mathrm{HOD}_{\{\Phi\}}[G] = \mathrm{HOD}_{\{\Phi\}}[\langle \pi_\delta[G]\rangle_{\mathbb{Q}_\delta}][G]$ by the discussion above about remainder forcing applied to the forcing projection π_δ. □

4. ∞-Borel Code Forcing

This section will present ∞-Borel code forcing (with ordinal definability constraints) on ${}^\omega\kappa$, its directed limit, and the analysis of natural determinacy models as symmetric collapse extension of HOD-type models. These results in various forms are due to Woodin and Ikegami-Trang ([25]). The uniformity of these results will be very important in the main result of the next section.

This section presents detailed proofs of the results in the generality needed in this paper. The results in this section are not very straightforward and simple generalizations of the classical OD ∞-Borel code forcing on $\mathbb{R}$ in $L(\mathbb{R})$ as presented in [8] which is uniquely simple. There are important differences between Vopěnka forcing and ∞-Borel code forcing and both are needed in the appropriate situation. The proofs of results for ${}^\omega\kappa$ when $\omega_1 \leq \kappa < \Theta$ require substantial changes from the result for just $\mathbb{R}$ or ${}^\omega\omega$. One needs to be very careful and explicit when relativizing from OD to OD_Z. From the author's experience, argument involving ordinal definability and Vopěnka-style forcings are very susceptible to hidden mistakes which lead to major errors.

To avoid frequent repetition, one makes the following definition.

Definition 4.1. A set Z is a parameter set if and only if $Z \in \mathrm{OD}_Z$ and there is an OD_Z wellordering of Z.

[9]Typically, one would use $\mathrm{Coll}(\omega, <\kappa)$ where κ is an inaccessible cardinal of an inner model of ZFC and κ is countable in the real world. This is the forcing used in [13] Theorem 5.7. Such an inaccessible cardinal exists under AD. However, the hypothesis of Theorem 3.11 cannot prove such an inaccessible exists. For instance, in the minimal Solovay model: Let κ be the least inaccessible cardinal of L. Let $G \subseteq \mathrm{Coll}(\omega, <\kappa)$ be generic over L. Then the Solovay model $L(\mathbb{R}^{L[G]})$ has the property that no countable ordinal is inaccessible in any inner model of ZFC.

After fixing a wellordering $\prec$ of Z which is OD_Z, one can defined a global wellordering of HOD_Z. Thus $\mathrm{HOD}_Z \models \mathsf{ZFC}$. This global wellordering will often be called the canonical wellordering of HOD_Z. Note also that there is a finite $F \subseteq Z$ so that $\prec \in \mathrm{OD}_F$. With $\prec$, all other elements of Z can be defined by referring to their rank in $\prec$. Thus $\mathrm{HOD}_Z = \mathrm{HOD}_F$.

Definition 4.2. Recall that $\mathscr{B}_\infty$ is the class of all ∞-Borel codes. (That is, $p \in \mathscr{B}_\infty$ if and only if p is a pair (S, φ) where S is a set of ordinals and φ is a $\mathscr{L}_{\mathrm{set}}$-formula.)

Let κ be an ordinal and Z be a parameter set. Let $n \in \omega$. Let $\mathcal{BC}_n^{\kappa;Z} \subseteq \mathscr{B}_\infty$ be the proper class of ∞-Borel codes $p = (S, \varphi)$ which belong to HOD_Z (that is, $S \in \mathrm{HOD}_Z$) and $\mathfrak{B}_p^{{}^n({}^\omega\kappa)} = \{y \in {}^n({}^\omega\kappa) : L[S, y] \models \varphi(S, y)\} \neq \emptyset$. (If $n = 0$, then ${}^0({}^\omega\kappa) = \{\emptyset\}$.) For each $n \in \omega$, define a proper class equivalence relation $\sim_n^{\kappa;Z}$ on $\mathcal{BC}_n^{\kappa;Z}$ by $p \sim_n^{\kappa;Z} q$ if and only if $\mathfrak{B}_p^{{}^n({}^\omega\kappa)} = \mathfrak{B}_q^{{}^n({}^\omega\kappa)}$. If $p \in \mathcal{BC}_n^{\kappa;Z}$, let $[p]_{\sim_n^{\kappa;Z}}$ be the $\sim_n^{\kappa;Z}$-equivalence class containing p (which is a proper class). Let $\mathbb{B}_n^{\kappa;Z}$ be the collection of all $p \in \mathcal{BC}_n^{\kappa;Z}$ such that p is the minimal element of $[p]_{\sim_n^{\kappa;Z}}$ according to the canonical wellordering of HOD_Z. (So all elements of $\mathbb{B}_n^{\kappa;Z}$ are ∞-Borel codes.) For $p, q \in \mathbb{B}_n^{\kappa;Z}$, define $p \leq_{\mathbb{B}_n^{\kappa;Z}} q$ if and only if $\mathfrak{B}_p^{{}^n({}^\omega\kappa)} \subseteq \mathfrak{B}_q^{{}^n({}^\omega\kappa)}$. Let $1_{\mathbb{B}_n^{\kappa;Z}}$ be the unique $p \in \mathbb{B}_n^{\kappa;Z}$ such that $\mathfrak{B}_p^{{}^n({}^\omega\kappa)} = {}^n({}^\omega\kappa)$. Note that $\mathbb{B}_n^{\kappa;Z}, \leq_{\mathbb{B}_n^{\kappa;Z}}, 1_{\mathbb{B}_n^{\kappa;Z}} \in \mathrm{HOD}_Z$. Let $\mathbb{B}_n^{\kappa;Z}$ also be used to refer to the forcing poset $(\mathbb{B}_n^{\kappa;Z}, \leq_{\mathbb{B}_n^{\kappa;Z}}, 1_{\mathbb{B}_n^{\kappa;Z}})$. $\mathbb{B}_n^{\kappa;Z}$ is called the n-dimensional OD_Z ∞-Borel code forcing on ${}^\omega\kappa$.

This forcing resembles the classical Vopěnka forcing of non-empty OD_Z subsets of ${}^n({}^\omega\kappa)$ defined later in Definition 4.8. The ∞-Borel code forcing will be very important tool in this paper, but the classical Vopěnka forcing will be needed in a few places to establish properties of the ∞-Borel code forcing. In many important cases in natural models of AD^+, these two forcings are the same but only after a careful analysis of ∞-Borel code.

Definition 4.3. Let κ be an ordinal. Let Z be a parameter set. For $0 \leq i < n$, $j \in \omega$, and $\alpha \in \kappa$, let $A_{n,i,j,\alpha} = \{y \in {}^n({}^\omega\kappa) : y(i)(j) = \alpha\}$ which clearly has an OD_Z ∞-Borel code. Let $p_{n,i,j,\alpha}^{\kappa;Z} \in \mathbb{B}_n^{\kappa;Z}$ be such that $A_{n,i,j,\alpha} = \mathfrak{B}_{p_{n,i,j,\alpha}^{\kappa;Z}}^{{}^n({}^\omega\kappa)}$. Let $\dot{\tau}^n = \{((i,j,\alpha), p_{n,i,j,\alpha}^{\kappa;Z}) : i < n \wedge j \in \omega \wedge \alpha \in \kappa\}$.[10] $\dot{\tau}^n$ is the canonical name for adding the generic element of ${}^n({}^\omega\kappa)$. For $i < n$, let $\dot{\tau}_i^n$ be the canonical name for the element of ${}^\omega\kappa$ which is the i^{th} term of $\dot{\tau}^n$. Note that $\dot{\tau}^n, \dot{\tau}_i^n \in \mathrm{HOD}_Z$.

Definition 4.4. If $\mathcal{J} \subseteq \mathscr{B}_\infty$ is a set of ∞-Borel codes, κ is an ordinal, and $x \in {}^n({}^\omega\kappa)$, then define $\mathcal{G}^{\kappa,n}(\mathcal{J}, x) = \{p \in \mathcal{J} : x \in \mathfrak{B}_p^{{}^n({}^\omega\kappa)}\}$.

Let Z be a parameter set. (Note that $\mathbb{B}_n^{\kappa;Z} \subseteq \mathscr{B}_\infty$.) Let $G_x^{\kappa;Z;n} = \mathcal{G}^{\kappa;n}(\mathbb{B}_n^{\kappa;Z}, x) = \{p \in \mathbb{B}_n^{\kappa;Z} : x \in \mathfrak{B}_p^{{}^n({}^\omega\kappa)}\}$. $G_x^{\kappa;Z;n}$ is the $\mathbb{B}_n^{\kappa;Z}$-filter derived from x.

Definition 4.5. Let κ be a cardinal. Let Z be a parameter set. Uniformly let $\hat{\mathbb{B}}^{\kappa;Z}$ be a set of ordinals in HOD_Z which codes $\mathbb{B}_n^{\kappa;Z}$, every $p \in \mathbb{B}_n^{\kappa;Z}$, $\leq_{\mathbb{B}_n^{\kappa;Z}}$, $1_{\mathbb{B}_n^{\kappa;Z}}$, $p_{n,i,j,\alpha}^{\kappa;Z}$, $\dot{\tau}^n$, and $\dot{\tau}_i^n$ for all $j, n \in \omega$ and $i < n$.

This coding can be done in a variety of ways to maintain the following important properties. Uniformly in Z means there is a fixed simple procedure that takes Z and returns $\hat{\mathbb{B}}^{\kappa;Z}$. From $\hat{\mathbb{B}}^{\kappa;Z}$, one should be able to extract all the conditions of $\mathbb{B}_n^{\kappa;Z}$ for all $n \in \omega$. The main property of $\hat{\mathbb{B}}^{\kappa;Z}$ is that in any inner model $M \subseteq \mathrm{HOD}_Z$ with $\hat{\mathbb{B}}^{\kappa;Z} \in M$, one can perform the relevant arguments which involve forcing with $\mathbb{B}_n^{\kappa;Z}$. Moreover, for any such inner model M and $y \in {}^n({}^\omega\kappa)$, $G_y^{\kappa;Z;n} = \mathcal{G}^{\kappa;z}(\mathbb{B}_n^{\kappa;Z}, y)$ can be defined inside of $M[y]$.

Fact 4.6. *Let κ be an ordinal, Z be a parameter set, and $n \in \omega$. Let M be an inner model of HOD_Z with $\hat{\mathbb{B}}^{\kappa;Z} \in M$. Let $y \in {}^n({}^\omega\kappa)$. Then $G_y^{\kappa;Z;n}$ is $\mathbb{B}_n^{\kappa;Z}$-generic over M and $\dot{\tau}^n[G_y^{\kappa;Z;n}] = y$. (In particular, this result applies to $M = \mathrm{HOD}_Z$.)*

Proof. Let $A \in M$ be a maximal antichain in M. Now work in the real world. Since $A \in M \subseteq \mathrm{HOD}_Z$, let $\langle (S_\alpha, \varphi_\alpha) : \alpha < \delta \rangle$ for some ordinal δ be the enumeration of A according to the canonical global wellordering of HOD_Z. Let $T = \{\langle \alpha, \beta \rangle : \alpha < \delta \wedge \beta \in S_\alpha\}$ where $\langle \alpha, \beta \rangle$ refers to the Gödel pairing function. Note that $T \in \mathrm{HOD}_Z$ and one can use T to create an OD_Z ∞-Borel code q_0 such that $\mathfrak{B}_{q_0}^{{}^n({}^\omega\kappa)} = \bigcup \{\mathfrak{B}_p^{{}^n({}^\omega\kappa)} : p \in A\}$. For the sake of contradiction, suppose $A \cap G_y^{\kappa;Z;n} = \emptyset$. Then $y \notin \mathfrak{B}_{q_0}^{{}^n({}^\omega\kappa)}$. From q_0, one can create an OD_Z ∞-Borel code q_1 so that $\mathfrak{B}_{q_1}^{{}^n({}^\omega\kappa)} = {}^\omega\kappa \setminus \mathfrak{B}_{q_0}^{{}^n({}^\omega\kappa)}$ which is a nonempty set as it contains y. Thus there is a

[10] $\dot{\tau}^n \in \mathbb{B}_n^{\kappa;Z}$ is formally a name for a subset of $n \times \omega \times \kappa$ but will be considered as adding the associated element of ${}^n({}^\omega\kappa)$.

$q \in \mathbb{B}_n^{\kappa;Z}$ such that $\mathfrak{B}_q^{n(^\omega\kappa)} = \mathfrak{B}_{q_1}^{n(^\omega\kappa)}$. Now return to M. Since $\hat{\mathbb{B}}^{\kappa;Z} \in M$, every condition of $\mathbb{B}_n^{\kappa;Z}$ belongs to M (in particular, $q \in M$) and the forcing ordering $\leq_{\mathbb{B}_n^{\kappa;Z}}$ belongs to M. Then M sees that $A \cup \{q\} \in M$ and is a larger antichain than A which contradicts the maximality of A. The second statement is clear. $\square$

One important advantage of $\mathbb{B}_n^{\kappa;Z}$ over the analogous classical Vopěnka forcing is the following fact.

Fact 4.7. *Let κ be an ordinal, Z be a parameter set, $n \in \omega$, and $y \in {}^n({}^\omega\kappa)$. Let M be an inner model of HOD_Z with $\hat{\mathbb{B}}^{\kappa;Z} \in M$. Then $M[y] = M[G_y^{\kappa;Z;n}]$. (In particular, $\mathrm{HOD}_Z[y] = \mathrm{HOD}[G_y^{\kappa;Z;n}]$.)*

Proof. Since $M \subseteq M[G_y^{\kappa;Z;n}]$, $\dot{\tau}^n \in M$ (since $\hat{\mathbb{B}}^{\kappa;Z} \in M$), and $y = \dot{\tau}^n[G_y^{\kappa;Z;n}] \in M[G_y^{\kappa;Z;n}]$, one has that $M[y] \subseteq M[G_y^{\kappa;Z;n}]$. Let $p \in \mathbb{B}_n^{\kappa;Z}$ and takes the form (S, φ) where S is a set of ordinals in HOD_Z and φ is a formula. $p \in G_y^{\kappa;Z;n}$ if and only if $L[S,y] \models \varphi(S,y)$ if and only if $M[y] \models L[S,y] \models \varphi(S,y)$ by the absoluteness of the satisfaction relation of $L[S,y]$. Thus $G_y^{\kappa;Z;n} \in M[y]$ by comprehension in M. This implies $M[G_y^{\kappa;Z;n}] \subseteq M[y]$. $\square$

The following is the Vopěnka forcing (on ω-sequences of ordinals) which will play a small but necessary role in this paper.

Definition 4.8. Let κ be an ordinal, $U \subseteq {}^\omega\kappa$, and Z be a parameter set with $U \in Z$. Let $n \in \omega$. Since Z is a parameter set, let $\prec$ be an OD_Z wellordering of Z. For each formula, tuple of ordinals, and tuple from Z (which will identified with an ordinal by its rank in $\prec$), one can associate a OD_Z set. In this way, there is a OD_Z definable class surjection $\mathfrak{S}_Z : \mathrm{ON} \to \mathrm{OD}_Z$. Let $\mathbb{V}_n^{U;Z}$ be the set of ordinals α such that $\mathfrak{S}_Z(\alpha)$ is a nonempty subset of ${}^nU$ and for all $\alpha' < \alpha$, $\mathfrak{S}_Z(\alpha') \neq \mathfrak{S}_Z(\alpha)$. Note that $\mathbb{V}_n^{U;Z} \in \mathrm{HOD}_Z$. For each $p, q \in \mathbb{V}_n^{U;Z}$, let $p \leq_{\mathbb{V}_n^{U;Z}} q$ if and only if $\mathfrak{S}_Z(p) \subseteq \mathfrak{S}_Z(q)$. Also note that $\leq_{\mathbb{V}_n^{U;Z}} \in \mathrm{HOD}_Z$. Let $1_{\mathbb{V}_n^{U;Z}}$ be the least ordinal α so that $\mathfrak{S}_Z(\alpha) = {}^nU$. $\mathbb{V}_n^{U;Z} = (\mathbb{V}_n^{U;Z}, \leq_{\mathbb{V}_n^{U;Z}}, 1_{\mathbb{V}_n^{U;Z}})$ is the n-dimensional OD_Z Vopěnka-forcing on U and $\mathbb{V}_n^{U;Z} \in \mathrm{HOD}_Z$.

If $y \in {}^nU$, let $\tilde{G}_y^{U;Z;n} = \{p \in \mathbb{V}_n^{U;Z} : y \in \mathfrak{S}_Z(p)\}$. This is the $\mathbb{V}_n^{U;Z}$-filter associated to y.

For $0 \leq i < n$, $j \in \omega$, and $\alpha \in \kappa$, let $A_{n,i,j,\alpha} = \{y \in {}^nU : y(i)(j) = \alpha\}$ which is clearly OD_Z. If $A_{n,i,j,\alpha} \neq \emptyset$, then let $p_{n,i,j,\alpha}^{U;Z} \in \mathbb{V}_n^{U;Z}$ be such that $\mathfrak{S}_Z(p_{n,i,j,\alpha}^{U;Z}) = A_{n,i,j,\alpha}$. Let $\dot{\tau}^n = \{((i,j,\alpha), p_{n,i,j,\alpha}^{U;Z}) : i < n \wedge j \in \omega \wedge \alpha \in \kappa \wedge A_{n,i,j,\alpha} \neq \emptyset\}$. $\dot{\tau}^n$ is the canonical name for adding the generic element of ${}^n({}^\omega\kappa)$. For $i < n$, let $\dot{\tau}_i^n$ be the canonical name for the element of ${}^\omega\kappa$ which is the i^{th} term of $\dot{\tau}^n$. Note that $\tau^n, \dot{\tau}_i^n \in \mathrm{HOD}_Z$. (The notation from the ∞-Borel code forcing has been reused, but this should not cause confusion in context.)

Let $\hat{\mathbb{V}}^{U;Z} \in \mathrm{HOD}_Z$ be a set of ordinals which codes all the conditions of $\mathbb{V}_n^{U;Z}$ for all $n \in \omega$, the set $\{(n,i,j,\alpha) : j,n \in \omega \wedge i < n \wedge \alpha \in \kappa \wedge A_{n,i,j,\alpha} \neq \emptyset\}$, the map $\langle p_{n,i,j,\alpha}^{U;Z} : j,n \in \omega \wedge i < n \wedge \alpha \in \kappa \wedge A_{n,i,j,\alpha} \neq \emptyset\rangle$.

Fact 4.9. *Let κ be an ordinal, $U \subseteq {}^\omega\kappa$, Z is a parameter set such that $U \in Z$, and $n \in \omega$. Let M be an inner model of HOD_Z with $\hat{\mathbb{V}}^{U;Z} \in M$. Let $y \in {}^nU$. Then $\tilde{G}_y^{U;Z;n}$ is $\mathbb{V}_n^{U;Z}$-generic over M and $\dot{\tau}^n[\tilde{G}_y^{U;Z;n}] = y$.*

Proof. Let $A \in M$ be a maximal antichain in M. Since $M \subseteq \mathrm{HOD}_Z$, $A \in \mathrm{OD}_Z$. Thus $\bigcup\{\mathfrak{S}_Z(p) : p \in A\}$ is an OD_Z subset of ${}^nU$. If $\tilde{G}_y^{U;Z;n} \cap A = \emptyset$, then $y \notin \bigcup\{\mathfrak{S}_Z(p) : p \in A\}$. So ${}^nU \setminus \bigcup\{\mathfrak{S}_Z(p) : p \in A\}$ is a nonempty OD_Z subset of ${}^nU$ since $y \in {}^nU$ and $U \in Z$. Thus there is a $p \in \mathbb{V}_n^{U;Z}$ so that $\mathfrak{S}_Z(p) = {}^nU \setminus \bigcup\{\mathfrak{S}_Z(p) : p \in A\}$. Then $A \cup \{p\}$ is a $\mathbb{V}_n^{U;Z}$-antichain which is larger than A. Contradiction. Thus there is some $p \in A$ so that $y \in \mathfrak{S}_Z(p)$. Thus $\tilde{G}_y^{U;Z;n} \cap A \neq \emptyset$. This shows that $\tilde{G}_y^{U;Z;n}$ is $\mathbb{V}_n^{U;Z}$-generic over M.

Since $\hat{\mathbb{V}}^{U;Z} \in M$, $\dot{\tau}^n \in M$. For all $i < n$, $j \in \omega$, and $\alpha \in \kappa$, $y(i)(j) = \alpha$ if and only if $y \in \mathfrak{S}_Z(p_{n,i,j,\alpha}^{U;Z})$ if and only if $p_{n,i,j,\alpha}^{U;Z} \in \tilde{G}_y^{U;Z;n}$. Thus $\dot{\tau}^n[\tilde{G}_y^{U;Z;n}] = y$. $\square$

As mentioned before Fact 4.7, $\mathbb{V}_n^{U;Z}$ is not as absolute as $\mathbb{B}_n^{\kappa;Z}$. Given $y \in {}^nU$ and $p \in \mathbb{V}_n^{U;Z}$, it is not clear how $M[y]$ can determine if $p \in \tilde{G}_y^{U;Z;n}$ or equivalently whether $y \in \mathfrak{S}_Z(p)$ since this requires knowing the truth about y, Z, and ordinals in the real world. So $M[y] \subseteq M[\tilde{G}_y^{U;Z;n}]$ but it is not clear if they are equal in general.

Definition 4.10. Let X be a set. Let $\mathscr{P}_{\omega_1}(X) = \{\sigma \in \mathscr{P}(X) : |\sigma| \leq \omega\}$, i.e. the set of all subsets σ of X such that ω surjects onto σ. Let ν be an ultrafilter on $\mathscr{P}_{\omega_1}(X)$.

- ν is a fine ultrafilter on $\mathscr{P}_{\omega_1}(X)$ if and only if for each $x \in X$, $A_x = \{\sigma \in \mathscr{P}_{\omega_1}(\kappa) : x \in \sigma\} \in \nu$.

- ν is a normal ultrafilter on $\mathscr{P}_{\omega_1}(X)$ if and only if for every function $\Phi : \mathscr{P}_{\omega_1}(X) \to \mathscr{P}_{\omega_1}(X)$ such that $\{\sigma \in \mathscr{P}_{\omega_1}(X) : \emptyset \neq \Phi(\sigma) \subseteq \sigma\} \in \nu$, there is an $x \in X$ so that $\{\sigma \in \mathscr{P}_{\omega_1}(X) : x \in \Phi(\sigma)\} \in \nu$.
- ν is a supercompact measure on $\mathscr{P}_{\omega_1}(X)$ if and only if ν is an ω_1-complete, fine, and normal ultrafilter on $\mathscr{P}_{\omega_1}(X)$. If there is a supercompact measure on X, then one says that ω_1 is X-supercompact.

If all sets of reals have the Baire property, then there are no nonprincipal ultrafilters on ω which implies ultrafilters on any set must be ω_1-complete. Thus AD implies all ultrafilters on any set are ω_1-complete.

Let $\leq_{\mathsf{Turing}}$ denote Turing reduction on $\mathbb{R}$ defined by $x \leq_{\mathsf{Turing}} y$ if and only if x is Turing computable from y. Let $\equiv_{\mathsf{Turing}}$ denote Turing equivalence on $\mathbb{R}$ defined by $x \equiv_{\mathsf{Turing}} y$ if and only if $x \leq_{\mathsf{Turing}} y$ and $y \leq_{\mathsf{Turing}} x$. Let $\mathcal{D}_{\mathsf{Turing}}$ be $\mathbb{R}/\equiv_{\mathsf{Turing}}$ which is the collection of Turing degrees (equivalence classes of $\equiv_{\mathsf{Turing}}$). One can extend $\leq_{\mathsf{Turing}}$ to $\mathcal{D}_{\mathsf{Turing}}$ by $X \leq_{\mathsf{Turing}} Y$ if and only if there exist $x \in X$ and $y \in Y$ so that $x \leq_{\mathsf{Turing}} y$. For $X \in \mathcal{D}_{\mathsf{Turing}}$, let $\mathcal{C}^X_{\mathsf{Turing}} = \{Y \in \mathcal{D}_{\mathsf{Turing}} : X \leq_{\mathsf{Turing}} Y\}$ which is called the Turing cone above X. Define the Martin filter on $\mathcal{D}_{\mathsf{Turing}}$ by $A \in \mu_{\mathsf{Turing}}$ if and only if there is an $X \in \mathcal{D}_{\mathsf{Turing}}$ such that $\mathcal{C}^X_{\mathsf{Turing}} \subseteq A$. Turing determinacy is the assertion that μ_{Turing} is an ultrafilter. Martin showed that under AD, μ_{Turing} is an ω_1-complete ultrafilter and thus μ_{Turing} is often called the Martin measure in this setting. Peng and Yu ([47]) showed even Turing determinacy alone implies $\mathsf{AC}^{\mathbb{R}}_\omega$ and hence μ_{Turing} is ω_1-complete.

Suppose K is a set which is a surjective image of $\mathbb{R}$. Let $\pi : \mathbb{R} \to K$ be a surjection. Let $\tilde{\pi} : \mathcal{D}_{\mathsf{Turing}} \to \mathscr{P}_{\omega_1}(K)$ be defined by $\tilde{\pi}(X) = \{\pi(z) : [z]_{\equiv_{\mathsf{Turing}}} \leq_{\mathsf{Turing}} X\}$. Let $\tilde{\pi}_*\mu_{\mathsf{Turing}}$ be the Rudin-Keisler pushforward of $\tilde{\pi}$ which is an ultrafilter on $\mathscr{P}_{\omega_1}(K)$ defined by $A \in \tilde{\pi}_*\mu_{\mathsf{Turing}}$ if and only if $\tilde{\pi}^{-1}[A] \in \mu_{\mathsf{Turing}}$. Let $x \in K$ and pick an $r \in \mathbb{R}$ such that $\pi(r) = x$. Let $A_x = \{\sigma \in \mathscr{P}_{\omega_1}(K) : x \in \sigma\}$. Let $R = [r]_{\equiv_{\mathsf{Turing}}}$. For all $Y \in \mathcal{C}_R^{\mathsf{Turing}}$, $x \in \tilde{\pi}(Y)$. This shows that $\mathcal{C}_R^{\mathsf{Turing}} \subseteq \tilde{\pi}^{-1}[A_x]$ and hence $A_x \in \pi_*\mu_{\mathsf{Turing}}$. $\pi_*\mu_{\mathsf{Turing}}$ is a fine measure on $\mathscr{P}_{\omega_1}(K)$. It has been shown that under AD, for every set K which is a surjective image of $\mathbb{R}$, there is an ω_1-complete fine measure on $\mathscr{P}_{\omega_1}(K)$.

If $\prod_{\mathcal{D}_{\mathsf{Turing}}} V/\mu_{\mathsf{Turing}}$ is wellfounded, then $\prod_{\mathscr{P}_{\omega_1}(K)} V/\tilde{\pi}_*\mu_{\mathsf{Turing}}$ is wellfounded. To see this: Define $\Psi : \prod_{\mathscr{P}_{\omega_1}(K)} V/\tilde{\pi}_*\mu_{\mathsf{Turing}} \to \prod_{\mathcal{D}_{\mathsf{Turing}}} V/\mu_{\mathsf{Turing}}$ by $\Psi([f]_{\tilde{\pi}_*\mu_{\mathsf{Turing}}}) = [f \circ \tilde{\pi}]_{\mu_{\mathsf{Turing}}}$. Ψ is a $\dot{\in}$-embedding with each ultrapower structure given the ultrapower interpretation of $\dot{\in}$. Thus it has been shown under AD that assuming $\prod_{\mathcal{D}_{\mathsf{Turing}}} V/\mu_{\mathsf{Turing}}$ is wellfounded, then there is an ω_1-complete fine measure ν on $\mathscr{P}_{\omega_1}(K)$ such that $\prod_{\mathscr{P}_{\omega_1}(K)} V/\nu$ is wellfounded. If DC holds, then $\prod_{\mathcal{D}_{\mathsf{Turing}}} V/\mu_{\mathsf{Turing}}$ is wellfounded. Solovay [55] 12.2 showed that the consistency of $\mathsf{AD}_{\mathbb{R}}$ implies the consistency of $\mathsf{AD} + \neg\mathsf{DC}$. Moreover Solovay [55] Lemma 2.3 states that if $\mathrm{cof}(\Theta) = \omega$, then DC fails. Solovay ([42] Theorem 9.2) showed that $\prod_{\mathcal{D}_{\mathsf{Turing}}} V/\mu_{\mathsf{Turing}}$ is wellfounded implies $\mathsf{DC}_{\mathbb{R}}$. Woodin ([56] Corollary 6) showed that AD^+ implies $\prod_{\mathcal{D}_{\mathsf{Turing}}} V/\mu_{\mathsf{Turing}}$ is wellfounded. This latter result of Woodin will not be needed here since one will work in the real world satisfying AD and $\mathsf{DC}_{\mathbb{R}}$ and drop into an inner model of determinacy satisfying DC.

To be explicit, if K is a set, then $L(K)$ is Gödel relative hierarchy built over the transitive closure of $\{K\}$. Unlike $L[K]$ which always satisfies AC, $L(K)$ may not satisfy AC. If $\{K_0, ..., K_{n-1}\}$ is finite, then one will write $L(K_1, ..., K_n)$ for $L(\{K_1, ..., K_n\})$.

If $\kappa \in \mathrm{ON}$ and $m \leq n < \omega$, then let $\pi^\kappa_{n,m} : {}^n({}^\omega\kappa) \to {}^m({}^\omega\kappa)$ be the projection onto the first m coordinates defined by $\pi^\kappa_{n,m}(\ell) = \ell \upharpoonright m$ for all $\ell \in {}^n({}^\omega\kappa)$. The following states that projections of ∞-Borel sets are ∞-Borel in a uniform manner. In various forms, this result is due to Woodin and Ikegami-Trang ([25] Claim 2 in the proof Theorem 5). A detailed proof in the form needed here will be provided and uniformity will be very important.

Fact 4.11. *(Woodin; Ikegami-Trang) Assume boldface* GCH *at* ω. *Let* κ *be an ordinal and assume there is a fine* ω_1*-complete measure* ν *on* $\mathscr{P}_{\omega_1}({}^\omega\kappa)$ *such that* $\prod_{\mathscr{P}_{\omega_1}({}^\omega\kappa)} V/\nu$ *is wellfounded. Then there is an* $\mathrm{OD}_{\{\nu\}}$ *function* Υ *so that for all* $m \leq n < \omega$ *and* ∞*-Borel code* b, $\pi^\kappa_{n,m}[\mathfrak{B}_b^{{}^n({}^\omega\kappa)}] = \mathfrak{B}_{\Upsilon(b,n,m)}^{{}^m({}^\omega\kappa)}$.

Proof. Without loss of generality suppose $m = 1$ and $n = 2$. Let b be an ∞-Borel code of the form $b = (S, \varphi)$. For any $\sigma \in \mathscr{P}_{\omega_1}({}^\omega\kappa)$, let $\mathbb{Q}_\sigma = (\mathbb{V}_2^{\sigma;\{b,\sigma\}})^{L(b,\sigma)}$ denote the 2-dimensional $\mathrm{OD}_{\{b,\sigma\}}$ Vopěnka forcing on σ defined in the inner model $L(b, \sigma)$. All sets in $L(b, \sigma)$ are ordinal definable from b, σ, and finitely many members from the countable set σ. After fixing a bijection of ω onto σ from the real world, one can construct externally a global wellordering of $L(b, \sigma)$ in the real world even though internally $L(b, \sigma)$ may fail to satisfy AC. Since σ is countable in the real world, let $B_0 : \sigma \to \omega$ be a bijection in the real world. Let $B_1 : \mathscr{P}(\sigma)^{L(b,\sigma)} \to \mathscr{P}(\omega)$ be an injection defined by $B_1(A) = B_0[A]$. Since $L(\sigma, b)$ has a global wellordering in the real world, $\mathscr{P}(\sigma)^{L(b,\sigma)}$ is wellorderable in the real world. Thus $B_1[\mathscr{P}(\sigma)^{L(b,\sigma)}]$ is a wellorderable subset of $\mathscr{P}(\omega)$ in the real world. Since boldface GCH at ω holds in the real world, $B_1[\mathscr{P}(\sigma)^{L(b,\sigma)}]$ is countable and hence $\mathscr{P}(\sigma)^{L(b,\sigma)}$ is countable in the real world. Let $\mathfrak{S}^{L(b,\sigma)}_{\{b,\sigma\}} : \mathrm{ON} \to \mathrm{OD}^{L(b,\sigma)}_{\{b,\sigma\}}$ be the surjection used to define $(\mathbb{V}_2^{\sigma;\{b,\sigma\}})^{L(b,\sigma)}$ (as in Definition 4.8). Then since the restriction $\mathfrak{S}^{L(b,\sigma)}_{\{b,\sigma\}} : (\mathbb{V}_2^{\sigma;\{b,\sigma\}})^{L(b,\sigma)} \to \mathscr{P}(\sigma)^{L(b,\sigma)}$ is an injection (note that $\mathbb{V}_2^{\sigma;\{b,\sigma\}} \subseteq \mathrm{ON}$ as in Definition 4.8), $\mathbb{Q}_\sigma$ is countable. Let $B_2 : \mathbb{Q}_\sigma \to \omega$ be a bijection. Let $B_3 : \mathscr{P}(\mathbb{Q}_\sigma)^{L(b,\sigma)} \to \mathscr{P}(\omega)$ be defined by $B_3(A) = B_2[A]$ which is a injection. Again by boldface GCH at ω, $\mathscr{P}(\mathbb{Q}_\sigma)^{L(b,\sigma)}$ is countable in the real world. Let $\mathbb{C}_\sigma = \mathrm{Coll}(\omega, \mathbb{Q}_\sigma)$ which is the forcing of partial functions from ω into $\mathbb{Q}_\sigma$ ordered by reverse extension. Since $\mathbb{Q}_\sigma$ is countable, $\mathbb{C}_\sigma$ is countable. Let $\hat{\mathbb{V}}^b_\sigma = (\hat{\mathbb{V}}^{\sigma;\{b,\sigma\}})^{L(b,\sigma)}$ which is the code set from Definition 4.8. Note that $\mathbb{Q}_\sigma, \mathbb{C}_\sigma \in L[b, \hat{\mathbb{V}}^b_\sigma]$.

By the fineness of ν, $\{\sigma \in \mathscr{P}_{\omega_1}({}^\omega\kappa) : x \in \sigma\} \in \nu$ for all $x \in {}^\omega\kappa$. Let $\prec_\sigma$ denote the constructibility wellordering of $L[b, \hat{\mathbb{V}}^b_\sigma]$. For any $x \in {}^\omega\kappa$, let $\prec_{\sigma,x}$ be the constructibility wellordering[11] on $L[b, \hat{\mathbb{V}}^b_\sigma, x]$. Using the maps $\sigma \mapsto \prec_\sigma$ and $\sigma \mapsto \prec_{\sigma,x}$ in the usual proof of Łoś's theorem, one has that $\prod_{\sigma \in \mathscr{P}_{\omega_1}({}^\omega\kappa)} L[b, \hat{\mathbb{V}}^b_\sigma]/\nu$ and $\prod_{\sigma \in \mathscr{P}_{\omega_1}({}^\omega\kappa)} L[b, \hat{\mathbb{V}}^b_\sigma, x]/\nu$ for all $x \in {}^\omega\kappa$ satisfy Łoś' theorem. By the hypothesis that $\prod_{\mathscr{P}_{\omega_1}({}^\omega\kappa)} V/\nu$ is wellfounded, one has that $\prod_{\sigma \in \mathscr{P}_{\omega_1}({}^\omega\kappa)} L[b, \hat{\mathbb{V}}^b_\sigma]/\nu$ and $\prod_{\sigma \in \mathscr{P}_{\omega_1}({}^\omega\kappa)} L[b, \hat{\mathbb{V}}^b_\sigma, x]/\nu$ for all $x \in {}^\omega\kappa$ are wellfounded. For any $v \in V$, let $c_v : \mathscr{P}_{\omega_1}({}^\omega\kappa) \to \{v\}$ be the constant function. Let $j_\nu : V \to \prod_{\mathscr{P}_{\omega_1}({}^\omega\kappa)} V/\nu$ be defined by $j_\nu(v) = [c_v]_\nu$. (Note that $\prod_{\mathscr{P}_{\omega_1}({}^\omega\kappa)} V/\nu$ may not satisfy ZF and j_ν may not be elementary since Łoś' theorem may fail for this ultrapower.) Define the following functions on $\mathscr{P}_{\omega_1}({}^\omega\kappa)$: $\Phi_{\mathbb{Q}}(\sigma) = \mathbb{Q}_\sigma$, $\Phi_{\mathbb{C}}(\sigma) = \mathbb{C}_\sigma$, $\Phi_{\hat{\mathbb{V}}^b}(\sigma) = \hat{\mathbb{V}}^b_\sigma$. Let $\mathbb{Q}_\infty = [\Phi_{\mathbb{Q}}]_\nu$, $\mathbb{C}_\infty = [\Phi_{\mathbb{C}}]_\nu$, and $\hat{\mathbb{V}}^b_\infty = [\Phi_{\hat{\mathbb{V}}^b}]_\nu$. Let $b_\infty = j_\nu(b)$ and $S_\infty = j_\nu(S)$ (where recall $b = (S, \varphi)$). By Lós theorem for $\prod_{\sigma \in \mathscr{P}_{\omega_1}({}^\omega\kappa)} L[b, \hat{\mathbb{V}}^b_\sigma]/\nu$, b_∞ is an ∞-Borel code of the form (S_∞, φ). Also by Łoś' theorem, $\hat{\mathbb{V}}^b_\infty$ is a set of ordinals and for all $x \in {}^\omega\kappa$, $\prod_{\sigma \in \mathscr{P}_{\omega_1}({}^\omega\kappa)} L[b, \hat{\mathbb{V}}^b_\sigma, x]/\nu = L[b_\infty, \hat{\mathbb{V}}^b_\infty, j_\nu(x)]$. $L[b_\infty, \hat{\mathbb{V}}^b_\infty, j_\nu(x)] \models$ "$j_\nu(x)$ is a function from $j_\nu(\omega)$ into $j_\nu(\kappa)$". Since ν is ω_1-complete, $j_\nu(\omega) = \omega$ and thus $j_\nu(x)$ is a function from ω into $j_\nu(\kappa)$. Thus by Łoś' theorem, for all $n \in \omega$, $j_\nu(x)(n) = j_\nu(x)(j_\nu(n)) = j_\nu(x(n))$. Thus $j_\nu(x) = j_\nu \circ x$ and hence $L[b_\infty, \hat{\mathbb{V}}^b_\infty, j_\nu(x)] = L[b_\infty, \hat{\mathbb{V}}^b_\infty, j_\nu \circ x]$. Let $j_* = j_\nu \upharpoonright \kappa = \{(\alpha, \beta) \in \kappa \times j_\nu[\kappa] : j(\alpha) = \beta\}$. Note that j_* is essentially an $\mathrm{OD}_{\{\nu\}}$ set of ordinals and $j_\nu(x) = j_\nu \circ x = j_* \circ x$. The maps $b \mapsto b_\infty$ and $b \mapsto \hat{\mathbb{V}}^b_\infty$ are $\mathrm{OD}_{\{\nu\}}$.

<u>Claim</u>: For all $x \in {}^\omega\kappa$,

$$x \in \pi^\kappa_{2,1}[\mathfrak{B}_b^{2({}^\omega\kappa)}] \Leftrightarrow L[b_\infty, \hat{\mathbb{V}}^b_\infty, j_* \circ x] \models 1_{\mathbb{C}_\infty} \Vdash_{\mathbb{C}_\infty} (\exists y)\Big(L[\check{S}_\infty, (j_* \circ x)\check{}, y] \models \varphi(\check{S}_\infty, (j_* \circ x)\check{}, y)\Big).$$

To see this claim: ($\Leftarrow$) Since $j_* \circ x = j_\nu(x)$, one has

$$L[b_\infty, \hat{\mathbb{V}}^b_\infty, j_\nu(x)] \models 1_{\mathbb{C}_\infty} \Vdash_{\mathbb{C}_\infty} (\exists y)\Big(L[\check{S}_\infty, j_\nu\check{(x)}, y] \models \varphi(\check{S}_\infty, j_\nu\check{(x)}, y)\Big).$$

Since $L[b_\infty, \hat{\mathbb{V}}^b_\infty, j_\nu(x)] = \prod_{\sigma \in \mathscr{P}_{\omega_1}({}^\omega\kappa)} L[b, \mathbb{V}^b_\sigma, x]/\nu$, one has that the following holds for ν-almost all σ by Łoś' theorem.

$$L[b, \hat{\mathbb{V}}^b_\sigma, x] \models 1_{\mathbb{C}_\sigma} \Vdash_{\mathbb{C}_\sigma} (\exists y)\Big(L[\check{S}, \check{x}, y] \models \varphi(\check{S}, \check{x}, y)\Big).$$

Fix such a σ. Since $\mathbb{C}_\sigma$ is countable in the real world and boldface GCH at ω holds, Fact 2.14 implies there is a $g \subseteq \mathbb{C}_\sigma$ (which belongs to the real world) and is $\mathbb{C}_\sigma$-generic over $L[b, \hat{\mathbb{V}}^b_\sigma, x]$. By the forcing theorem,

$$L[b, \hat{\mathbb{V}}^b_\sigma, x][g] \models (\exists y)\Big(L[S, x, y] \models \varphi(S, x, y)\Big).$$

There is a $y \in {}^\omega\kappa \cap L[b, \hat{\mathbb{V}}^b_\sigma, x][g]$ such that

$$L[b, \hat{\mathbb{V}}^b_\sigma, x][g] \models (L[S, x, y] \models \varphi(S, x, y)$$

and note that y belongs to the real world since g belonged to the real world. By absoluteness, $L[S, x, y] \models \varphi(S, x, y)$. Since $b = (S, \varphi)$, this implies that $(x, y) \in \mathfrak{B}_b^{2({}^\omega\kappa)}$. Hence $x \in \pi^\kappa_{2,1}[\mathfrak{B}_b^{2({}^\omega\kappa)}]$.

($\Rightarrow$) Now suppose $x \in \pi^\kappa_{2,1}[\mathfrak{B}_b^{2({}^\omega\kappa)}]$. There is a $z \in {}^\omega\kappa$ so that $(x, z) \in \mathfrak{B}_b^{2({}^\omega\kappa)}$. So $L[S, x, z] \models \varphi(S, x, z)$. By fineness of ν, there is an $A \in \nu$ so that $x, z \in \sigma$ for all $\sigma \in A$. Fix $\sigma \in A$. Let $H_{\sigma,x,z}$ be the $\mathbb{Q}_\sigma$-generic

[11] The constructibility ordering is not independent of the model $L[b, \hat{\mathbb{V}}^b_\sigma, x]$. To be explicit, let $\prec_{\sigma,x}$ be the constructibility ordering using predicate b, $\hat{\mathbb{V}}^b_\sigma$, and x.

associated to (x,z) which is $\mathbb{Q}_\sigma$-generic over $L[b,\hat{\mathbb{V}}^b_\sigma]$ by Fact 4.9 (note that $(x,z)\in {}^2\sigma$ and $L[b,\hat{\mathbb{V}}^b_\sigma]$ is an inner model of $\mathrm{HOD}^{L(\sigma,b)}_{\{b,\sigma\}}$). By absoluteness,

$$L[b,\hat{\mathbb{V}}^b_\sigma][H_{\sigma,x,z}]\models L[S,x,z]\models\varphi(S,x,z).$$

In $L[b,\mathbb{V}^b_\sigma]$, $\mathbb{Q}_\sigma$ regularly embeds into $\mathbb{C}_\sigma$ by [31] Corollary 26.8. There is some $g\subseteq\mathbb{C}_\sigma$ so that $H_{\sigma,x,z}\in L[b,\hat{\mathbb{V}}^b_\sigma][g]$. Again by absoluteness,

$$L[b,\hat{\mathbb{V}}^b_\sigma][g]\models L[S,x,z]\models\varphi(S,x,z).$$

Since $x\in L[b,\hat{\mathbb{V}}^b_\sigma][g]$, there is an $h\subseteq\mathbb{C}_\sigma$ such that h is $\mathbb{C}_\sigma$-generic over $L[b,\hat{\mathbb{V}}^b_\sigma][x]$ and $L[b,\hat{\mathbb{V}}^b_\sigma][g]=L[b,\hat{\mathbb{V}}^b_\sigma][x][h]$ by [31] Corollary 26.10. Thus one has

$$L[b,\hat{\mathbb{V}}^b_\sigma,x][h]\models L[S,x,z]\models\varphi(S,x,z).$$

Then one has

$$L[b,\hat{\mathbb{V}}^b_\sigma,x][h]\models(\exists y)\Big(L[S,x,y]\models\varphi(S,x,y)\Big).$$

By the forcing theorem, there is a $p\in\mathbb{C}_\sigma$ so that

$$L[b,\hat{\mathbb{V}}^b_\sigma,x]\models p\Vdash_{\mathbb{C}_\sigma}(\exists y)\Big(L[\check{S},\check{x},y]\models\varphi(\check{S},\check{x},y)\Big).$$

By the homogeneity of $\mathbb{C}_\sigma$, one has

$$L[b,\hat{\mathbb{V}}^b_\sigma,x]\models 1_{\mathbb{C}_\sigma}\Vdash_{\mathbb{C}_\sigma}(\exists y)\Big(L[\check{S},\check{x},y]\models\varphi(\check{S},\check{x},y)\Big).$$

Since σ was an arbitrary element of $A\in\nu$, one has the following by Łoś' theorem.

$$L[b_\infty,\hat{\mathbb{V}}^b_\infty,j_\nu(x)]\models 1_{\mathbb{C}_\infty}\Vdash_{\mathbb{C}_\infty}(\exists y)\Big(L[\check{S}_\infty,\check{j_\nu}(x),y]\models\varphi(\check{S}_\infty,\check{j_\nu}(x),y)\Big).$$

Since $j_\nu(x)=j_*\circ x$,

$$L[b_\infty,\hat{\mathbb{V}}^b_\infty,j_*\circ x]\models 1_{\mathbb{C}_\infty}\Vdash_{\mathbb{C}_\infty}(\exists y)\Big(L[\check{S}_\infty,(j_*\circ x)\check{},y]\models\varphi(\check{S}_\infty,(j_*\circ x)\check{},y)\Big).$$

This completes the proof of the claim.

Note that $j_*\circ x$ can be obtained from j_* and x. By the claim and absoluteness, $x\in\pi^\kappa_{2,1}[\mathfrak{B}_b^{2^{({}^\omega\kappa)}}]$ is equivalent to

$$L[b_\infty,\hat{\mathbb{V}}^b_\infty,j_*,x]\models L[b_\infty,\hat{\mathbb{V}}^b_\infty,j_*\circ x]\models 1_{\mathbb{C}_\infty}\Vdash_{\mathbb{C}_\infty}(\exists y)\Big(L[\check{S}_\infty,(j_*\circ x)\check{},y]\models\varphi(\check{S}_\infty,(j_*\circ x)\check{},y)\Big).$$

Let $\langle b_\infty,\hat{\mathbb{V}}^b_\infty,j_*\rangle$ be some uniform method of coding three sets of ordinals into one set of ordinals. Let ψ be a formula such that $\psi(\langle b_\infty,\hat{\mathbb{V}}^b_\infty,j_*\rangle,x)$ asserts the inner expression above (behind the first satisfaction symbol). $(\langle b_\infty,\hat{\mathbb{V}}^b_\infty,j_*\rangle,\psi)$ is an ∞-Borel code for $\pi^\kappa_{2,1}[\mathfrak{B}_b^{2^{({}^\omega\kappa)}}]$. Define $\Upsilon(b)=(\langle b_\infty,\hat{\mathbb{V}}^b_\infty,j_*\rangle,\psi)$. Υ is $\mathrm{OD}_{\{\nu\}}$ since it was observed above that $j_*\in\mathrm{HOD}_{\{\nu\}}$ and the mappings $b\mapsto b_\infty$ and $b\mapsto\hat{\mathbb{V}}^b_\infty$ are $\mathrm{OD}_{\{\nu\}}$. ☐

Remark 4.12. The following are some remark about the structure of the proof and important subtle points of Fact 4.11: In the direction ($\Leftarrow$), it was important that $\mathbb{C}_\sigma$ was countable in the real world in order to find $\mathbb{C}_\sigma$-generic filters over $L[b,\hat{\mathbb{V}}_\sigma,x]$. Note that in the real world, the cardinality of $\mathbb{C}_\sigma$ is the same cardinality as $\mathbb{Q}_\sigma$. In ($\Rightarrow$), the relevant property of $\mathbb{Q}_\sigma$ is that it needs to make the pair (x,z)-generic over the relevant inner model of ZFC. There are two forcings mentioned above with this property, the ∞-Borel code forcing and the Vopěnka forcing. The Vopěnka forcing and ∞-Borel code forcing defined in the real world are never countable. In a first attempt to make this forcing countable, one goes into $L(\sigma,b)$ which has an external global wellordering. If one was proving the Fact 4.11 for $\mathbb{R}$ or ${}^\omega\omega$, then the corresponding Vopěnka and ∞-Borel code forcing defined in $L(b,\sigma)$ are both already countable. (So one could have more simply used the ∞-Borel code forcing and even use the more concrete Martin's measure on the Turing degrees for $\mathbb{R}$ or ${}^\omega\omega$.) When $\kappa>\omega$, the Vopěnka and ∞-Borel code forcing on ${}^\omega\kappa$ defined in $L(b,\sigma)$ are still uncountable in the real world. To resolve this, one restricts the forcing to subsets of the countable set σ rather than all of ${}^\omega\kappa$. However, it is not clear if σ is ∞-Borel in $L(\sigma,\mathbb{R})$ with an $\mathrm{OD}^{L(b,\sigma)}_{\{b,\sigma\}}$ ∞-Borel code (or ∞-Borel at all). Thus it is unclear how one can create an ∞-Borel code forcing which only considers subsets of σ. This problem was resolved by instead using the Vopěnka forcing $\mathbb{Q}_\sigma=(\mathbb{V}_2^{\sigma;\{b,\sigma\}})^{L(b,\sigma)}$.

Definition 4.13. A set Z is a good parameter set if and only if Z if a parameter set and for all $\tilde{z}$ in the transitive closure of $\{Z\}$, there is an $r \in \mathbb{R}$ so that $\tilde{z}$ is $\mathrm{OD}_{Z \cup \{r\}}$.

Recall for any set K, $L(K)$ is the relativized constructibility starting with the transitive closure $\{K\}$. So in $L(Z, \mathbb{R})$, every set is ordinal definable from finitely many parameters from $\mathbb{R}$ and elements of the transitive closure of Z. If Z is a good parameter set, then this can be improved: every set in $L(Z, \mathbb{R})$ is ordinal definable from finitely many reals and finitely many elements of Z. The purpose of the concept of a good parameter set is for Fact 4.14 below.

Fact 4.14. *Assume* AD *and* $\mathsf{DC}_{\mathbb{R}}$. *Let* Z *be a set such that* $V = L(Z, \mathbb{R})$ *and* Z *is a good parameter set. Let* $\kappa < \Theta$. *Then there is an* OD_Z *function* Υ *so that for all* $m \leq n < \omega$ *and* ∞*-Borel code* p, $\pi^\kappa_{n,m}[\mathfrak{B}_p^{n(^\omega\kappa)}] = \mathfrak{B}^{m(^\omega\kappa)}_{\Upsilon(p,n,m)}$.

Proof. Fix $\kappa < \Theta$. Since $V = L(Z, \mathbb{R})$ and Z is a good parameter set, every set is $\mathrm{OD}_{Z\cup\{r\}}$ for some $r \in \mathbb{R}$ (by the discussion above). For each $r \in \mathbb{R}$, let π_r denote the least $\mathrm{OD}_{Z\cup\{r\}}$ surjective function $\pi : \mathbb{R} \to \kappa$ according to the canonical wellordering of $\mathrm{OD}_{Z\cup\{r\}}$ if such a surjection exists and otherwise let $\pi_r : \mathbb{R} \to \kappa$ be the constant 0 function. Let $\varpi(r) = \pi_{r^{[0]}}(r^{[1]})$ (where recall $r^{[n]}$ comes from the coding of ω-many reals by a single real defined before Fact 2.24) . ϖ is an OD_Z surjection of $\mathbb{R}$ onto κ. Let $\pi : \mathbb{R} \to {^\omega\kappa}$ be defined by $\pi(r)(n) = \varpi(r^{[n]})$. π is an OD_Z surjection of $\mathbb{R}$ onto $^\omega\kappa$. Let $\tilde{\pi} : \mathcal{D}_{\mathsf{Turing}} \to \mathscr{P}_{\omega_1}(^\omega\kappa)$ be defined by $\tilde{\pi}(X) = \{\pi(z) : [z]_{\equiv_{\mathsf{Turing}}} \leq_{\mathsf{Turing}} X\}$. $\tilde{\pi}$ is OD_Z. As shown above, $\tilde{\pi}_*\mu_{\mathsf{Turing}}$ is an OD_Z fine ω_1-complete measure on $\mathscr{P}_{\omega_1}(^\omega\kappa)$. Since $L(Z,\mathbb{R}) \models \mathsf{DC}_{\mathbb{R}}$, one has that $L(Z,\mathbb{R}) \models \mathsf{DC}$ and hence $\prod_{\mathscr{P}_{\omega_1}(^\omega\kappa)} V/\tilde{\pi}_*\mu_{\mathsf{Turing}}$ is wellfounded. Let Υ be the $\mathrm{OD}_{\{\tilde{\pi}_*\mu_{\mathsf{Turing}}\}}$ function given by Fact 4.14 applied to $\tilde{\pi}_*\mu_{\mathsf{Turing}}$. Since $\pi_*\mu_{\mathsf{Turing}} \in \mathrm{OD}_Z$, $\Upsilon \in \mathrm{OD}_Z$. ☐

Definition 4.15. Assume AD and $\mathsf{DC}_{\mathbb{R}}$. Let Z be a good parameter set such that $V = L(Z,\mathbb{R})$. Let $\kappa < \Theta$. Let Υ be the OD_Z function obtained from Fact 4.14.

Define $\Pi^{\kappa;Z}_{n,m} : \mathbb{B}^{\kappa;Z}_n \to \mathbb{B}^{\kappa;Z}_m$ by $\Pi^{\kappa;Z}_{n,m}(p)$ is the least member of the $[\Upsilon(p,n,m)]_{\sim^{\kappa;Z}_m}$ according to the canonical wellordering of HOD_Z. By Fact 4.14, $\pi^\kappa_{n,m}[\mathfrak{B}_p^{n(^\omega\kappa)}] = \mathfrak{B}^{m(^\omega\kappa)}_{\Pi^{\kappa;Z}_n(p)}$. If $q_0 \leq_{\mathbb{B}^{\kappa;Z}_n} q_1$, then $\mathfrak{B}^{n(^\omega\kappa)}_{q_0} \subseteq \mathfrak{B}^{n(^\omega\kappa)}_{q_1}$. Thus $\pi^\kappa_{n,m}[\mathfrak{B}^{n(^\omega\kappa)}_{q_0}] \subseteq \pi^\kappa_{n,m}[\mathfrak{B}^{n(^\omega\kappa)}_{q_1}]$ and hence $\Pi^{\kappa;Z}_{n,m}(q_0) \leq_{\mathbb{B}^{\kappa;Z}_m} \Pi^{\kappa;Z}_{n,m}(q_1)$. Let $q \in \mathbb{B}^{\kappa;Z}_n$ and $p \leq_{\mathbb{B}^{\kappa;Z}_m} \Pi^{\kappa;Z}_{n,m}(q)$. Thus $\mathfrak{B}_p^{m(^\omega\kappa)} \subseteq \mathfrak{B}^{m(^\omega\kappa)}_{\Pi^{\kappa;Z}_{n,m}(q)}$. Suppose $p = (S_0, \varphi_0)$ and $q = (S_1, \varphi_1)$. Let $\psi(A_0, A_1, v)$ be the formula which asserts "$L[A_0, v \restriction m] \models \varphi_0(A_0, v \restriction m)$ and $L[A_1, v] \models \varphi_1(A_1, v)$". Let $\tilde{r} = (\langle S_0, S_1\rangle, \psi)$ and note that $\tilde{r}$ is OD_Z. Then $f \in \mathfrak{B}^{n(^\omega\kappa)}_{\tilde{r}}$ if and only if $L[S_0, S_1, f] \models \psi(S_0, S_1, f)$ if and only if $L[S_0, f \restriction m] \models \varphi_0(S_0, f \restriction m)$ and $L[S_1, f] \models \varphi_1(S_1, f)$ if and only if $f \restriction m \in \mathfrak{B}_p^{m(^\omega\kappa)}$ and $f \in \mathfrak{B}_q^{n(^\omega\kappa)}$. Let $r \in \mathbb{B}^{\kappa;Z}_n$ be such that $r \sim^{\kappa;Z}_n \tilde{r}$. Then $r \leq_{\mathbb{B}^{\kappa;Z}_n} q$ and $\Pi^{\kappa;Z}_{n,m}(r) = p$. It has been shown that $\Pi^{\kappa;Z}_{n,m} : \mathbb{B}^{\kappa;Z}_n \to \mathbb{B}^{\kappa;Z}_m$ is a forcing projection.

$\langle \mathbb{B}^{\kappa;Z}_m, \Pi^{\kappa;Z}_{n,m} : m \leq n < \omega\rangle$ is a direct system of forcing projection. Let $\mathbb{B}^{\kappa;Z}_\omega$ be the collection of all functions $\mathfrak{p} \in \prod_{n\in\omega} \mathbb{B}^{\kappa;Z}_n$ such that the following holds:

- $m \leq n < \omega$, $\Pi^{\kappa;Z}_{n,m}(\mathfrak{p}(n)) = \mathfrak{p}(m)$.
- There is an $N < \omega$ so that for all $N < k < \omega$, $\mathfrak{B}^{k(^\omega\kappa)}_{\mathfrak{p}(k)} = \mathfrak{B}^{n(^\omega\kappa)}_{\mathfrak{p}(N)} \times {^{k-N}(^\omega\kappa)}$. The least such N will be denoted $\dim(\mathfrak{p})$ and is called the dimension of $\mathfrak{p}$.

Let $\leq_{\mathbb{B}^{\kappa;Z}_\omega}$ be defined by $\mathfrak{p} \leq_{\mathbb{B}^{\kappa;Z}_\omega} \mathfrak{q}$ if and only if for all $n \in \omega$, $\mathfrak{p}(n) \leq_{\mathbb{B}^{\kappa;Z}_n} \mathfrak{q}(n)$. Let $1_{\mathbb{B}^{\kappa;Z}_\omega}$ be defined by $1_{\mathbb{B}^{\kappa;Z}_\omega}(n) = 1_{\mathbb{B}^{\kappa;Z}_n}$ for all $n \in \omega$. Note that the directed system $\langle \mathbb{B}^{\kappa;Z}_m, \Pi^{\kappa;Z}_{n,m} : m \leq n < \omega\rangle$ and $\mathbb{B}^{\kappa;Z}_\omega = (\mathbb{B}^{\kappa;Z}_\omega, \leq_{\mathbb{B}^{\kappa;Z}_\omega}, 1_{\mathbb{B}^{\kappa;Z}_\omega})$ belong to HOD_Z. This forcing is called the finite support direct limit of the OD_Z ∞-Borel code forcing on $^\omega\kappa$. For each $n \in \omega$, let $\Pi^{\kappa;Z}_{\omega,n} : \mathbb{B}^{\kappa;Z}_\omega \to \mathbb{B}^{\kappa;Z}_n$ be defined by $\Pi^{\kappa;Z}_{\omega,n}(\mathfrak{p}) = \mathfrak{p}(n)$. $\Pi^{\kappa;Z}_{\omega,n}$ is a forcing projection.

Let $\dot{G}_\omega$ be the canonical $\mathbb{B}^{\kappa;Z}_\omega$-name for the generic filter. For each $n \in \omega$, let $\dot{G}_n$ be the $\mathbb{B}^{\kappa;Z}_\omega$-name such that $1_{\mathbb{B}^{\kappa;Z}_\omega} \Vdash_{\mathbb{B}^{\kappa;Z}_\omega} \dot{G}_n = \langle \check{\Pi}^{\kappa;Z}_{\omega,n}[\dot{G}_\omega]\rangle_{\check{\mathbb{B}}^{\kappa;Z}_n}$ (that is, $\dot{G}_n$ is the canonical $\mathbb{B}^{\kappa;Z}_\omega$-name for the generic filter on $\mathbb{B}^{\kappa;Z}_n$ induced by the forcing projection $\Pi^{\kappa;Z}_{\omega,n}$). Let $\dot{\tau}^\omega_n$ be the $\mathbb{B}^{\kappa;Z}_\omega$-name such that $1_{\mathbb{B}^{\kappa;Z}_\omega} \Vdash_{\mathbb{B}^{\kappa;Z}_\omega} \dot{\tau}^\omega_n = \dot{\tau}^n[\dot{G}_n]$ where $\dot{\tau}^n$ here refers to the canonical $\mathbb{B}^{\kappa;Z}_n$-name for the generic element of $^n(^\omega\kappa)$ naturally understood as a $\mathbb{B}^{\kappa;Z}_\omega$-name. Let $\dot{\tau}^\omega$ be the $\mathbb{B}^{\kappa;Z}_\omega$-name such that for all $n \in \omega$, $1_{\mathbb{B}^{\kappa;Z}_\omega} \Vdash_{\mathbb{B}^{\kappa;Z}_\omega} \dot{\tau}^\omega \restriction \check{n} = \dot{\tau}^\omega_n$. $\dot{\tau}^\omega$ names the generic element of $^\omega(^\omega\kappa)$ added by $\mathbb{B}^{\kappa;Z}_\omega$. Let $^\omega\dot{\kappa}^{\mathrm{sym}}$ be a $\mathbb{B}^{\kappa;Z}_\omega$-name such that $1_{\mathbb{B}^{\kappa;Z}_\omega} \Vdash_{\mathbb{B}^{\kappa;Z}_\omega} {^\omega\dot{\kappa}^{\mathrm{sym}}} = \{\dot{\tau}^\omega(n) : n \in \omega\}$.

(If $\kappa = \omega$, one usually writes $\dot{\mathbb{R}}^{\mathrm{sym}}$ rather than ${}^{\omega}\dot{\omega}^{\mathrm{sym}}$.) Also let $\dot{\mathbb{R}}^{\kappa;\mathrm{sym}}$ be a $\mathbb{B}^{\kappa;Z}_{\omega}$-name for the collection of reals inside ${}^{\omega}\dot{\kappa}^{\mathrm{sym}}$. (Formally $\dot{\mathbb{R}}^{\mathrm{sym}}$ is only defined for $\mathbb{B}^{\omega;Z}_{\omega}$ and names the terms of the generic ω-sequence of $\mathbb{B}^{\omega;Z}_{\omega}$.) Since $\Pi^{\kappa;Z}_{\omega,n}$ is a forcing projection, whenever G is a $\mathbb{B}^{\kappa;Z}_{\omega}$ filter, let $G_n = \dot{G}_n[G] = \langle \Pi^{\kappa;Z}_{\omega,n}[G]\rangle_{\mathbb{B}^{\kappa;Z}_n}$ which is a filter on $\mathbb{B}^{\kappa;Z}_n$. One can form the corresponding remainder forcing, $\mathbb{B}^{\kappa;Z}_{\omega}/_{\Pi^{\kappa;Z}_{\omega,n}} G_n$. ${}^{\omega}\dot{\kappa}^{\mathrm{sym}}$ and $\dot{\mathbb{R}}^{\kappa;\mathrm{sym}}$ can naturally be considered a $(\mathbb{B}^{\kappa;Z}_{\omega}/_{\Pi^{\kappa;Z}_{\omega,n}} G_n)$-name.

Similarly to Definition 4.5, let $\vec{\mathbb{B}}^{\kappa;Z}$ be an OD_Z set of ordinals which codes $\langle \mathbb{B}^{\kappa;Z}_n, \Pi^{\kappa;Z}_{n,m}, \Pi^{\kappa;Z}_{\omega,m} : m \leq n < \omega\rangle$, $\mathbb{B}^{\kappa;Z}_{\omega}$, $\dot{\tau}^{\omega}$, $\dot{\tau}^{\omega}_n$, and ${}^{\omega}\dot{\kappa}^{\mathrm{sym}}$. Again there are various ways to define $\vec{\mathbb{B}}^{\kappa;Z}$ but it should be uniform in Z. Its main property is that any inner model $M \subseteq \mathrm{HOD}_Z$ with $\vec{\mathbb{B}}^{\kappa;Z} \in M$ can see the direct system of forcing projections.

If $h : \omega \to ({}^{\omega}\kappa)^V$ (which may not necessarily belong to the real world V), then let $G^{\kappa;Z;\omega}_h = \{\mathfrak{p} \in \omega : (\forall n \in \omega)(h \restriction n \in \mathfrak{B}^{{}^{n}({}^{\omega}\kappa)}_{\mathfrak{p}(n)})\}$. Note that $\Pi^{\kappa;Z}_{\omega,n}[G^{\kappa;Z;\omega}_h] = G^{\kappa;Z;n}_{h\restriction n}$. Thus $\dot{\tau}^{\omega}_n[G^{\kappa;Z;\omega}_h] = h \restriction n$. Hence $\dot{\tau}^{\omega}[G^{\kappa;Z;\omega}_h] = h$ and ${}^{\omega}\dot{\kappa}^{\mathrm{sym}}[G^{\kappa;Z;\omega}_h] = h[\omega]$.

Fact 4.16. *(Woodin) Assume* AD *and* $\mathsf{DC}_{\mathbb{R}}$*. Let Z be a good parameter set such that $V = L(Z,\mathbb{R})$. Let $\kappa < \Theta$. Let $G \subseteq \mathrm{Coll}(\omega, {}^{\omega}\kappa)$ be $\mathrm{Coll}(\omega, {}^{\omega}\kappa)$-generic over the true universe V. Let $g : \omega \to {}^{\omega}\kappa$ be the generic surjection added by G; namely $g = \bigcup G$. Let M be an inner model of HOD_Z with $\vec{\mathbb{B}}^{\kappa;Z} \in M$. Then $G^{\kappa;Z;\omega}_g$ is $\mathbb{B}^{\kappa;Z}_{\omega}$-generic over M.*

Proof. Let $D \subseteq \mathbb{B}^{\kappa;Z}_{\omega}$ be a dense subset of $\mathbb{B}^{\kappa;Z}_{\omega}$ in M. Although $\mathrm{Coll}(\omega, {}^{\omega}\kappa)$ is usually defined as the collection of finite partial functions from ω into ${}^{\omega}\kappa$, here $\mathrm{Coll}(\omega, {}^{\omega}\kappa)$ will be identified with it dense subforcing of finite partial function s from ω into ${}^{\omega}\kappa$ such that $\mathrm{dom}(s) \in \omega$ (i.e. a finite sequence in ${}^{\omega}\kappa$). Let $\tilde{D} \subseteq \mathrm{Coll}(\omega, {}^{\omega}\kappa)$ be the collection of $t \in \mathrm{Coll}(\omega, {}^{\omega}\kappa)$ such that there is a $\mathfrak{p} \in D$ with $\dim(\mathfrak{p}) \leq |t|$ and $t \in \mathfrak{B}^{{}^{|t|}({}^{\omega}\kappa)}_{\mathfrak{p}(|t|)}$. First, one will show that $\tilde{D}$ is dense in $\mathrm{Coll}(\omega, {}^{\omega}\kappa)$. Fix $s \in \mathrm{Coll}(\omega, {}^{\omega}\kappa)$ and let $n = |s|$. Let $E \subseteq \mathbb{B}^{\kappa;Z}_n$ consists of those $u \in \mathbb{B}^{\kappa;Z}_n$ such that there exists a $\mathfrak{q} \in D$ with $\Pi^{\kappa;Z}_{\omega,n}(\mathfrak{q}) = u$. As an intermediate step, one will show that E is dense in $\mathbb{B}^{\kappa;Z}_n$. Let $r \in \mathbb{B}^{\kappa;Z}_n$. Let $\mathfrak{r}$ be a function on ω such that $\mathfrak{r}(k) \in \mathbb{B}^{\kappa;Z}_k$ for all $k \in \omega$, $\mathfrak{r}(k) = \Pi^{\kappa;Z}_{n,k}(r)$ for all $k \leq n$, and $\mathfrak{B}^{{}^{k}({}^{\omega}\kappa)}_{\mathfrak{r}(k)} = \mathfrak{B}^{{}^{n}({}^{\omega}\kappa)}_{r} \times {}^{k-n}({}^{\omega}\kappa)$ for all $n < k < \omega$. Thus $\mathfrak{r} \in \mathbb{B}^{\kappa;Z}_{\omega}$ and $\Pi^{\kappa;Z}_{\omega,n}(\mathfrak{r}) = r$. Since D is dense in $\mathbb{B}^{\kappa;Z}_{\omega}$, there is a $\mathfrak{q} \in D$ with $\mathfrak{q} \leq_{\mathbb{B}^{\kappa;Z}_{\omega}} \mathfrak{r}$. Then $\Pi^{\kappa;Z}_{\omega,n}(\mathfrak{q}) \leq_{\mathbb{B}^{\kappa;Z}_n} r$ and $\Pi^{\kappa;Z}_{\omega,n}(\mathfrak{q}) \in E$ since $\mathfrak{q} \in D$. Since $r \in \mathbb{B}^{\kappa;Z}_n$ was arbitrary, E has been shown to be dense in $\mathbb{B}^{\kappa;Z}_n$. $G^{\kappa;Z;n}_s$ is $\mathbb{B}^{\kappa;Z}_n$-generic over M by Fact 4.6. Thus there is some $u \in G^{\kappa;Z;n}_s \cap E$. By definition of $u \in E$, there is a $\mathfrak{q} \in D$ with $\Pi^{\kappa;Z}_{\omega,n}(\mathfrak{q}) = u$. Since $u \in G^{\kappa;Z;n}_s$, $s \in \mathfrak{B}^{{}^{n}({}^{\omega}\kappa)}_{u}$. Let $m = \max\{n, \dim(\mathfrak{q})\}$. By definition of $\mathfrak{q} \in \mathbb{B}^{\kappa;Z}_{\omega}$, $\Pi^{\kappa;Z}_{m,n}(\mathfrak{q}(m)) = \mathfrak{q}(n) = \Pi^{\kappa;Z}_{\omega,n}(\mathfrak{q}) = u$. Thus there is some $t \in \mathfrak{B}^{{}^{m}({}^{\omega}\kappa)}_{\mathfrak{q}(m)}$ with $\pi^{\kappa}_{m,n}(t) = t \restriction n = s$. Hence $t \leq_{\mathrm{Coll}(\omega, {}^{\omega}\kappa)} s$ and $t \in \tilde{D}$ since $\mathfrak{q} \in D$, $\dim(\mathfrak{q}) \leq m = |t|$, and $t \in \mathfrak{B}^{{}^{|t|}({}^{\omega}\kappa)}_{\mathfrak{q}(|t|)}$. Since $s \in \mathrm{Coll}(\omega, {}^{\omega}\kappa)$ was arbitrary, this shows that $\tilde{D}$ is dense in $\mathrm{Coll}(\omega, {}^{\omega}\kappa)$. Since $G \subseteq \mathrm{Coll}(\omega, {}^{\omega}\kappa)$ is $\mathrm{Coll}(\omega, {}^{\omega}\kappa)$-generic over V, there is some $s \in G \cap \tilde{D}$. By definition of $s \in \tilde{D}$, there is a $\mathfrak{p} \in D$ with $\dim(\mathfrak{p}) \leq |s|$ and $s \in \mathfrak{B}^{{}^{|s|}({}^{\omega}\kappa)}_{\mathfrak{p}(|s|)}$. Since the generic surjection $g : \omega \to {}^{\omega}\kappa$ is defined by $g = \bigcup G$, $s \subseteq g$. For all $k \leq |s|$, $g \restriction k = s \restriction k \in \pi^{\kappa}_{|s|,k}[\mathfrak{B}^{{}^{|s|}({}^{\omega}\kappa)}_{\mathfrak{p}(|s|)}] = \mathfrak{B}^{{}^{k}({}^{\omega}\kappa)}_{\mathfrak{p}(k)}$ since $\Pi^{\kappa;Z}_{|s|,k}(\mathfrak{p}(|s|)) = \mathfrak{p}(k)$ by definition of $\mathbb{B}^{\kappa;Z}_{\omega}$. Since $\dim(\mathfrak{p}) \leq |s|$, for all $k > |s|$, $g \restriction k \in \mathfrak{B}^{{}^{|s|}({}^{\omega}\kappa)}_{\mathfrak{p}(|s|)} \times \mathbb{R}^{k-|s|} = \mathfrak{B}^{{}^{k}({}^{\omega}\kappa)}_{\mathfrak{p}(k)}$. By definition of $G^{\kappa;Z;\omega}_g$, $\mathfrak{p} \in G^{\kappa;Z;\omega}_g$. Since $\mathfrak{p} \in D$, $G^{\kappa;Z;\omega}_g \cap D \neq \emptyset$. Since D was an arbitrary dense subset of $\mathbb{B}^{\kappa;Z}_{\omega}$ in M, $G^{\kappa;Z;\omega}_g$ is $\mathbb{B}^{\kappa;Z}_{\omega}$-generic over M. ☐

The following fact will be an important tool for proving the necessary capturing of set in the next section (Lemma 5.9).

Fact 4.17. *(Woodin) Assume* AD *and* $\mathsf{DC}_{\mathbb{R}}$*. Let Z be a good parameter set such that $V = L(Z,\mathbb{R})$. Let $\kappa < \Theta$. Let M be an inner model of HOD_Z such that $\vec{\mathbb{B}}^{\kappa;Z} \in M$. Let $n \in \omega$, $x \in {}^{n}({}^{\omega}\kappa)$, $Y \in M[x]$, $a \in M[x] \cap L(Y, {}^{\omega}\kappa)$, and φ is a $\mathscr{L}_{\mathrm{set}}$-formula. Then*

$$L(Y, {}^{\omega}\kappa) \models \varphi(a) \text{ if and only if } M[x] \Vdash 1_{\mathbb{B}^{\kappa;Z}_{\omega}/_{\Pi^{\kappa;Z}_{\omega,n}} G^{\kappa;Z;n}_x} \Vdash_{\mathbb{B}^{\kappa;Z}_{\omega}/_{\Pi^{\kappa;Z}_{\omega,n}} G^{\kappa;Z;n}_x} L(\check{Y}, {}^{\omega}\dot{\kappa}^{\mathrm{sym}}) \models \varphi(\check{a}).$$

(Note that if $\kappa < \Theta^{L(Y,\mathbb{R})}$, then $L(Y,{}^{\omega}\kappa) = L(Y,\mathbb{R})$ by the Moschovakis coding lemma, Fact 2.15. Thus in the above, ${}^{\omega}\kappa$ and ${}^{\omega}\dot{\kappa}^{\mathrm{sym}}$ can be replaced with $\mathbb{R}$ and $\dot{\mathbb{R}}^{\kappa;\mathrm{sym}}$, respectively.)

Proof. ($\Rightarrow$) Suppose $L(Y,{}^{\omega}\kappa) \models \varphi(a)$. Let $\mathfrak{p} \in \mathbb{B}^{\kappa;Z}_{\omega}/_{\Pi^{\kappa;Z}_{\omega,n}} G^{\kappa;Z;n}_x$. By the definition of the quotient forcing, this means that $\mathfrak{p}(n) = \Pi^{\kappa;Z}_{\omega,n}(\mathfrak{p}) \in G^{\kappa;Z;n}_x$ and thus $x \in \mathfrak{B}^{{}^{n}({}^{\omega}\kappa)}_{\mathfrak{p}(n)}$. Let $m = \max\{n, \dim(\mathfrak{p})\}$. Pick $s \in {}^{m}({}^{\omega}\kappa)$ such that $s \upharpoonright n = x$ and $s \in \mathfrak{B}^{{}^{m}({}^{\omega}\kappa)}_{\mathfrak{p}(m)}$. Let $G \subseteq \mathrm{Coll}(\omega, {}^{\omega}\kappa)$ be $\mathrm{Coll}(\omega,{}^{\omega}\kappa)$-generic over the true universe V with $s \in G$. (The generic does not exist in the real world. One will use the usual convenient forcing argument to conclude a fact about the forcing relation via the forcing theorem.) Let $g : \omega \to {}^{\omega}\kappa$ be the generic surjection added by G. $G^{\kappa;Z;\omega}_g$ is $\mathbb{B}^{\kappa;Z}_{\omega}$-generic over M by Fact 4.16. Since $s \in G$, $s \in \mathfrak{B}^{{}^{m}({}^{\omega}\kappa)}_{\mathfrak{p}(m)}$, and $\dim(\mathfrak{p}) \leq m$, one has that $g \upharpoonright k \in \mathfrak{B}^{{}^{k}({}^{\omega}\kappa)}_{\mathfrak{p}(k)}$ for all $k \in \omega$. Thus $\mathfrak{p} \in G^{\kappa;Z;\omega}_g$. Note $G^{\kappa;Z;n}_x = \langle \Pi^{\kappa;Z}_{\omega,n}[G^{\kappa;Z;\omega}_g]\rangle_{\mathbb{B}^{\kappa;Z}_n}$ and $G^{\kappa;Z;\omega}_g$ is $\mathbb{B}^{\kappa;Z}_{\omega}/_{\Pi^{\kappa;Z}_{\omega,n}} G^{\kappa;Z;n}_x$-generic over $M[G^{\kappa;Z;n}_x]$ by the properties of forcing projections discussed in Definition 3.7. Since ${}^{\omega}\dot{\kappa}^{\mathrm{sym}}[G^{\kappa;Z;\omega}_g] = {}^{\omega}\kappa$ and $L(Y,{}^{\omega}\kappa) \models \varphi(a)$, one has

$$M[G^{\kappa;Z;n}_x][G^{\kappa;Z;\omega}_g] \models L(\check{Y}, {}^{\omega}\dot{\kappa}^{\mathrm{sym}}[G^{\kappa;Z;\omega}_g]) \models \varphi(\check{a})$$

Since $M[x] = M[G^{\kappa;Z;n}_x]$ by Fact 4.7, $\mathfrak{p} \in G^{\kappa;Z;\omega}_g$, and by using the forcing theorem, there is a $\mathfrak{q} \leq_{\mathbb{B}^{\kappa;Z}_{\omega}/_{\Pi^{\kappa;Z}_{\omega,n}} G^{\kappa;Z;n}_x}$ $\mathfrak{p}$ such that

$$M[x] \models \mathfrak{q} \Vdash_{\mathbb{B}^{\kappa;Z}_{\omega}/_{\Pi^{\kappa;Z}_{\omega,n}} G^{\kappa;Z;n}_x} L(\check{Y}, {}^{\omega}\dot{\kappa}^{\mathrm{sym}}) \models \varphi(\check{a}).$$

Since $\mathfrak{p} \in \mathbb{B}^{\kappa;Z}_{\omega}/_{\Pi^{\kappa;Z}_{\omega,n}} G^{\kappa;Z;n}_x$ was arbitrary, one has that within $M[x]$, $1_{\mathbb{B}^{\kappa;Z}_{\omega}/_{\Pi^{\kappa;Z}_{\omega,n}} G^{\kappa;Z;n}_x}$ forces the inner statement.

($\Leftarrow$) Considering x as a condition of $\mathrm{Coll}(\omega,{}^{\omega}\kappa)$, pick $G \subseteq \mathrm{Coll}(\omega,{}^{\omega}\kappa)$ which is $\mathrm{Coll}(\omega,{}^{\omega}\kappa)$-generic over V with $x \in G$. Let g be the generic surjection associated to G and let $G^{\kappa;Z;\omega}_g$ be the associated $\mathbb{B}^{\kappa;Z}_{\omega}$-generic filter over M by Fact 4.16. $G^{\kappa;Z;n}_x = \langle \Pi^{\kappa;Z}_{\omega,n}[G^{\kappa;Z;\omega}_g]\rangle_{\mathbb{B}^{\kappa;Z}_n}$ is $\mathbb{B}^{\kappa;Z}_n$-generic over M by Fact 4.6. Again $M[x] = M[G^{\kappa;Z;n}_x]$ by Fact 4.7. As before using general properties of forcing projections from Definition 3.7, $G^{\kappa;Z;\omega}_g$ is $\mathbb{B}^{\kappa;Z}_{\omega}/_{\Pi^{\kappa;Z}_{\omega,n}} G^{\kappa;Z;n}_x$ generic over $M[x]$. By the hypothesis and the forcing theorem, $M[x][G^{\kappa;Z;\omega}_g] \models L(Y, {}^{\omega}\dot{\kappa}^{\mathrm{sym}}[G^{\kappa;Z;\omega}_g]) \models \varphi(a)$. Since ${}^{\omega}\dot{\kappa}^{\mathrm{sym}}[G^{\kappa;Z;\omega}_g] = {}^{\omega}\kappa$, one has $L(Y,{}^{\omega}\kappa) \models \varphi(a)$. ☐

The significance of Fact 4.17 is that for any inner model M of HOD_Z with $\vec{\mathbb{B}}^{\kappa;Z} \in M$, $x \in {}^{\omega}\kappa$, and $Y \in M[x]$, the truth about objects $a \in M[x] \cap L(Y,{}^{\omega}\kappa)$ within the larger choiceless universe $L(Y,{}^{\omega}\kappa)$ can be determined within the ZFC model $M[x]$. There are many applications of Fact 4.17.

In applications below, the parameter set Z may contain elements which do not belong to HOD_Z (but they are of course OD_Z). For example, if boldface GCH at ω holds and there is an uncountable $A \in \mathscr{P}(\mathbb{R})$ such that $A \in Z$, then $A \in \mathrm{OD}_Z$ but $A \notin \mathrm{HOD}_Z$. Another very relevant example is if Σ is an iteration strategy for a countable mouse and $\Sigma \in Z$, then $\Sigma \in \mathrm{OD}_Z$ but $\Sigma \notin \mathrm{HOD}_Z$. However, in many cases, there is a set $Y \in \mathrm{HOD}_Z$ so that $\mathrm{OD}_Z = \mathrm{OD}_Y$. In this situation, Fact 4.17 can be used to more fully understand HOD_Z as in the following results.

Definition 4.18. A set Z is a very good parameter set if and only if Z is a good parameter set and there is a set Y so that $Y \in \mathrm{HOD}_Z$ and $\mathrm{HOD}_Z = \mathrm{HOD}_Y$. A set Y as above is called a very good parameter-witness for Z.

Suppose $V = L(Z,\mathbb{R})$ and Z is a very good parameter set with very good parameter-witness Y. Since Z is a good parameter set, every element of $L(Z,\mathbb{R})$ is a ordinal definable from finitely many elements of Z and finitely many reals. Since Y is a very good parameter-witness to Z, every element of $L(Z,\mathbb{R})$ is ordinal definable from finitely many parameters from Y and finitely many reals. Thus Y is also a good parameter set and $L(Z,\mathbb{R}) \subseteq L(Y,\mathbb{R})$. Since $V = L(Z,\mathbb{R})$ and hence $Y \in L(Z,\mathbb{R})$, one has $L(Z,\mathbb{R}) = L(Y,\mathbb{R})$. Again, one can always find a finite very good parameter-witness Y. Moreover, since elements of $y \in Y$ belong to $\mathrm{HOD}_Z \models \mathsf{ZFC}$, after choosing a bijection π of an ordinal δ onto the transitive closure of y, one can code y as a set of ordinals. Thus the very good parameter-witness can be chosen to be a finite set of sets of ordinals.

Formally, one can always replace a very good parameter Z with its very good parameter-witness Y. In practice, as in the proof of main result of the paper (Theorem 6.7), one naturally has a good parameter set

Z which contains elements that do not belong to HOD_Z but one must show that Z is a very good parameter set by finding its very good parameter-witness Y.

Fact 4.19. *Assume* AD *and* $\mathsf{DC}_{\mathbb{R}}$. *Let* Z *be a very good parameter set such that* $V = L(Z, \mathbb{R})$. *Let* $\kappa < \Theta$. *If* $n \in \omega$ *and* $x \in {}^n({}^\omega\kappa)$, *then* $\mathrm{HOD}_Z[x] = \mathrm{HOD}_{Z \cup \{x\}}$.

Proof. Let Y be a very good parameter-witness for Z. Since $\mathrm{OD}_Z = \mathrm{OD}_Y$, $\mathrm{HOD}_Z[x] = \mathrm{HOD}_Y[x]$ and $\mathrm{HOD}_{Z\cup\{x\}} = \mathrm{HOD}_{Y\cup\{x\}}$. Thus one will work with Y rather than Z. Also note that $Y \in \mathrm{HOD}_Z = \mathrm{HOD}_Y$. Since $x \in \mathrm{HOD}_{Y\cup\{x\}}$, $\mathrm{HOD}_Y[x] \subseteq \mathrm{HOD}_{Y\cup\{x\}}$. Let $U \in \mathrm{HOD}_{Y\cup\{x\}}$. Let W be the transitive closure of $\{U\}$ and note that $W \in \mathrm{HOD}_{Y\cup\{x\}}$. Since $\mathrm{HOD}_{Y\cup\{x\}} \models \mathsf{AC}$, let $\pi : \delta \to W$ be a bijection where $\delta \in \mathrm{ON}$ and $\pi \in \mathrm{HOD}_{Y\cup\{x\}}$. Define $R \subseteq \delta \times \delta$ by $R(\alpha, \beta)$ if and only if $\pi(\alpha) \in \pi(\beta)$. Note that $R \in \mathrm{HOD}_{Y\cup\{x\}}$. Thus there is a finite tuple $\vec{y}$ in Y, a finite tuple of ordinals $\vec{\xi}$, and a $\mathscr{L}_{\mathrm{set}}$-formula φ so that $R(\alpha,\beta)$ if and only if $\varphi(\vec{y}, \vec{\xi}, x, \alpha, \beta)$. Since $Y \in \mathrm{HOD}_Z = \mathrm{HOD}_Y$, all the elements of the tuple $\vec{y}$ also belong to $\mathrm{HOD}_Y \subseteq \mathrm{HOD}_Y[x]$. By Fact 4.17, for all $\alpha, \beta \in \delta$, $\varphi(\vec{y}, \vec{\xi}, x, \alpha, \beta)$ if and only if $L(Y, \mathbb{R}) \models \varphi(\vec{y}, \vec{\xi}, x, \alpha, \beta)$ if and only if

$$\mathrm{HOD}_Y[x] \models 1_{\mathbb{B}_\omega^{\kappa;Z}/_{\Pi_{\omega,n}^{\kappa;Z}} G_x^{\kappa;Z;n}} L(\check{Y}, \dot{\mathbb{R}}^{\kappa;\mathrm{sym}}) \models \varphi(\check{\vec{y}}, \check{\vec{\xi}}, \check{x}, \check{\alpha}, \check{\beta}).$$

(Note $\vec{\mathbb{B}}^{\kappa;Z} \in \mathrm{HOD}_Y = \mathrm{HOD}_Z$ although one could also have used $\vec{\mathbb{B}}^{\kappa;Y}$.) By comprehension inside $\mathrm{HOD}_Y[x]$, one has that $R \in \mathrm{HOD}_Y[x]$. $W \in \mathrm{HOD}_Y[x]$ since W is the Mostowski collapse of (δ, R). Then $U \in \mathrm{HOD}_Y[x]$, since U is the unique element of highest rank in W. Since $U \in \mathrm{HOD}_{Y\cup\{x\}}$ was arbitrary, this shows that $\mathrm{HOD}_{Y\cup\{x\}} \subseteq \mathrm{HOD}_Y[x]$. Thus $\mathrm{HOD}_Y[x] = \mathrm{HOD}_{Y\cup\{x\}}$. □

An argument similar to the proof of Fact 4.19 gives the next result.

Fact 4.20. *Assume* AD *and* $\mathsf{DC}_{\mathbb{R}}$. *Let* Z *be a very good parameter set with good parameter witness* Y *such that* $V = L(Z, \mathbb{R})$. *Then* $\mathrm{HOD}_Z = L[\vec{\mathbb{B}}^{\kappa;Z}, Y]$.

In particular, $\mathrm{HOD}^{L(\mathbb{R})} = L[\vec{\mathbb{B}}^{\kappa;\emptyset}]$ *for any* $\kappa < \Theta^{L(\mathbb{R})}$ *and in particular* $\mathrm{HOD}^{L(\mathbb{R})} = L[\vec{\mathbb{B}}^{\omega;\emptyset}]$. *By using Fact 4.19, for any* $\kappa < \Theta^{L(\mathbb{R})}$, $n \in \omega$, *and* $x \in {}^n({}^\omega\kappa)$, $\mathrm{HOD}^{L(\mathbb{R})}_{\{x\}} = L[\vec{\mathbb{B}}^{\kappa;\emptyset}, x]$.

Fact 4.17 can be used to show all subsets ${}^\omega\kappa$ in $L(Z, \mathbb{R})$ have ∞-Borel codes in a uniform manner.

Fact 4.21. *Assume* AD *and* $\mathsf{DC}_{\mathbb{R}}$. *Let* Z *be a very good parameter set such that* $V = L(Z, \mathbb{R})$. *Let* $\kappa < \Theta$. *All subsets of* ${}^\omega\kappa$ *have* ∞*-Borel codes. Moreover, if* Y *is a very good parameter-witness for* Z *and* $A \subseteq {}^\omega\kappa$ *is* $\mathrm{OD}_{Y\cup\{x\}}$ *for some* $x \in \mathbb{R}$, *then* A *has an* ∞*-Borel code which is* $\mathrm{OD}_{Y\cup\{x\}}$ *(and this* ∞*-Borel code can be obtained uniformly in the sense described in the proof).*

Proof. Let $A \subseteq {}^\omega\kappa$. Let Y be a very good parameter-witness for Z. Since $\mathrm{OD}_Z = \mathrm{OD}_Y$ and $V = L(Z, \mathbb{R}) = L(Y, \mathbb{R})$, there is a $\mathscr{L}_{\mathrm{set}}$ formula φ, finite tuple $\vec{y}$ from Y, finite tuple of ordinals $\vec{\xi}$, and a finite tuple x in $\mathbb{R}$ (say n is the length of the tuple x) so that for all $a \in {}^\omega\kappa$, $a \in A$ if and only if $\varphi(\vec{y}, \vec{\xi}, x, a)$. Fact 4.17 asserts that $a \in A$ if and only if $L(Y, \mathbb{R}) \models \varphi(\vec{y}, \vec{\xi}, x, a)$ if and only if

$$L[\vec{\mathbb{B}}^{\kappa;Z}, Y, \vec{y}, x, a] \models 1_{\mathbb{B}_\omega^{\kappa;Z}/_{\Pi_{\omega,n+1}^{\kappa;Z}} G_{x\hat{\ }a}^{\kappa;Z;n+1}} \Vdash_{\mathbb{B}_\omega^{\kappa;Z}/_{\Pi_{\omega,n+1}^{\kappa;Z}} G_{x\hat{\ }a}^{\kappa;Z;n+1}} L(\check{Y}, \dot{\mathbb{R}}^{\kappa;\mathrm{sym}}) \models \varphi(\check{\vec{y}}, \check{\vec{\xi}}, \check{x}, \check{a}).$$

The inner expression can be written as a formula $\psi(\vec{\mathbb{B}}^{\kappa;Z}, Y, \vec{y}, \vec{\xi}, x, a)$. One has shown that $a \in A$ if and only if $L[\vec{\mathbb{B}}^{\kappa;Z}, Y, \vec{y}, \vec{\xi}, x, a] \models \psi(\vec{\mathbb{B}}^{\kappa;Z}, Y, \bar{\xi}, x, a)$. Hence $(\langle \mathbb{B}^{\kappa;Z}, Y, x, \bar{\xi}\rangle, \psi)$ is an ∞-Borel code for A. (Since $Y \in \mathrm{HOD}_Z = \mathrm{HOD}_Y$, $\langle \mathbb{B}^{\kappa;Z}, Y, \vec{y}, x, \bar{\xi}\rangle$ is essentially a set of ordinals and is $\mathrm{OD}_{Y\cup\{x\}}$.) □

Fact 4.22. *Assume* AD, $\mathsf{DC}_{\mathbb{R}}$, *and all subsets of* $\mathbb{R}$ *has* ∞*-Borel codes. Let* $\kappa < \Theta$. *All subsets of* ${}^\omega\kappa$ *have* ∞*-Borel code.*

Proof. Let $\kappa < \Theta$ and let $\pi : \mathbb{R} \to \kappa$ be a surjection. Let $\preceq$ be a prewellordering on $\mathbb{R}$ defined by $x \preceq y$ if and only if $\pi(x) \leq \pi(x)$. Let $\varpi : \mathbb{R} \to {}^\omega\kappa$ be a surjection defined by $\varpi(x)(n) = \pi(x^{[n]})$. Let $A \subseteq {}^\omega\kappa$. Define $\tilde{A} \subseteq \mathbb{R}$ by $\tilde{A} = \{x \in \mathbb{R} : \varpi(x) \in A\}$. By the hypothesis, let S be a set of ordinals and φ_0, φ_1 be two formulas such that (S, φ_0) is an ∞-Borel code for $\tilde{A}$ and (S, φ_1) is an ∞-Borel code for $\preceq$. Then $\tilde{A}, \preceq \in L(S, \mathbb{R})$. Within $L(S, \mathbb{R})$, $\preceq$ can be used to recover π. $L(S, \mathbb{R})$ can define ϖ. Using $\tilde{A}$, one can show that $A \in L(S, \mathbb{R})$. In the inner model $L(S, \mathbb{R})$, $\{S\}$ is a very good parameter set. By Fact 4.21, A has an ∞-Borel code in $L(S, \mathbb{R})$ and thus A has an ∞-Borel code in the real world. □

A very important observation from the proof of Fact 4.21 is that there is an explicit procedure that transforms the definition of A (using parameter Y, finitely many elements from Y, finitely many reals, and finitely many ordinals) to an ∞-Borel code for A. This uniform transformation is used for various combinatorial results under natural models of AD^+ such as in [13] and [10]. This uniformity is especially powerful when combined with the proof of Woodin's perfect set dichotomy ([8] Theorem 8.5, [4]).

Fact 4.17 is also used to show that in models of the form $L(Z,\mathbb{R})$ as above, $\vec{\mathbb{B}}^{\omega;Z}$ can be used to define intermediate cardinalities which provide counterexamples to the cardinality structure below $\mathscr{P}(\omega_1)$ proved under $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ (or equivalently $\mathsf{AC}^{\mathbb{R}}_{\mathbb{R}}$). Under $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$, [13] Corollary 7.6 showed that the only uncountable cardinality below $|\mathbb{R}\times\omega_1|$ are $|\mathbb{R}|$, $|\omega_1|$, $|\mathbb{R}\sqcup\omega_1|$, and $|\mathbb{R}\times\omega_1|$. [13] Theorem 7.33 isolated a very large family of cardinalities between $|\mathbb{R}|$ and $|\mathbb{R}\times\omega_1|$ which do not have an injective copy of ω_1 in $L(\mathbb{R})$. In $L(\mathbb{R})$, let $N_1 = \bigsqcup_{r\in\mathbb{R}}((\omega_1^{L(\mathbb{R})})^+)^{L[\vec{\mathbb{B}}^{\omega;\emptyset},r]}$. [13] Theorem 4.10 shows $|\mathbb{R}\times\omega_1| < |N_1| < |\mathbb{R}\times\omega_2|$, $|N_1| < |\mathscr{P}(\omega_1)|$, $\neg(|N_1| \leq |[\omega_1]^{<\omega_1}|)$, $\neg(|[\omega_1]^\omega| \leq |N_1|)$, and $|[\omega_1]^{<\omega_1}| < |N_1\cup[\omega_1]^{<\omega_1}| < |\mathscr{P}(\omega_1)|$. In $L(\mathbb{R})$, let $M_1 = \bigsqcup_{f\in[\omega_1]^{<\omega_1}}((\omega_1^{L(\mathbb{R})})^+)^{L[\vec{\mathbb{B}}^{\omega;\emptyset},f]}$. Then in $L(\mathbb{R})$, $|[\omega_1]^{<\omega_1}| < |M_1| < |\mathscr{P}(\omega_1)|$ by similar arguments. Woodin [65] under $\mathsf{AD}_\mathbb{R}$ and DC and later the author under $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ showed that if $|X| \leq |[\omega_1]^\omega|$, then either $|X|\leq|\mathbb{R}\times\omega_1|$ or $|[\omega_1]^\omega| \leq |X|$. It was open whether a counterexample existed if $\mathsf{AC}^{\mathbb{R}}_{\mathbb{R}}$ failed. As an application of the Fact 4.17, the next result will produce the required intermediate cardinality and illustrate how $\vec{\mathbb{B}}^{\kappa;Z}$ is used to diagonalized against injections.[12] This will motivate analogous constructions of intermediate cardinalities below $\mathscr{P}_B(\kappa)$ in $L(\mathbb{R})$ using the techniques of the next section. (See Theorem 6.27.)

If Z is a good parameter set and $V = L(Z,\mathbb{R})$ satisfies boldface GCH at ω, then $L(Z,\mathbb{R})$ fails to satisfy $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ or equivalently $\mathsf{AC}^{\mathbb{R}}_{\mathbb{R}}$. This follows from the usual example of Kechris and Solovay: Let $R\subseteq\mathbb{R}\times\mathbb{R}$ be defined by $R(x,y)$ if and only if $y\notin \mathrm{OD}_{Z\cup\{x\}}$. Since Z is a parameter set, $\mathrm{HOD}_Z\models\mathsf{ZFC}$ and hence $\mathbb{R}\cap\mathrm{HOD}_Z$ is wellorderable. By boldface GCH at ω, $\mathbb{R}\cap\mathrm{OD}_Z$ is countable. Thus for all $x\in\mathbb{R}$ and $y\notin\mathrm{OD}_{Z\cup\{x\}}$, $R(x,y)$. That is, $\mathrm{dom}(R)=\mathbb{R}$. Suppose $\Phi:\mathbb{R}\to\mathbb{R}$ is a uniformization for R. Since Z is a good parameter set, Φ is $\mathrm{OD}_{Z\cup\{x\}}$ for some $x\in\mathbb{R}$. Then $\Phi(x)\in\mathrm{OD}_{Z\cup\{x\}}$ and $R(x,\Phi(x))$ which is a contradiction.

Theorem 4.23. *Assume* AD *and* $\mathsf{DC}_\mathbb{R}$. *Let* Z *be a very good parameter set such that* $V = L(Z,\mathbb{R})$. *Let* Y *be a very good parameter-witness for* Z *which is a finite set of sets of ordinals. Let* $\vec{y}$ *be a finite enumeration of* Y.[13] *Let* $E_2^{\vec{y}} = \bigsqcup_{r\in\mathbb{R}}{}^\omega(\omega_2^{L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},r]}) = \{(r,f) : r\in\mathbb{R}\wedge f\in{}^\omega(\omega_2^{L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},r]})\}$. *Then* $|E_2^{\vec{y}}| \leq |{}^\omega\omega_1|$, $\neg(|\omega_1|\leq|E_2^{\vec{y}}|)$, $\neg(|E_2^{\vec{y}}|\leq|\mathbb{R}\times\omega_1|)$, *and* $|\mathbb{R}\times\omega_1| < |(\mathbb{R}\times\omega_1)\sqcup E_2^{\vec{y}}| < |{}^\omega\omega_1|$.

Proof. Since $\omega_2^{L[\mathbb{B}^{\omega_1;\vec{y}},\vec{y},r]} < \omega_1$ for all $r\in\mathbb{R}$ by boldface GCH at ω (which holds under just AD as noted in Section 2), $E_2^{\vec{y}}\subseteq\mathbb{R}\times{}^\omega\omega_1$ and thus $|E_2^{\vec{y}}|\leq|{}^\omega\omega_1|$.

Suppose $\Phi_0:\omega_1\to E_2^{\vec{y}}$ be an injection. Let $\pi_1:\mathbb{R}\times{}^\omega\omega_1\to\mathbb{R}$ be the projection onto the first coordinate. $\pi_1[\Phi_0[\omega_1]]\subseteq\mathbb{R}$ is a wellorderable subset of $\mathbb{R}$ and thus must be countable by boldface GCH at ω. Since ω_1 is regular by $\mathsf{AC}^{\mathbb{R}}_{\omega}$, there is an $r\in\mathbb{R}$ so that $|(\pi_1\circ\Phi_0)^{-1}[\{r\}]| = \omega_1$. So $\Phi_0 : (\pi_1\circ\Phi_0)^{-1}[\{r\}] \to \{r\}\times{}^\omega(\omega_2^{L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},r]})$ is an injection. Since $\omega_2^{L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},r]} < \omega_1$, $|{}^\omega(\omega_2^{L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},r]})| = |\mathbb{R}|$. Thus Φ_0 induces an injection of ω_1 into $\mathbb{R}$ which violates boldface GCH at ω. This shows $\neg(|\omega_1|\leq|E_2^{\vec{y}}|)$.

Suppose $\Phi_1 : E_2^{\vec{y}}\to\mathbb{R}\times\omega_1$ is an injection. Identify $\mathbb{R}$ with $\mathscr{P}(\omega)$. If $(r,\alpha)\in\mathbb{R}\times\omega_1$, then let $P_{(r,\alpha)}\subseteq\omega\times\omega_1$ be defined by $P_{r,\alpha} = \{(n,\alpha) : n\in r\}$. If $(s,f)\in\mathbb{R}\times{}^\omega\omega_1$, then let $Q_{(s,f)}\subseteq\omega\times\omega\times\omega_1$ be defined by $Q_{(s,f)} = \{(m,n,\beta) : m\in s\wedge f(n)=\beta\}$. Let $\mathrm{Gr}_{\Phi_1}\subseteq\mathbb{R}\times{}^\omega\omega_1\times\omega\times\omega_1$ be defined by $\mathrm{Gr}_{\Phi_1} = \{(r,f,m,\alpha) : (r,f)\in E_2^{\vec{y}}\wedge(m,\alpha)\in P_{\Phi_1(r,f)}\}$. Let $\mathrm{Gr}_{\Phi_0^{-1}}\subseteq\mathbb{R}\times\alpha\times\omega\times\omega\times\omega_1$ be defined by $\mathrm{Gr}_{\Phi_1^{-1}} = \{(s,\alpha,m,n,\beta) : (s,\alpha)\in\Phi_1[\mathbb{R}\times\omega_1]\wedge(m,n,\beta)\in Q_{\Phi_1^{-1}(s,\alpha)}\}$. Since $V = L(Z,\mathbb{R}) = L(Y,\mathbb{R}) = L(\vec{y},\mathbb{R})$, there are formulas φ_0,φ_1, finite tuple of ordinals $\vec{\xi_0},\vec{\xi_1}$, and $e_0,e_1\in\mathbb{R}$ so that for all $(x,m,\alpha)\in(\mathbb{R}\times{}^\omega\omega_1)\times\omega\times\omega_1$ and $(y,m,n,\beta)\in(\mathbb{R}\times\omega_1)\times\omega\times\omega\times\omega_1$, $\mathrm{Gr}_{\Phi_1}(x,m,\alpha)\Leftrightarrow\varphi_0(x,m,\alpha,\vec{y},e_0,\vec{\xi_0})$ and

[12]It is worth noting that one advantage of Fact 4.17 is that one can use $\vec{\mathbb{B}}^{\omega_1;Z}$ rather than $\vec{\mathbb{B}}^{\omega;Z}$ which yields a more elegant proof of Theorem 4.23. The result can still be proved using the classical $\vec{\mathbb{B}}^{\omega;Z}$; however, one would need to introduce a coding of ${}^\omega\omega_1$ by reals and the collapse forcing much like in the proof of Fact 2.24.

[13]By the discussion above, if Z is a very good parameters set, there will always be a very good parameter-witness Y which is a finite set of sets of ordinals. It is necessary to use a finite enumeration $\vec{y}$ of Y rather than the finite set Y itself. For example, if r is a nonconstructible reals, then $L[\{r\}] = L$. However, since Y is a finite set of sets of ordinals, Y can be canonically wellordered by the lexicographic ordering on the powerset of some sufficiently large ordinal. Thus each member of Y is OD_Y.

$\mathrm{Gr}_{\Phi_1^{-1}}(y,m,n,\beta) \Leftrightarrow \varphi_1(y,m,n,\beta,\vec{y},e_1,\vec{\xi_1})$. Woodin ([40] Corollary 5.10) showed that there is an $e \in \mathbb{R}$ so that $e_0, e_1 \in L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e]$ and $L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e] \models \mathsf{CH}$. In $L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e]$, let $\mathbb{P}$ be the Namba forcing ([31] Theorem 28.10) consisting of $p \subseteq {}^{<\omega}\omega_2$ which are nonempty trees on ω_2 so that each node of p has ω_2 many extensions. Let $\leq_{\mathbb{P}}$ be the subset relation. $\mathbb{P}$ is countable in the real world and there is a $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over $L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e]$ and exists in the real world by Fact 2.14. Since $L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e] \models \mathsf{CH}$, by [31] Theorem 28.10, $\mathbb{P}$ adds a generic cofinal function $g : \omega \to \omega_2^{L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e]}$ induced by G and $\mathbb{P}$ adds no new reals. By Fact 4.17 and the fact that $e_0 \in L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e]$, $(m,\alpha) \in P_{\Phi_1(e,g)}$ if and only if $\mathrm{Gr}_{\Phi_1}((e,g),m,\alpha)$ if and only if $\varphi_0((e,g),m,\alpha,\vec{y},e_0,\vec{\xi_0})$ if and only if

$$L[\vec{\mathbb{B}}^{\omega;\vec{y}},\vec{y},e][g] \models 1_{\mathbb{B}_{\omega}^{\omega_1;\vec{y}}/_{\Pi_{\omega,1}^{\omega_1;\vec{y}}G_g^{\omega_1;\vec{y};1}}} \Vdash_{\mathbb{B}_{\omega}^{\omega_1;\vec{y}}/_{\Pi_{\omega,1}^{\omega_1;\vec{y}}G_g^{\omega_1;\vec{y};1}}} L(\check{\vec{y}},{}^{\omega}\dot{\omega}_1{}^{\mathrm{sym}}) \models \varphi_0(\check{e},\check{g},\check{m},\check{\alpha},\check{\vec{y}},\check{e}_0,(\vec{\xi_0})\check{}).$$

Thus $P_{\Phi_1(e,g)} \in L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e][g]$ by comprehension and hence $\Phi_1(e,g) \in L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e][g]$. Since $\Phi_1(e,g) \in \mathbb{R}\times\omega_1$ and $\mathbb{P}$ adds no new reals, $\Phi_1(e,g) \in L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e]$. By Fact 4.17, $(m,n,\beta) \in Q_{(e,g)}$ if and only if $(m,n,\beta) \in Q_{\Phi_1^{-1}(\Phi_1(e,g))}$ if and only if $\mathrm{Gr}_{\Phi_1^{-1}}(\Phi_1(e,g),m,n,\beta)$ if and only if $\varphi_1(\Phi_1(e,g),m,n,\beta,\vec{y},e_1,\vec{\xi_1})$ if and only if

$$L[\vec{\mathbb{B}}^{\omega;\vec{y}},\vec{y},e] \models 1_{\mathbb{B}_{\omega}^{\omega_1;\vec{y}}} \Vdash_{\mathbb{B}_{\omega}^{\omega_1;\vec{y}}} L(\check{\vec{y}},{}^{\omega}\dot{\omega}_1{}^{\mathrm{sym}}) \models \varphi_0((\Phi(e,g))\check{},\check{m},\check{n},\check{\beta},\check{\vec{y}},\check{e}_1,(\vec{\xi_1})\check{}).$$

Thus by comprehension, $Q_{(e,g)} \in L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e]$ and thus $g \in L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e]$. This is impossible since by absoluteness, $L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e] \models$ "$g : \omega \to \omega_2$ is cofinal". Since $L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e] \models \mathsf{AC}$, $L[\vec{\mathbb{B}}^{\omega_1;\vec{y}},\vec{y},e] \models \mathrm{cof}(\omega_2) = \omega_2$. Contradiction.

Since $E_2^{\vec{y}}$ does not inject into $\mathbb{R}\times\omega_1$, one has that $|\mathbb{R}\times\omega_1| < |(\mathbb{R}\times\omega_1)\sqcup E_2^{\vec{y}}|$. Since $|{}^{\omega}\omega_1| = |[\omega_1]^{\omega}| = |[\omega_1]^{\omega}_*|$, suppose there is an injection $\Phi_2 : [\omega_1]^{\omega}_* \to (\mathbb{R}\times\omega_1)\sqcup E_2^{\vec{y}}$. Since AD implies $\omega_1 \to_* (\omega_1)^{\omega}_2$, $\mu^{\omega}_{\omega_1}$ is an ultrafilter on $[\omega_1]^{\omega}_*$. Thus either $\Phi_2^{-1}[\mathbb{R}\times\omega_1] \in \mu^{\omega}_{\omega_1}$ or $\Phi_2^{-1}[E_2^{\vec{y}}] \in \mu^{\omega}_{\omega_1}$. Thus $|\Phi_2^{-1}[\mathbb{R}\times\omega_1]| = |{}^{\omega}\omega_1|$ or $|\Phi_2^{-1}[E_2^{\vec{y}}]| = |{}^{\omega}\omega_1|$. Thus Φ_2 either induces an injection of ${}^{\omega}\omega_1$ into $\mathbb{R}\times\omega_1$ or an injection of ${}^{\omega}\omega_1$ into $E_2^{\vec{y}}$. Thus there is an injection of ${}^{\omega}\omega_1$ into $\mathbb{R}\times\omega_1$ or an injection of ${}^{\omega}\omega_1$ into $E_2^{\vec{y}}$. The first is impossible by Fact 2.10 or Fact 3.5. The second is impossible since $|\omega_1| \leq |{}^{\omega}\omega_1|$ and it was shown above that $\neg(|\omega_1| \leq |E_2^{\vec{y}}|$. It has been shown that $|(\mathbb{R}\times\omega_1)\sqcup E_2^{\vec{y}}| < |{}^{\omega}\omega_1|$. ☐

Woodin showed that natural models of determinacy satisfying AD^+, $V = L(\mathscr{P}(\mathbb{R}))$, and $\neg\mathsf{AD}_{\mathbb{R}}$ must take the form $L(J,\mathbb{R})$ where J is a set of ordinals. Thus Woodin's dichotomy (the statement that for all $X \subseteq [\omega_1]^{\omega}$, $|X| \leq |\mathbb{R}\times\omega_1|$ or $|[\omega_1]^{\omega}| \leq |X|$) must always fails under AD^+, $V = L(\mathscr{P}(\mathbb{R}))$, and $\neg\mathsf{AD}_{\mathbb{R}}$ (with counterexamples derived from $E_2^{\langle J\rangle}$ when $V = L(J,\mathbb{R})$).

5. Amenable Surjections, Generic Filter Existence, and Capturing

This section will prove the basic tools needed for the main result concerning the cardinality of $\mathscr{P}_B(\kappa)$ when $\kappa < \Theta$.

The proof of Fact 2.25 and Theorem 3.3 serve as the basic template for the main argument. In Fact 2.25, the relevant forcing was $\mathrm{Coll}(\omega^+,\omega^{++})^M$ for some inner model M of ZFC. Since boldface GCH at ω was assumed to hold in the real world, Fact 2.14 asserts that there is a $\mathrm{Coll}(\omega^+,\omega^{++})^M$-generic filter over M which exists in the real world.

For motivation, assume $V = L(\mathbb{R})$ and for the sake of contradiction there is an injection $\Phi : \mathscr{P}_B(\kappa) \to {}^{\epsilon}\mathrm{ON}$ for $\epsilon < \kappa < \Theta$. Φ is ordinal definable from some $x \in \mathbb{R}$, but for simplicity assume $x = \emptyset$ and thus $\Phi \in \mathrm{OD}^{L(\mathbb{R})}$. The analogous forcing will be $\mathbb{P} = \mathrm{Coll}(\epsilon^+,\epsilon^{++})^{\mathrm{HOD}^{L(\mathbb{R})}}$. The first step will be to show a $\mathbb{P}$-generic filter over $\mathrm{HOD}^{L(\mathbb{R})}$ exists in the real world. Since Steel ([60] Corollary 8.22) showed $\mathrm{HOD}^{L(\mathbb{R})} \models \mathsf{GCH}$, $\mathrm{HOD}^{L(\mathbb{R})} \models |\mathscr{P}(\mathbb{P})| = 2^{\epsilon^{++}} = \epsilon^{+++}$. The collection $\mathcal{D}$ of all dense subsets of $\mathbb{P}$ in $\mathrm{HOD}^{L(\mathbb{R})}$ is internally in bijection with $(\epsilon^{+++})^{\mathrm{HOD}^{L(\mathbb{R})}}$. Since Steel ([60] Theorem 8.26) showed boldface GCH holds below Θ in $\mathrm{HOD}^{L(\mathbb{R})}$, there is a bijection of $(\epsilon^+)^{\mathrm{HOD}^{L(\mathbb{R})}}$ (even of ϵ or $|\epsilon|^{L(\mathbb{R})}$) onto $(\epsilon^{+++})^{\mathrm{HOD}^{L(\mathbb{R})}}$ externally in the real world. Thus there is a surjection $h : (\epsilon^+)^{\mathrm{HOD}^{L(\mathbb{R})}} \to \mathcal{D}$. It is tempting to use that $\mathrm{HOD}^{L(\mathbb{R})} \models$ "$\mathbb{P} = \mathrm{Coll}(\epsilon^+,\epsilon^{++})$ is $< \epsilon^+$-closed" to extend conditions in an ϵ^+-length construction to meet the dense set $h(\alpha)$ at stage α. This construction can only continue for $(\epsilon^+)^{\mathrm{HOD}^{L(\mathbb{R})}}$-stages if $h \restriction \alpha \in \mathrm{HOD}^{L(\mathbb{R})}$ for each $\alpha < \epsilon^+$ so that the α^{th}-stage of the construction can be internalized into $\text{HOD}^{L(\mathbb{R})}$ to use the fact that $\text{HOD}^{L(\mathbb{R})} \models$ "$\mathbb{P}$ is $< \epsilon^+$-closed". The existence of such a surjection h which is "amenable" relative to $\text{HOD}^{L(\mathbb{R})}$ will be a consequence of one further important consequence ([60] Lemma 8.25) of Steel's directed system analysis of $\text{HOD}^{L(\mathbb{R})}$: namely, for every $\epsilon < \Theta$, $\text{cof}^{L(\mathbb{R})}((\epsilon^+)^{\text{HOD}^{L(\mathbb{R})}}) = \omega$.

Let $M \models \mathsf{ZF}$ be an inner model. A set $A \subseteq M$ is M-amenable if and only if for all $x \in M$, $A \cap x \in M$. Here, one will be most concerned with functions $f : \epsilon \to M$ where $\epsilon \in \text{ON}$. A function $f : \epsilon \to M$ is M-amenable if and only if for all $\alpha < \epsilon$, $f \upharpoonright \alpha = \{(\xi, f(\xi)) : \xi < \alpha\} \in M$.

The next lemma gives a criterion for the existence of certain M-amenable surjections which are uniform in certain objects. A single application of the lemma will only need an appropriate form of countable choice (see Lemma 5.2). However, the uniformity will be important for applying the lemma to obtain uncountably many M-amenable surjections simultaneously which will be needed in Section 7.

Lemma 5.1. *Let $\omega < \kappa < \lambda$ and M is an inner model of* ZF *such that $M \models$ "κ and λ are cardinals". Suppose ρ_0, ρ_1, and ϖ have the following properties.*

(1) $\rho_0 : \omega \to \kappa$ is an increasing cofinal function through κ.
(2) $\rho_1 : \omega \to \lambda$ is an increasing cofinal function through λ.
(3) ϖ is a function on ω so that for all $n \in \omega$, $\varpi(n) : \kappa \to \rho_1(n)$ is an M-amenable surjection of κ onto $\rho_1(n)$.

Then there is a M-amenable surjection of κ onto λ which is produced uniformly from ρ_0, ρ_1, and ϖ with the above property.[14]

Proof. Since $M \models$ "κ is a cardinal", by modifying ρ_0 (uniformly) if necessary, one may assume ρ_0 takes values among the indecomposable limit ordinals. Let $I_0 = \{\alpha : 0 \leq \alpha < \rho_0(0)\}$ and for all $n > 0$, let $I_n = \{\alpha : \rho_0(n-1) \leq \alpha < \rho_0(n)\}$. Note that for all $n \in \omega$, $\text{ot}(I_n) = \rho_0(n)$ by the indecomposability of $\rho_0(n)$. Let $\mathfrak{m}_n : I_n \to \rho_0(n)$ be the Mostowski collapse of I_n. Note that $I_n, \mathfrak{m}_n \in M$ for each $n \in \omega$. For each $n \in \omega$, let $b_n : \rho_0(n) \to \rho_0(n) \times (n+1)$ be the standard bijection of $\rho_0(n)$ onto $\rho_0(n) \times (n+1)$ which belongs to M (or any inner model of ZF). Let $\pi_1 : \kappa \times \omega \to \kappa$ and $\pi_2 : \kappa \times \omega \to \omega$ be the projection on the first and second coordinate, respectively. For each $\alpha < \kappa$, let n_α be the unique n so that $\alpha \in I_n$. For each $n \in \omega$, let $f_n : \kappa \to \rho_1(n)$ be an M-amenable surjection defined by $f_n = \varpi(n)$. Define $\Phi : \kappa \to \lambda$ by $\Phi(\alpha) = f_{\pi_2(b_{n_\alpha}(\mathfrak{m}_{n_\alpha}(\alpha)))}(\pi_1(b_{n_\alpha}(\mathfrak{m}_{n_\alpha}(\alpha))))$.

Let $\beta < \lambda$. Let n_0 be the least $n \in \omega$ so that $\beta < \rho_1(n)$. Let n_1 be the least $n \in \omega$ so that $n \geq n_0$ and there is a $\gamma < \rho_0(n)$ so that $f_{n_0}(\gamma) = \beta$. Let γ_0 be the least γ such that $\gamma < \rho_0(n_1)$ and $f_{n_0}(\gamma) = \beta$. Let $\alpha = \mathfrak{m}_{n_1}^{-1}(b_{n_1}^{-1}(\gamma_0, n_0))$. Note that $\alpha \in I_{n_1}$ and therefore $n_1 = n_\alpha$. Thus

$$\Phi(\alpha) = f_{\pi_2(b_{n_\alpha}(\mathfrak{m}_{n_\alpha}(\alpha)))}(\pi_1(b_{n_\alpha}(\mathfrak{m}_{n_\alpha}(\alpha)))) = f_{\pi_2(b_{n_1}(\mathfrak{m}_{n_1}(\alpha)))}(\pi_1(b_{n_1}(\mathfrak{m}_{n_1}(\alpha))))$$

$$= f_{\pi_2(b_{n_1}(\mathfrak{m}_{n_1}(\mathfrak{m}_{n_1}^{-1}(b_{n_1}^{-1}(\gamma_0,n_0)))))}(\pi_1(b_{n_1}(\mathfrak{m}_{n_1}(\mathfrak{m}_{n_1}^{-1}(b_{n_1}^{-1}(\gamma_0, n_0)))))) = f_{\pi_2(\gamma_0,n_0)}(\pi_1(\gamma_0, n_0)) = f_{n_0}(\gamma_0) = \beta$$

Thus Φ is a surjection. Note that the definition of Φ is uniform in ρ_0, ρ_1, and ϖ.

Fix $n \in \omega$. Note that $\rho_0 \upharpoonright n+1$, $\langle \mathfrak{m}_k : k < n+1 \rangle$, $\langle I_k : k < n+1 \rangle$, and $\langle b_k : k < n+1 \rangle$ are elements of M (since these are finite functions into M). Since for all $k \leq n$, f_k is an M-amenable function, one has that $f_k \upharpoonright \rho_0(n) \in M$ and thus $\langle f_k \upharpoonright \rho_0(n) : k < n+1 \rangle \in M$. Note that $\Phi \upharpoonright \rho_0(n)$ can be defined using $\rho_0 \upharpoonright n+1$, $\langle \mathfrak{m}_k : k < n+1 \rangle$, $\langle I_k : k < n+1 \rangle$, $\langle b_k : k < n+1 \rangle$, and $\langle f_k \upharpoonright \rho_0(n) : k < n+1 \rangle$ using the definition of Φ above restricted to $\alpha < \rho_0(n)$. Since $M \models \mathsf{ZF}$, $\Phi \upharpoonright \rho_0(n) \in M$. Since ρ_0 is cofinal through κ, Φ is an M-amenable function. ☐

Since AD implies $\mathsf{AC}_\omega^{\mathbb{R}}$, one obtains the following using the Moschovakis coding lemma.

Lemma 5.2. *Assume* AD. *Let $\omega < \kappa < \lambda < \Theta$ with $\text{cof}(\kappa) = \text{cof}(\lambda) = \omega$ and M be an inner model of* ZF *such that $M \models$ "κ and λ are cardinals". Suppose for all $\delta < \lambda$, there is an M-amenable surjection of κ onto δ. Then there is an M-amenable surjection of κ onto λ.*

Proof. Since $\text{cof}(\kappa) = \text{cof}(\lambda) = \omega$, let $\rho_0 : \omega \to \kappa$ and $\rho_1 : \omega \to \lambda$ be increasing cofinal function through κ and λ, respectively. By the Moschovakis coding lemma (Fact 2.15), let $\varsigma : \mathbb{R} \to \mathscr{P}(\lambda)$ be a surjection. Define

[14] This means that there is a function Υ so that whenever ρ_0, ρ_1, and ϖ have the required properties, $\Upsilon(\rho_0, \rho_1, \varpi)$ is an M-amenable surjection from κ onto λ.

$R \subseteq \omega \times \mathbb{R}$ by $R(n,x)$ if and only if $\varsigma(x)$ codes the graph of an M-amenable surjection of κ onto $\rho_1(n)$. Observe that $\mathrm{dom}(R) = \omega$ by the hypothesis. By $\mathsf{AC}^{\mathbb{R}}_\omega$, there is a sequence $\langle x_n : n \in \omega\rangle$ so that $R(n, x_n)$ holds for all $n \in \omega$. Let $f_n : \kappa \to \rho_1(n)$ be the M-amenable surjection coded by $\varsigma(x_n)$. Let $\varpi(n) = f_n$. The result now following from applying Lemma 5.1 to ρ_0, ρ_1, and ϖ. □

Lemma 5.3. *Assume* AD. *Let $\kappa < \Theta$ with $\mathrm{cof}^V(\kappa) = \omega$ and M be an inner model of* ZF *such that $M \models$ "κ is a cardinal". Suppose $\delta < (\kappa^{+\omega_1^V})^M$ and has the property that for all λ with $\kappa \le \lambda \le \delta$, if $M \models$ "λ is a successor cardinal", then $\mathrm{cof}^V(\lambda) = \omega$. Then there is an M-amenable surjection of κ onto δ.*

Proof. Note that if δ satisfies this hypothesis, then every $\gamma < \delta$ also satisfies this hypothesis. By induction, it will be shown that for every $\gamma \le \delta$, there is an M-amenable surjection of κ onto γ.

If $\gamma < (\kappa^+)^M$, then there is a surjection of κ onto γ which is M-amenable (and even is a member of M). Now suppose $(\kappa^+)^M \le \gamma \le \delta$ and for all $\beta < \gamma$, there is an M-amenable surjection of κ onto β.

(Case I) γ is not a cardinal of M. There is a $g \in M$ so that $g : |\gamma|^M \to \gamma$ is a bijection. By the induction hypothesis, there is an M-amenable surjection $k : \kappa \to |\gamma|^M$. Then $g \circ k : \kappa \to \gamma$ is an M-amenable surjection.

(Case II) γ is a successor cardinal of M. By the hypothesis, $\mathrm{cof}^V(\gamma) = \omega$. Since $\mathrm{cof}^V(\kappa) = \omega$ and the induction hypothesis, Lemma 5.2 implies that there is an M-amenable surjection of κ onto γ.

(Case III) γ is a limit cardinal of M. Since $\gamma \le \delta < (\kappa^{+\omega_1^V})^M$, $\gamma = (\kappa^{+\alpha})^M$ for some limit ordinal $\alpha < \omega_1^V$. So $\mathrm{cof}^V(\alpha) = \omega$ and thus $\mathrm{cof}^V(\gamma) = \omega$. Again Lemma 5.2 implies that there is an M-amenable surjection of κ onto γ. □

Frequently, one will need an M-amenable surjection of a cardinal of cofinality ω onto the double successor of the cardinal. The following lemma isolates a uniform procedure to obtain such M-amenable surjection from ω-cofinal functions into the original cardinal, the successor, and the double successor which will be useful in the Section 7.

Lemma 5.4. *Suppose M is an inner model of* ZF *and $\kappa \in \mathrm{ON}$ is such that $M \models$ "κ is a cardinal". Suppose $\tau_0 : \omega \to \kappa$, $\tau_1 : \omega \to (\kappa^+)^M$, and $\tau_2 : \omega \to (\kappa^{++})^M$ are increasing cofinal map into κ, $(\kappa^+)^M$, and $(\kappa^{++})^M$, respectively. Suppose $\prec$ is a wellordering of $(\mathscr{P}((\kappa^{++})^M))^M$. Then there is an M-amenable surjection of κ onto $(\kappa^{++})^M$ which is obtained uniformly from τ_0, τ_1, τ_2, and $\prec$.*

Proof. Let $\rho_0^0 = \tau_0$. Let $\rho_1^0 = \tau_1$. Since for all $n \in \omega$, $\tau_1(n) < (\kappa^+)^M$, there is a surjection of κ onto $\tau_1(n)$ which is a member of M. Let $\varpi^0(n)$ be the $\prec$-least surjection of κ onto $\tau_1(n) = \rho_1^0(n)$ and note that ϖ^0 is produced uniformly in τ_0, τ_1, and $\prec$. Lemma 5.1 gives an M-amenable surjection $h_0 : \kappa \to (\kappa^+)^M$ which is produced uniformly in ρ_0^0, ρ_1^0, and ϖ^0. Let $\rho_0^1 = \tau_0$ and $\rho_1^1 = \tau_2$. For all $n \in \omega$, $\tau_2(n) < (\kappa^{++})^M$ and hence there is a surjection of $(\kappa^+)^M$ onto $\tau_2(n)$ which belong to M. Let $\sigma_n : (\kappa^+)^M \to \tau_2(n)$ be the $\prec$-least surjection of $(\kappa^+)^M$ onto $\tau_2(n)$. Let $\varpi^1(n) = \sigma_n \circ h_0$ which is an M-amenable surjection of κ into $\rho_1^1(n)$. Lemma 5.1 gives an M-amenable surjection $h_1 : \kappa \to (\kappa^{++})^M$ which is obtained uniformly in ρ_0^1, ρ_1^1, and ϖ^1. Thus h_1 is obtained uniformly in τ_0, τ_1, τ_2, and $\prec$. □

The following lemma uses a closedness condition of a forcing inside of an inner model M of ZFC and the existence of an appropriate M-amenable surjection to create generic filters over M which exist in the real world. Section 7 will use amenable surjections to create a large set of filters for a type of Namba forcing (which is not countably closed) generic over an inner model M by using fusion sequences.

Lemma 5.5. *Assume* AD. *Let M be an inner model of* ZFC *and $\mathbb{P} \in M$ be a forcing. Let $\mathcal{D}$ denote the $\mathbb{P}$-dense subsets of $\mathbb{P}$ in M. Let $\kappa \le \delta < \Theta$ be such that $\mathrm{cof}^V(\kappa) = \omega$, $M \models$ "κ is a cardinal, $|\mathcal{D}| \le \delta$, and $\mathbb{P}$ is $< \kappa$-closed", and there is an M-amenable surjection of κ onto δ. Then there is a $G \subseteq \mathbb{P}$ so that G is $\mathbb{P}$-generic over M.*

Proof. Since $M \models |\mathcal{D}| \le \delta$, let $f \in M$ be such that $f : \delta \to \mathcal{D}$ is a surjection. Let $g : \kappa \to \delta$ be an M-amenable surjection given by the hypothesis. Then $h : \kappa \to \mathcal{D}$ defined by $h = f \circ g$ is an M-amenable surjection. Since $M \models \mathsf{ZFC}$, let $\sqsubset \in M$ be such that $M \models$ "$\sqsubset$ is a wellordering of $\mathbb{P}$". Since for each $\gamma < \kappa$, $h \restriction \gamma \in M$ because h is M-amenable and $M \models$ "$\mathbb{P}$ is $< \kappa$-closed", there is a unique sequence $\langle p_\alpha^\gamma : \alpha < \gamma\rangle \in M$ so that for all $\alpha < \gamma$, p_α^γ is the $\sqsubset$-least element $p \in \mathbb{P}$ so that for all $\beta < \alpha$, $p \le_{\mathbb{P}} p_\beta^\gamma$ and p belongs to the dense set $h(\alpha)$. The uniqueness implies the following continuity: for all $\alpha < \gamma_1 \le \gamma_2 < \kappa$, $p_\alpha^{\gamma_0} = p_\alpha^{\gamma_1}$. For all $\alpha < \kappa$, let $p_\alpha = p_\alpha^{\alpha+1}$. Let $G = \langle \{p_\alpha : \alpha < \kappa\}\rangle_{\mathbb{P}} = \{p \in \mathbb{P} : (\exists \alpha < \kappa) : p_\alpha \leq_{\mathbb{P}} p\}$. Since for all $\alpha < \kappa$, $p_\alpha \in h(\alpha)$ and $h : \kappa \to \mathcal{D}$ is a surjection, G is a $\mathbb{P}$-generic filter over M. □

Lemma 5.6. *Assume* AD. *Let M be an inner model of* ZFC. *Let ϵ be an infinite ordinal. Suppose* $\text{cof}^V((\epsilon^+)^M) = \text{cof}^V((\epsilon^{++})^M) = \text{cof}^V((\epsilon^{+++})^M) = \omega$ *and* $M \models 2^{\epsilon^+} = \epsilon^{++} \wedge 2^{\epsilon^{++}} = \epsilon^{+++}$. *Let* $\text{Coll}(\epsilon^+, \epsilon^{++})^M$ *be the forcing of partial function $p : \epsilon^+ \to \epsilon^{++}$ of size less than ϵ^+ ordered by reverse function extension as defined in M. Then there is a $G \subseteq \text{Coll}(\epsilon^+, \epsilon^{++})^M$ which is $\text{Coll}(\epsilon^+, \epsilon^{++})^M$-generic over M.*

Proof. Since $\text{cof}((\epsilon^+)^M) = \text{cof}((\epsilon^{++})^M) = \text{cof}((\epsilon^{+++})^M) = \omega$, Lemma 5.3 implies that there is an M-amenable surjection $f : (\epsilon^+)^M \to (\epsilon^{+++})^M$.

Work in M for the moment. For all $\delta < \epsilon^{++}$, $|{}^{\epsilon^+}\delta| = 2^{\epsilon^+} = \epsilon^{++}$. $\text{Coll}(\epsilon^+, \epsilon^{++})$ injects into ${}^{\epsilon^+}\epsilon^{++}$, the collection of all functions from ϵ^+ into ϵ^{++}. Since ϵ^{++} is regular (as $M \models$ ZFC), ${}^{\epsilon^+}\epsilon^{++} = \bigcup_{\delta < \epsilon^{++}} {}^{\epsilon^+}\delta$. Then $|\text{Coll}(\epsilon^+, \epsilon^{++})| = \epsilon^{++}$. So $|\mathscr{P}(\text{Coll}(\epsilon^+, \epsilon^{++}))| = 2^{\epsilon^{++}} = \epsilon^{+++}$. Let $\mathcal{D}$ be the collection of dense subsets of $\text{Coll}(\epsilon^+, \epsilon^{++})$. Then $|\mathcal{D}| \leq |\mathscr{P}(\text{Coll}(\epsilon^+, \epsilon^{++}))| = \epsilon^{+++}$.

One has that $\text{cof}^V((\epsilon^+)^M) = \omega$, $M \models$ "$|\mathcal{D}| \leq \epsilon^{+++}$", and there is an M-amenable surjection of $(\epsilon^+)^M$ onto $(\epsilon^{+++})^M$. The result follows from Lemma 5.5. □

A tree on a set X is a set $T \subseteq {}^{<\omega}X$ which is closed under subsequences. The body of the tree T is $[T] = \{f \in {}^\omega X : (\forall n \in \omega)(f \upharpoonright n \in T)\}$. A set $A \subseteq {}^\omega\omega$ is Suslin if and only if there is a cardinal κ and a tree T on $\omega \times \kappa$ so that $A = \pi_1[[T]] = \{x \in {}^\omega\omega : (\exists f \in {}^\omega\kappa)((x, f) \in [T])\}$. A set $A \subseteq {}^\omega\omega$ is coSuslin if ${}^\omega\omega \setminus A$ is Suslin. If $A = \pi_1[[T]]$ where T is a tree on $\omega \times \kappa$, then one says that A is κ-Suslin. Let $\mathcal{S}_\kappa$ denote the collection of κ-Suslin set. A cardinal κ is a Suslin cardinal if and only if there is an $A \subseteq {}^\omega\omega$ which is κ-Suslin but not δ-Suslin for any $\delta < \kappa$. Let SN denote the class of all Suslin cardinals. A Suslin cardinal κ is a called a limit Suslin cardinal if and only if κ is a limit of Suslin cardinals. A Suslin cardinal which is not a limit Suslin cardinal is called a successor Suslin cardinal. Steel and Woodin showed that SN is a closed subset of sup(SN) under AD and $\mathsf{DC}_{\mathbb{R}}$. Under AD and $\mathsf{DC}_{\mathbb{R}}$, sup(SN) is a limit of Suslin cardinals. To see this, suppose for the sake of contradiction that $\lambda = \sup(\text{SN})$ is not a limit of Suslin cardinals. Then λ must be a successor Suslin cardinal. Suppose S_λ is not closed under $\forall^{\mathbb{R}}$. Then $\forall^{\mathbb{R}}\mathcal{S}_\lambda$ would have the scale property by the second periodicity theorem of Moschovakis ([45] 6C.3). Thus there is a set $A \in \forall^{\mathbb{R}}\mathcal{S}_\lambda \setminus S_\lambda$ which is Suslin. A must be λ'-Suslin for some $\lambda' > \lambda$. Hence λ is not the largest Suslin cardinal which is a contradiction. This shows S_λ must be closed under $\forall^{\mathbb{R}}$. Then λ must a regular limit Suslin cardinal by [30] Lemma 3.6 which contradict the assumption that λ is not even a limit of Suslin cardinals. This completes the argument that sup(SN) is a limit of Suslin cardinals. In fact, sup(SN) is a limit of measurable cardinals and even measurable successor Suslin cardinals. To see this, let $\kappa < \sup(\text{SN})$. Since sup(SN) is a limit of Suslin cardinals, let $\langle \kappa_n : n \in \omega\rangle$ be a increasing sequence of Suslin cardinals above κ and let $\kappa_\infty = \sup\{\kappa_n : n \in \omega\}$ which is a limit Suslin cardinal of cofinality ω. Let $\mathcal{S}_{<\kappa} = \bigcup_{\delta<\kappa} \mathcal{S}_\delta$. Let $\boldsymbol{\Sigma}_0$ be the pointclass of countable union of sets from $\mathcal{S}_{<\kappa}$. $\boldsymbol{\Sigma}_0$ has the scale property with norms in κ (see [30] Theorem 3.28, Case I), closed under countable union, and even closed under $\exists^{\mathbb{R}}$. By the second periodicity theorem of Moschovakis ([45] Theorem 6C.3), $\boldsymbol{\Pi}_1 = \forall^{\mathbb{R}}\boldsymbol{\Sigma}_0$ has the scale property with norm in $\boldsymbol{\delta}(\boldsymbol{\Pi}_1)$. By the Kunen-Martin Theorem, one can show that $\boldsymbol{\delta}(\boldsymbol{\Pi}_1) = \kappa^+$ and κ^+ is the next Suslin cardinal after κ (see [30] Theorem 3.28, Case I). Since $\boldsymbol{\Pi}_1$ is closed under $\forall^{\mathbb{R}}$, $\wedge$, $\vee$, and has the prewellordering property (even the scale property), $\kappa^+ = \boldsymbol{\delta}(\boldsymbol{\Pi}_1)$ is measurable and even satisfies $\kappa^+ \to_* (\kappa^+)^\epsilon_2$ for all $\epsilon < \omega_1$ by [30] Theorem 2.36 (and thus the ω-club filter $\mu^1_{\kappa^+}$ is a normal measure on κ^+). One can even show that sup(SN) is a supremum of regular limit Suslin cardinals and such cardinals have the strong partition property by [37]. Under AD^+, if $\sup(\text{SN}) < \Theta$, then $\sup(\text{SN}) \in \text{SN}$, i.e. sup(SN) is also a Suslin cardinal and SN has a largest element. (See [42] Corollary 11.23 and Remark 11.24.)

Solovay [55] showed that under $\mathsf{AD}_{\mathbb{R}}$, ω_1 is $\mathbb{R}$-supercompact. If ν is a supercompact measure on $\mathscr{P}_{\omega_1}(\mathbb{R})$ and $\kappa < \Theta$, then an appropriate Rudin-Keisler pushforward of ν will define a supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$. However, AD^+ cannot prove that ω_1 is $\mathbb{R}$-supercompact since $L(\mathbb{R}) \models$ "ω_1 is not $\mathbb{R}$-supercompact" because Solovay [55] showed this hypothesis implies the existence of $\mathbb{R}^\sharp$. Under AD and $\mathsf{DC}_{\mathbb{R}}$, Harrington and Kechris [24] showed that if κ is below a Suslin cardinal, then ω_1 is κ-supercompact. Woodin [66] showed the supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$ is unique if κ is below a Suslin cardinal. (Also see [42] Theorem 13.15.) Uniqueness will be very important as it implies that the supercompact measure ν on $\mathscr{P}_{\omega_1}(\kappa)$ is ordinal definable when κ is below a Suslin cardinal. Under AD^+, for all $\kappa < \Theta$, Woodin showed that ω_1 is

κ-supercompact and Neeman showed that there is a unique supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$. Except for Theorem 6.27, this result can be avoided by reflecting a counterexample below the largest Suslin cardinal.

Fact 5.7. *(Neeman [46] Theorem 2.19; Woodin [61] Lemma 2.19) Assume AD^+. For any $\kappa < \Theta$, there is a unique supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$.*

Fact 5.8. *Let X be a set and let ν be a supercompact measure on $\mathscr{P}_{\omega_1}(X)$. Let $j_\nu : V \to \prod_{\mathscr{P}_{\omega_1}(X)} V/\nu$ be the ultrapower map. Let $A \subseteq X$ and let $\Phi : \mathscr{P}_{\omega_1}(X) \to \mathscr{P}_{\omega_1}(X)$ be defined by $\Phi(\sigma) = A \cap \sigma$. Then $[\Phi]_\nu = j_\nu[A]$.*

Proof. Suppose $\Psi : \mathscr{P}_{\omega_1}(X) \to V$ and $[\Psi]_\nu \in [\Phi]_\nu$. By definition of the interpretation of the ultrapower membership relation, $B = \{\sigma \in \mathscr{P}_{\omega_1}(X) : \Psi(\sigma) \in \Phi(\sigma)\} \in \nu$. Define $\Upsilon : B \to \mathscr{P}_{\omega_1}(X)$ by $\Upsilon(\sigma) = \{\Psi(\sigma)\}$. Then for all $\sigma \in B$, $\emptyset \neq \Upsilon(\sigma) \subseteq \Phi(\sigma) \subseteq \sigma$. By normality, there is an $x \in X$ so that $C = \{\sigma \in \mathscr{P}_{\omega_1}(X) : x \in \Upsilon(\sigma)\} \in \nu$. Since for ν-almost all σ, $\Upsilon(\sigma)$ is the singleton $\Psi(\sigma)$, one has that for ν-almost all σ, $\Psi(\sigma) = x$. Thus $[\Psi]_\nu = j_\nu(x)$. Since for ν-almost all $\sigma \in \mathscr{P}_{\omega_1}(X)$, $\Psi(\sigma) \in \Phi(\sigma) = A \cap \sigma$, one has that $x \in A$. This shows that $[\Phi]_\nu \subseteq j_\nu[A]$. For any $x \in A$, $\{\sigma \in \mathscr{P}_{\omega_1}(X) : x \in A \cap \sigma = \Phi(\sigma)\} = \{\sigma \in \mathscr{P}_{\omega_1}(X) : x \in \sigma\} \in \nu$ by fineness. By Łoś' theorem, $j_\nu(x) \in [\Phi]_\nu$. Since $x \in A$ was arbitrary, this shows that $j_\nu[A] \subseteq [\Phi]_\nu$. One can conclude $j_\nu[A] = [\Phi]_\nu$. □

Let $\mathbb{P} = (\mathbb{P}, \leq_\mathbb{P}, 1_\mathbb{P})$ be a forcing and $F \subseteq \mathbb{P}$ is a $\mathbb{P}$-filter. F is said to be countably generated if and only if there is a countable $\sigma \subseteq F$ such that $F = \langle \sigma \rangle_\mathbb{P} = \{p \in \mathbb{P} : (\exists q \in \sigma)(q \leq_\mathbb{P} p)\}$. The next result says in certain circumstances, the truth concerning countably generated filter in the real world can be expressed over a HOD-type model. This result will give the required capturing necessary in the main result.

Lemma 5.9. *Assume AD and $\mathsf{DC}_\mathbb{R}$. Let Z be a very good parameter set and $V = L(Z,\mathbb{R})$. Let $\mathbb{P}$ be a forcing belonging to HOD_Z such that there is an ordinal κ with the property that there is a unique supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$ and $\text{HOD}_Z \models |\mathbb{P}| = |\kappa|$. Let F be a countable generated $\mathbb{P}$-filter. Then $\text{HOD}_Z[F] = \text{HOD}_{Z\cup\{F\}}$.*

In particular, under AD and $\mathsf{DC}_\mathbb{R}$, the hypothesis on κ holds if κ is below a Suslin cardinal. Under AD^+, this hypothesis on κ holds for all $\kappa < \Theta$.

Proof. Let Y be a very good parameter-witness for Z. By the hypothesis, let $\tilde{\nu}$ be the unique supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$. (If κ is below a Suslin cardinal, then the results of Harrington-Kechris and Woodin mentioned above imply the existence of such a unique supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$.) Thus $\tilde{\nu} \in \text{OD}$ and hence $\tilde{\nu} \in \text{OD}_Z$. By the hypothesis, let $\varpi : \kappa \to \mathbb{P}$ be a bijection with $\varpi \in \text{HOD}_Z$. Define a supercompact measure ν on $\mathscr{P}_{\omega_1}(\mathbb{P})$ by $A \in \nu$ if and only if $\{\varpi^{-1}[\sigma] : \sigma \in A\} \in \tilde{\nu}$. Note that $\nu \in \text{OD}_Z$. Since $V = L(Z,\mathbb{R}) = L(Y,\mathbb{R})$ and $\mathsf{DC}_\mathbb{R}$ hold, one has that DC holds. Hence $\prod_{\mathscr{P}_{\omega_1}(\mathbb{P})} V/\nu$ is wellfounded. Let $j_\nu : V \to \prod_{\mathscr{P}_{\omega_1}(\mathbb{P})} V/\nu$ be the ultrapower map. For each $\sigma \in \mathscr{P}_{\omega_1}(\mathbb{P})$, there is uniform procedure that assigns σ to a global wellordering of $L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, F \cap \sigma]$. Thus $\prod_{\sigma \in \mathscr{P}_{\omega_1}(\mathbb{P})} L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, F \cap \sigma]/\nu$ is wellfounded and satisfies Łoś' theorem. Let $\Phi_F : \mathscr{P}_{\omega_1}(\mathbb{P}) \to \mathscr{P}_{\omega_1}(\mathbb{P})$ be defined by $\Phi_F(\sigma) = \sigma \cap F$. By Fact 5.8, $[\Phi_F]_\nu = j_\nu[F]$. By Łoś' theorem, $\prod_{\sigma \in \mathscr{P}_{\omega_1}(\mathbb{P})} L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, F \cap \sigma]/\nu = L[j_\nu(\vec{\mathbb{B}}^{\kappa;Z}), j_\nu(\mathbb{P}), j_\nu(Y), [\Phi_F]_\nu] = L[j_\nu(\vec{\mathbb{B}}^{\kappa;Z}), j_\nu(\mathbb{P}), j_\nu(Y), j_\nu[F]]$. For each $\sigma \in \mathscr{P}_{\omega_1}(\kappa)$, let $\text{Coll}(\omega, F \cap \sigma)$ be the forcing of finite partial function from ω into $F \cap \sigma$. Let Φ_{Coll} be defined by $\Phi_{\text{Coll}}(\sigma) = \text{Coll}(\omega, F \cap \sigma)$. By Łoś' theorem, $[\Phi_{\text{Coll}}]_\nu = \text{Coll}(\omega, j_\nu[F])$. For each σ, let $\dot{g}_\sigma$ be the canonical $\text{Coll}(\omega, F \cap \sigma)$-name for the generic surjection of ω onto $F \cap \sigma$. Let $\Phi_{\dot{g}}$ be defined by $\Phi_{\dot{g}}(\sigma) = \dot{g}_\sigma$. Let $\dot{h} = [\Phi_{\dot{g}}]_\nu$. By Łoś' theorem, $\dot{h}$ is the canonical $\text{Coll}(\omega, j_\nu[F])$-name for the generic surjection of ω onto $j_\nu[F]$. Thus $\dot{h}$ belongs to any inner model of ZF containing $j_\nu[F]$. By Los' theorem applied in $\prod_{\sigma \in \mathscr{P}_{\omega_1}(\mathbb{P})} L[\vec{\mathbb{B}}^{\kappa,Z}, \mathbb{P}, Y, F \cap \sigma]/\nu$, $j_\nu(\vec{\mathbb{B}}^{\kappa;Z})$ codes a direct system of forcing projection and $j_\nu(\mathbb{B}^{\kappa;Z}_n) \subseteq \mathcal{B}_\infty$ for all $n \in \omega$ (that is, the conditions of the forcing $j_\nu(\mathbb{B}^{\kappa;Z}_n)$ are ∞-Borel codes or equivalently pairs of ordinals and formulas). Recall the operation $\mathcal{G}^{\kappa,1}$ from Definition 4.4.

<u>Claim</u>: Let $\varphi(v,w)$ be a $\mathscr{L}_{\text{set}}$-formula and $a \in \text{HOD}_Z$. Let $\tilde{\varphi}$ be the formula defined by $\tilde{\varphi}(x,y,z)$ is "y is a forcing, x is a function from ω into y, and $\varphi(\langle x[\omega]\rangle_y, z)$". Then

$$\varphi(F,a) \Leftrightarrow L[j_\nu(\vec{\mathbb{B}}^{\kappa;Z}), j_\nu(\mathbb{P}), j_\nu(Y), j_\nu(a), j_\nu[F]] \models 1_{\text{Coll}(\omega, j_\nu[F])} \Vdash L[j_\nu(\vec{\mathbb{B}}^{\kappa;Z}), j_\nu(\mathbb{P}), j_\nu(Y), j_\nu(a), \dot{h}] \models$$

$$1_{j_\nu(\mathbb{B}^{\kappa;Z}_\omega)/_{j_\nu(\Pi^{\kappa;Z}_{\omega,1})} \mathcal{G}^{j_\nu(\kappa),1}(j_\nu(\mathbb{B}^{\kappa;Z}_1), \dot{h})} \Vdash L(j_\nu(Y), j_\nu(\dot{\mathbb{R}}^{\kappa;\text{sym}})) \models \tilde{\varphi}(\dot{h}, j_\nu(\mathbb{P}), j_\nu(a)).$$

To see the claim: Since F is countably generated, there is a countable $\tilde{\sigma} \subseteq F$ so that $\langle \tilde{\sigma} \rangle_{\mathbb{P}} = F$. Since ν is fine and countably complete, $A = \{\sigma \in \mathscr{P}_{\omega_1}(\nu) : \tilde{\sigma} \subseteq \sigma\} \in \nu$.

($\Rightarrow$) Suppose $\varphi(F, a)$. Fix $\sigma \in A$. Let $p \in \mathrm{Coll}(\omega, F \cap \sigma)$. Since σ is countable, $\mathrm{Coll}(\omega, F \cap \sigma)$ is a countable forcing in the real world. By Fact 2.14, let $K \subseteq \mathrm{Coll}(\omega, F \cap \sigma)$ be $\mathrm{Coll}(\omega, F \cap \sigma)$-generic over $L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, F \cap \sigma]$ with $p \in K$. Let $k : \omega \to F \cap \sigma$ be the associated generic surjection. Note that $\langle k[\omega] \rangle_{\mathbb{P}} = \langle F \cap \sigma \rangle_{\mathbb{P}} \supseteq \langle \tilde{\sigma} \rangle_{\mathbb{P}} = F$ since $\tilde{\sigma} \subseteq \sigma$ because $\sigma \in A$. Hence $\langle k[\omega] \rangle_{\mathbb{P}} = F$. Since $L(Z, \mathbb{R}) = L(Y, \mathbb{R})$, one has that $L(Y, \mathbb{R}) \models \varphi(\langle k[\omega] \rangle_{\mathbb{P}}, a)$. Thus $L(Y, \mathbb{R}) \models \tilde{\varphi}(k, \mathbb{P}, a)$. Note that $L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a] \subseteq \mathrm{HOD}_Z$. Since $k \in {}^{\omega}\mathbb{P}$, Fact 4.17 (applied to $M = L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a]$) gives that

$$L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a, k] \models 1_{\mathbb{B}^{\kappa;Z}_{\omega} / \Pi^{\kappa;Z}_{\omega,n} G^{\kappa;Z;1}_k} \Vdash L(Y, \dot{\mathbb{R}}^{\kappa;\mathrm{sym}}) \models \tilde{\varphi}(k, \mathbb{P}, a).$$

By absoluteness and since $G^{\kappa;Z;1}_k = \mathcal{G}^{\kappa,1}(\mathbb{B}^{\kappa;Z}_1, k)$,

$$L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a, F \cap \sigma][K] \models L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a, k] \models 1_{\mathbb{B}^{\kappa;Z}_{\omega} / \Pi^{\kappa;Z}_{\omega,n} \mathcal{G}^{\kappa,1}(\mathbb{B}^{\kappa;Z}_1, k)} \Vdash L(Y, \dot{\mathbb{R}}^{\kappa;\mathrm{sym}}) \models \tilde{\varphi}(k, \mathbb{P}, a).$$

Since this holds for all $p \in \mathrm{Coll}(\omega, F \cap \sigma)$ and all $\mathrm{Coll}(\omega, F \cap \sigma)$-generic filter K over $L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a, F \cap \sigma]$ with $p \in K$, one has the following by the forcing theorem.

$$L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a, F \cap \sigma] \models 1_{\mathrm{Coll}(\omega, F \cap \sigma)} \Vdash L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a, \dot{g}_{\sigma}] \models 1_{\mathbb{B}^{\kappa;Z}_{\omega} / \Pi^{\kappa;Z}_{\omega,n} \mathcal{G}^{\kappa,1}(\mathbb{B}^{\kappa;Z}_1, \dot{g}_{\sigma})} \Vdash L(Y, \dot{\mathbb{R}}^{\kappa;\mathrm{sym}}) \models \tilde{\varphi}(\dot{g}_{\sigma}, \mathbb{P}, a).$$

Since $\sigma \in A \in \nu$ was arbitrary, the right side of the claim follows from Łoś' theorem.

($\Leftarrow$) Assume the right side of the claim. By Łoś' theorem, there is a $B \in \nu$ such that for all $\sigma \in B$

$$L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a, F \cap \sigma] \models 1_{\mathrm{Coll}(\omega, F \cap \sigma)} \Vdash L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a, \dot{g}_{\sigma}] \models 1_{\mathbb{B}^{\kappa;Z}_{\omega} / \Pi^{\kappa;Z}_{\omega,n} \mathcal{G}^{\kappa,1}(\mathbb{B}^{\kappa;Z}_1, \dot{g}_{\sigma})} \Vdash L(Y, \dot{\mathbb{R}}^{\kappa;\mathrm{sym}}) \models \tilde{\varphi}(\dot{g}_{\sigma}, \mathbb{P}, a).$$

Since $A \in \nu$, $A \cap B \in \nu$. Pick any $\sigma \in A \cap B$. Since $\mathrm{Coll}(\omega, F \cap \sigma)$ is countable in the real world, let $K \subseteq \mathrm{Coll}(\omega, F \cap \sigma)$ be $\mathrm{Coll}(\omega, F \cap \sigma)$-generic over $L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a, F \cap \sigma]$ using Fact 2.14. Let k be the generic surjection of ω onto $F \cap \sigma$ induced from K and note that $\dot{g}_{\sigma}[K] = k$. By the forcing theorem, one has the following.

$$L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a, F \cap \sigma][K] \models L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a, k] \models 1_{\mathbb{B}^{\kappa;Z}_{\omega} / \Pi^{\kappa;Z}_{\omega,n} \mathcal{G}^{\kappa,1}(\mathbb{B}^{\kappa;Z}_1, k)} \Vdash L(Y, \dot{\mathbb{R}}^{\kappa;\mathrm{sym}}) \models \tilde{\varphi}(k, \mathbb{P}, a).$$

By absoluteness,

$$L[\vec{\mathbb{B}}^{\kappa;Z}, \mathbb{P}, Y, a, k] \models 1_{\mathbb{B}^{\kappa;Z}_{\omega} / \Pi^{\kappa;Z}_{\omega,n} \mathcal{G}^{\kappa,1}(\mathbb{B}^{\kappa;Z}_1, k)} \Vdash L(Y, \dot{\mathbb{R}}^{\kappa;\mathrm{sym}}) \models \tilde{\varphi}(k, \mathbb{P}, a).$$

Since $k \in {}^{\omega}\mathbb{P}$, Fact 4.17 implies $L(Y, \mathbb{R}) \models \tilde{\varphi}(k, \mathbb{P}, a)$. By the definition of $\tilde{\varphi}$, $L(Y, \mathbb{R}) \models \varphi(\langle k[\omega] \rangle_{\mathbb{P}}, a)$. Since $\sigma \in A \cap B \subseteq A$ and thus $\tilde{\sigma} \subseteq F \cap \sigma = k[\omega]$, $\langle k[\omega] \rangle_{\mathbb{P}} = F$. So $L(Y, \mathbb{R}) \models \varphi(F, a)$. Finally since $V = L(Z, \mathbb{R}) = L(Y, \mathbb{R})$, one has $\varphi(F, a)$. This completes the proof of the claim.

Now to complete the proof of the lemma: It is clear that $\mathrm{HOD}_Z[F] \subseteq \mathrm{HOD}_{Z \cup \{F\}}$. Let $U \in \mathrm{HOD}_{Z \cup \{F\}}$. Let W be the transitive closure of $\{U\}$. Since $\mathrm{HOD}_{Z \cup \{F\}} \models \mathsf{ZFC}$, there is a bijection $\pi : \delta \to W$ where δ is an ordinal and $\pi \in \mathrm{HOD}_{Z \cup \{F\}}$. Within $\mathrm{HOD}_{Z \cup \{F\}}$, define $R \subseteq \delta \times \delta$ by $R(\alpha, \beta)$ if and only if $\pi(\alpha) \in \pi(\beta)$. By the hypothesis that $\mathrm{OD}_Z = \mathrm{OD}_Y$, one has that $\mathrm{OD}_{Z \cup \{F\}} = \mathrm{OD}_{Y \cup \{F\}}$. There is a finite sequence $\vec{y}$ in Y (and note that all the elements in $\vec{y}$ belong to HOD_Z since $Y \in \mathrm{HOD}_Z = \mathrm{HOD}_Y$), a finite sequence of ordinals $\vec{\xi}$, and a $\mathscr{L}_{\mathrm{set}}$-formula φ so that $R(\alpha, \beta)$ if and only if $\varphi(F, \vec{y}, \vec{\xi}, \alpha, \beta)$. Note that $j_{\nu}(\vec{\mathbb{B}}^{\kappa;Z}), j_{\nu}(\mathbb{P}), j_{\nu}(Y), j_{\nu}(\vec{y}), j_{\nu} \restriction \delta \in \mathrm{HOD}_Z$. Since $j_{\nu} \restriction \mathbb{P} \in \mathrm{HOD}_Z$, $j_{\nu}[F] \in \mathrm{HOD}_Z[F]$. Also note that $\dot{h}$ belongs to any inner model of ZF containing $j_{\nu}[F]$. Thus by the above claim and absoluteness, for all $\alpha, \beta \in \delta$, $R(\alpha, \beta)$ if and only if

$$\mathrm{HOD}_Z[F] \models L[j_{\nu}(\vec{\mathbb{B}}^{\kappa;Z}), j_{\nu}(\mathbb{P}), j_{\nu}(Y), j_{\nu}(\vec{y}), j_{\nu}[F]] \models 1_{\mathrm{Coll}(\omega, j_{\nu}[F])} \Vdash L[j_{\nu}(\mathbb{B}^{\kappa;Z}), j_{\nu}(\mathbb{P}), j_{\nu}(Y), j_{\nu}(\vec{y}), \dot{h}] \models$$

$$1_{j_{\nu}(\mathbb{B}^{\kappa;Z}_{\omega}) / j_{\nu}(\Pi^{\kappa;Z}_{\omega,1}) \mathcal{G}^{j_{\nu}(\kappa),1}(j_{\nu}(\mathbb{B}^{\kappa;Z}_1), \dot{h})} \Vdash L(j_{\nu}(Y), j_{\nu}(\dot{\mathbb{R}}^{\kappa;\mathrm{sym}})) \models \tilde{\varphi}(\dot{h}, j_{\nu}(\mathbb{P}), j_{\nu}(\vec{y}), j_{\nu}(\vec{\xi}), j_{\nu}(\alpha), j_{\nu}(\beta)).$$

Since $j_{\nu} \restriction \delta \in \mathrm{HOD}_Z[F]$, the above formula is obtained uniformly from (α, β) in $\mathrm{HOD}_Z[F]$. By comprehension in $\mathrm{HOD}_Z[F]$, $R \in \mathrm{HOD}_Z[F]$. Thus $W \in \mathrm{HOD}_Z[F]$ since it is the Mostowski collapse of the wellfounded relation (δ, R). Then $U \in \mathrm{HOD}_Z[F]$ since it is the element of highest rank in W. Since $U \in \mathrm{HOD}_{Z \cup \{F\}}$ was arbitrary, this shows that $\mathrm{HOD}_{Z \cup F} \subseteq \mathrm{HOD}_Z[F]$. This completes the proof of the lemma. □

Corollary 5.10. *Assume* AD *and* $\mathsf{DC}_\mathbb{R}$. *Let* Z *be a very good parameter set. Let* $\delta \leq \kappa < \Theta$ *be such that* κ *is below a Suslin cardinal,* $\mathrm{HOD}_Z \models$ *"*δ *is a cardinal", and* $\mathrm{cof}^V(\delta) = \omega$. *Suppose* $f : \delta \to \kappa$ *is a function with the property that for all* $\alpha < \delta$, $f \upharpoonright \delta \in \mathrm{HOD}_Z$. *Then* $\mathrm{HOD}_Z[f] = \mathrm{HOD}_{Z \cup \{f\}}$.

Proof. Let $\mathbb{P} = \mathrm{Coll}(\delta, \kappa)^{\mathrm{HOD}_Z}$ be the forcing of partial functions from δ into κ of cardinality less than δ. Since κ is less than a Suslin cardinal and the supremum of the Suslin cardinals is a limit of measurable Suslin cardinals, κ is a below a measurable Suslin cardinal λ. Since λ is measurable, there is no injection of λ into $\mathscr{P}(\kappa)$ by Fact 2.3. Because $\mathbb{P}$ may be regarded as a wellorderable subset of $\mathscr{P}(\kappa)$, one must have $|\mathbb{P}|^{\mathrm{HOD}_Z} < |\lambda|$. By the hypothesis that for all $\alpha < \delta$, $f \upharpoonright \alpha \in \mathrm{HOD}_Z$, one has that $F = \langle\{f \upharpoonright \alpha : \alpha < \delta\}\rangle_\mathbb{P}$ is a $\mathbb{P}$-filter, $\mathrm{HOD}_Z[F] = \mathrm{HOD}_Z[f]$, and $\mathrm{HOD}_{Z\cup\{F\}} = \mathrm{HOD}_{Z\cup\{f\}}$. By the hypothesis that $\mathrm{cof}^V(\delta) = \omega$, let $\rho : \omega \to \delta$ be an increasing cofinal function. Let $\sigma = \{f \upharpoonright \rho(n) : n \in \omega\}$. Then $\langle\sigma\rangle_\mathbb{P} = F$. This shows that F is a countable generated $\mathbb{P}$-filter. Thus by Lemma 5.9, $\mathrm{HOD}_Z[f] = \mathrm{HOD}_Z[F] = \mathrm{HOD}_{Z\cup\{F\}} = \mathrm{HOD}_{Z\cup\{f\}}$. ☐

6. Cardinality of the Set of Bounded Subsets of a Cardinal

This section will prove the eponymous main result of this paper.

Suppose $\Phi : \mathscr{P}_B(\kappa) \to {}^\epsilon\mathrm{ON}$ where $\epsilon < \kappa$ is an injection. To follow the template of Fact 2.25 to prove the main result, it remains to find a HOD-type model M for which Lemma 5.6 can be used to produce generics for $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^M$ in the real world and the HOD-type model M codes Φ is some way so that Lemma 5.9 can be used to capture the function in the $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^M$-generic extension. The model and its requisite properties come from the direct system analysis of HOD-type inner models of certain AD^+ models. Woodin and Steel used this analysis of HOD-type models to show boldface GCH below Θ and the existence and uniqueness of the supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$ for $\kappa < \Theta$ under AD^+ which were mentioned earlier.

First, the main result will be proved for $L(\mathbb{R})$ under AD which has the most familiar HOD-analysis. Finally, the main result under AD^+ will be given which uses the theory of Γ-Woodin mice, their iteration strategy, and hybrid strategy mouse built over these Γ-Woodin mice.

For $L(\mathbb{R}) \models \mathsf{AD}$, the necessary HOD-type model comes from the original analysis of $\mathrm{HOD}^{L(\mathbb{R})}$. $\boldsymbol{\delta}^2_1$ is the supremum of the $\boldsymbol{\Delta}^2_1$ prewellorderings on $\mathbb{R}$. By [40] Theorem 2.28, $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$ is also the least Σ_1-stable ordinal of $L(\mathbb{R})$ (that is, the least ordinal δ such that $L_\delta(\mathbb{R}) \prec_1 L(\mathbb{R})$ in the language of set theory with constant symbols for reals and the set of reals). By the analysis of scales in $L(\mathbb{R})$ ([43], [57]), $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$ is also the largest Suslin cardinal of $L(\mathbb{R})$. The following will be the relevant properties.

Fact 6.1. *(Steel, Woodin;* [58],[64]*) Assume* AD. *Let* $x \in \mathbb{R}$.

(1) $\mathrm{HOD}^{L(\mathbb{R})}_{\{x\}} \models \mathsf{GCH}$.

(2) For all $\kappa < \Theta^{L(\mathbb{R})}$, $\mathrm{cof}^{L(\mathbb{R})}((\kappa^+)^{\mathrm{HOD}^{L(\mathbb{R})}_{\{x\}}}) = \omega$.

Only the simpler analysis of $\mathrm{HOD}^{L(\mathbb{R})}_{\{x\}} \cap V_{(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}}$ is necessary for the first statement of Fact 6.1 (see remarks following the proof of [60] Corollary 8.22). This simpler analysis is also sufficient to prove the second statement of Fact 6.1 for $\kappa < (\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$ (see remarks following [60] Lemma 8.25) which will be enough for the proof of Theorem 6.2. These are proved by showing that a portion of $\mathrm{HOD}^{L(\mathbb{R})}_{\{x\}}$ is a limit of a directed system of mice with certain suitable properties. Moreover, if one assumes $M_1(x)^\sharp$ exists, then the proof can be even further simplified by using an external directed system of mice as in [60]. The following is the main theorem in $L(\mathbb{R})$.

Theorem 6.2. *Assume* AD. $L(\mathbb{R}) \models$ *"For all cardinals* κ *with* $\omega < \kappa < \Theta$ *and any* $\epsilon < \kappa$*, there is no injection of* $\mathscr{P}_B(\kappa)$ *into* ${}^\epsilon\mathrm{ON}$*".*

Proof. By Kechris [35], if AD holds, then $L(\mathbb{R}) \models \mathsf{DC}_\mathbb{R}$. For the rest of the proof, work within $L(\mathbb{R})$ or assume $V = L(\mathbb{R})$. Assume the theorem is false. Since $\kappa \geq \omega_1$, $|\mathbb{R}| \leq |\mathscr{P}_B(\kappa)|$. If $\epsilon < \omega$, then ${}^\epsilon\mathrm{ON}$ has a definable bijection with ON. An injection $\Phi : \mathscr{P}_B(\kappa) \to {}^\epsilon\mathrm{ON}$ when $\epsilon < \omega$ would imply there is an injection of $\mathbb{R}$ into ON. This is impossible since AD implies there are no wellorderings of $\mathbb{R}$. Thus any failure of the theorem must take the form $\Phi : \mathscr{P}_B(\kappa) \to {}^\epsilon\mathrm{ON}$ where ϵ is infinite. Let ϕ be the $\mathscr{L}_{\mathrm{set}}$ sentence asserting "there exists $\epsilon < \kappa < \Theta$ and injection $\Phi : \mathscr{P}_B(\kappa) \to {}^\epsilon\mathrm{ON}$". Let ZF^* denote some sufficiently large finite fragment of ZF. Let φ be the formula in the language $\mathscr{L}_{\mathrm{set}}$ augmented with a symbol for $\mathbb{R}$ asserting "there exist an ordinal α such that $L_\alpha(\mathbb{R}) \models \mathsf{ZF}^*$, AD, and ϕ". Note that φ is Σ_1 in this augmented language. Note that

$L(\mathbb{R}) \models \phi$. By reflection applied to the hierarchy $\langle L_\alpha(\mathbb{R}) : \alpha \in \mathrm{ON}\rangle$, there is an $\alpha \in \mathrm{ON}$ so that $L_\alpha(\mathbb{R}) \models \mathsf{ZF}^*$, AD, and ϕ. So $L(\mathbb{R}) \models \varphi$. Since $L_{\boldsymbol{\delta}^2_1}(\mathbb{R}) \prec_1 L(\mathbb{R})$, $L_{\boldsymbol{\delta}^2_1}(\mathbb{R}) \models \varphi$. Thus there is an $\alpha < \boldsymbol{\delta}^2_1$, $L_\alpha(\mathbb{R}) \models \phi$. So there are $\epsilon < \kappa < \Theta^{L_\alpha(\mathbb{R})} < \boldsymbol{\delta}^2_1$ and a function $\Phi \in L_\alpha(\mathbb{R})$ such that $L_\alpha(\mathbb{R}) \models$ "$\Phi : \mathscr{P}_B(\kappa) \to {}^\epsilon\mathrm{ON}$ is an injection". Since $\kappa < \Theta^{L_\alpha(\mathbb{R})}$, let $\psi : \mathbb{R} \to \kappa$ be a surjection in $L_\alpha(\mathbb{R})$. Let $\preceq \in L_\alpha(\mathbb{R})$ be the prewellordering on $\mathbb{R}$ of length κ associated to ψ defined by $x \preceq y$ if and only if $\psi(x) \leq \psi(y)$. Let Γ be a pointclass in $L_\alpha(\mathbb{R})$ closed under $\exists^\mathbb{R}$ and $\wedge$ with $\preceq \in \Gamma$. (For instance, $\Gamma = \boldsymbol{\Sigma}^1_1(\preceq)$.) Let $U \subseteq \mathbb{R}^3$ be universal for Γ and note that $U \in L_\alpha(\mathbb{R})$. Let $\vec{0}$ and $\vec{1}$ be the constant 0 and 1 function, respectively. For each $e \in \mathbb{R}$, let $A_e = \{\alpha \in \kappa : (\exists x)(\psi(x) = \alpha \wedge U(e, x, \vec{1}))\}$. For each $A \in \mathscr{P}(\kappa)$, define $T_A \subseteq \mathbb{R} \times \mathbb{R}$ by $(x, y) \in T_A$ if and only if $\psi(x) \in A \Rightarrow y = \vec{1}$ and $\psi(x) \notin A \Rightarrow y = \vec{0}$. By the Moschovakis coding lemma (Fact 2.15), there is a $S \in \Gamma$ so that $S \subseteq T_A$ and for all $\alpha < \kappa$, there are $x, y \in \mathbb{R}$ such that $\psi(x) = \alpha$ and $S(x, y)$. Since U is universal, there is an $e \in \mathbb{R}$ so that $S = U_e = \{(x, y) : U(e, x, y)\}$. Then $A = A_e$. It has been shown that for all $A \in \mathscr{P}(\kappa)$, there is an $e \in \mathbb{R}$ such that $A = A_e$. Since $\psi, \preceq, U \in L_\alpha(\mathbb{R})$, the absoluteness of the map $e \mapsto A_e$ implies that $\mathscr{P}(\kappa)^{L_\alpha(\mathbb{R})} = \mathscr{P}(\kappa)^{L(\mathbb{R})}$. Thus $\mathscr{P}_B(\kappa)^{L_\alpha(\mathbb{R})} = \mathscr{P}_B(\kappa)^{L(\mathbb{R})}$. By absoluteness, $\Phi : \mathscr{P}_B(\kappa) \to {}^\epsilon\mathrm{ON}$ is an injection in $L(\mathbb{R})$. All sets in $L(\mathbb{R})$ are ordinal definable from a real. Let $x \in \mathbb{R}$ be such that $\Phi \in \mathrm{OD}_x$. $\mathrm{HOD}_{\{x\}} \models \mathsf{GCH}$ and $\mathrm{cof}^{L(\mathbb{R})}((\epsilon^+)^{\mathrm{HOD}_{\{x\}}}) = \mathrm{cof}^{L(\mathbb{R})}((\epsilon^{++})^{\mathrm{HOD}_{\{x\}}}) = \mathrm{cof}^{L(\mathbb{R})}((\epsilon^{+++})^{\mathrm{HOD}_{\{x\}}}) = \omega$ by Fact 6.1. Thus there is a $G \subseteq \mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$ (in the real world) which is $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$-generic over $\mathrm{HOD}_{\{x\}}$ by Lemma 5.6. Let $g : (\epsilon^+)^{\mathrm{HOD}_{\{x\}}} \to (\epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$ be the generic surjection, namely $g = \bigcup G$. Since $\mathrm{cof}^{L(\mathbb{R})}((\epsilon^+)^{\mathrm{HOD}_{\{x\}}}) = \omega$, let $\rho : \omega \to (\epsilon^+)^{\mathrm{HOD}_{\{x\}}}$ be an increasing cofinal map. Let $\sigma = \{g \upharpoonright \rho(n) : n \in \omega\}$ and note that $\langle \sigma \rangle_{\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_{\{x\}}}} = G$. Thus G is a countably generated $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$ filter. Also $|(\epsilon^{++})^{\mathrm{HOD}_{\{x\}}}|$ is below a Suslin cardinal (for example, $\boldsymbol{\delta}^2_1$) and $\mathrm{HOD}_{\{x\}} \models |\mathrm{Coll}(\epsilon^+, \epsilon^{++})| = \epsilon^{++}$. Thus by Lemma 5.9, $\mathrm{HOD}_{\{x\}}[G] = \mathrm{HOD}_{\{x, G\}}$. Let $\tilde{g} = \{\langle \alpha, g(\alpha)\rangle : \alpha < (\epsilon^+)^{\mathrm{HOD}_{\{x\}}}\}$ (where $\langle \cdot, \cdot \rangle$ refers to the Gödel pairing function). Note that $\tilde{g} \subseteq (\epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$ and thus $\tilde{g} \in \mathscr{P}_B(\kappa)$ since $(\epsilon^{++})^{\mathrm{HOD}_{\{x\}}} < \kappa$ by Fact 2.18. Since Φ is $\mathrm{OD}_{\{x\}}$ and $\tilde{g} \in \mathrm{OD}_{\{G\}}$, one has that $\Phi(\tilde{g}) \in \mathrm{OD}_{\{x, G\}}$. Therefore, $\Phi(\tilde{g}) \in \mathrm{HOD}_{\{x, G\}} = \mathrm{HOD}_{\{x\}}[G]$. Since $\Phi(\tilde{g}) \in {}^\epsilon\mathrm{ON} \cap \mathrm{HOD}_{\{x\}}[G]$ and $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$ add no new functions from ϵ into $\mathrm{HOD}_{\{x\}}$ since $\mathrm{HOD}_{\{x\}} \models$ "$\mathrm{Coll}(\epsilon^+, \epsilon^{++})$ is $< \epsilon^+$-closed", one has that $\Phi(\tilde{g}) \in \mathrm{HOD}_{\{x\}}$. Since $\Phi(\tilde{g}) \in \mathrm{OD}_x$, $\Phi^{-1} \in \mathrm{OD}_{\{x\}}$, Φ is an injection, and $\tilde{g} = \Phi^{-1}(\Phi(\tilde{g}))$, one has that $\tilde{g} \in \mathrm{OD}_{\{x\}}$ and hence $\tilde{g} \in \mathrm{HOD}_{\{x\}}$. By applying the inverse of the Gödel pairing function, $g \in \mathrm{HOD}_{\{x\}}$ and thus $G \in \mathrm{HOD}_{\{x\}}$. This is impossible since G is a $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$-generic over $\mathrm{HOD}_{\{x\}}$ or alternatively by absoluteness, $\mathrm{HOD}_{\{x\}} \models$ "$g : \epsilon^+ \to \epsilon^{++}$ is a surjection". ☐

Theorem 6.3. *Assume* AD. *Within* $L(\mathbb{R})$*, the following holds: Let* $\kappa < \Theta = \Theta^{L(\mathbb{R})}$ *be a singular cardinal and let* $\epsilon < \kappa$ *be such that that* $\mathrm{cof}(\epsilon) = \mathrm{cof}(\kappa)$*. Then*

(1) $|\mathscr{P}_B(\kappa)| < |[\kappa]^\epsilon \cup \mathscr{P}_B(\kappa)|$.
(2) $|[\kappa]^\epsilon| < |[\kappa]^\epsilon \cup \mathscr{P}_B(\kappa)|$.
(3) $|[\kappa]^\epsilon \cup \mathscr{P}_B(\kappa)| < |\mathscr{P}(\kappa)|$.

Proof. (1) If $\Phi : [\kappa]^\epsilon \cup \mathscr{P}_B(\kappa) \to \mathscr{P}_B(\kappa)$ is an injection, then $\Phi \upharpoonright [\kappa]^\epsilon : [\kappa]^\epsilon \to \mathscr{P}_B(\kappa)$ is an injection. This violates Theorem 2.2 which applies since boldface GCH below $\Theta^{L(\mathbb{R})}$ holds by Fact 2.22 and $L(\mathbb{R}) \models \mathsf{AD}^+$ or more explicitly by [60] Theorem 8.26.

(2) If $\Phi : [\kappa]^\epsilon \cup \mathscr{P}_B(\kappa) \to [\kappa]^\epsilon$ is an injection, then $\Phi \upharpoonright \mathscr{P}_B(\kappa) : \mathscr{P}_B(\kappa) \to [\kappa]^\epsilon$ is an injection. This violates Theorem 6.2 since $\epsilon < \kappa$.

(3) Assume that statement (3) is false. By a reflection argument as in the beginning of the proof of Theorem 6.2, there is a singular cardinal $\kappa < \boldsymbol{\delta}^2_1$ and an injection $\Phi : \mathscr{P}(\kappa) \to [\kappa]^\epsilon \cup \mathscr{P}_B(\kappa)$. Since $V = L(\mathbb{R})$, there is an $x \in \mathbb{R}$ so that Φ is $\mathrm{OD}_{\{x\}}$. By Fact 6.1, $\mathrm{HOD}_{\{x\}} \models \mathsf{GCH}$ and $\mathrm{cof}((\epsilon^+)^{\mathrm{HOD}_{\{x\}}}) = \mathrm{cof}((\epsilon^{++})^{\mathrm{HOD}_{\{x\}}}) = \mathrm{cof}((\epsilon^{++})^{\mathrm{HOD}_{\{x\}}}) = \omega$. By Lemma 5.6, let $G \subseteq \mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$ (which belongs to $L(\mathbb{R})$) be $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$-generic over $\mathrm{HOD}_{\{x\}}$. Since $\mathrm{cof}^{L(\mathbb{R})}((\epsilon^+)^{\mathrm{HOD}_{\{x\}}}) = \omega$, G is a countably generated $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$ filter. Since $\mathrm{HOD}_{\{x\}} \models |\mathrm{Coll}(\epsilon^+, \epsilon^{++})| = |\epsilon^{++}|$ and $(\epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$ is less than a Suslin cardinal, $\mathrm{HOD}_{\{x\}}[G] = \mathrm{HOD}_{\{x, G\}}$ by Lemma 5.9. Let $g : (\epsilon^+)^{\mathrm{HOD}_{\{x\}}} \to (\epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$ be the generic surjection associated to G. Let $\tilde{g} = \{\langle \alpha, g(\alpha)\rangle : \alpha < (\epsilon^+)^{\mathrm{HOD}_{\{x\}}}\}$ where $\langle \cdot, \cdot \rangle$ is the Gödel pairing function. So $\tilde{g} \subseteq (\epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$, $\tilde{g} \in \mathscr{P}_B(\kappa)$, and $\tilde{g} \in \mathrm{HOD}_{\{x\}}[G]$. In $\mathrm{HOD}_{\{x\}}[G]$, define $A = \{\tilde{g} \cup K : K \in \mathscr{P}(\kappa) \wedge \min(K) > (\epsilon^{++})^{\mathrm{HOD}_{\{x\}}}\}$. Note that $\mathrm{HOD}_{\{x\}}[G] \models |A| = |\mathscr{P}(\kappa)|$. Let $\tilde{\Phi} = \Phi \upharpoonright \mathrm{HOD}_{\{x\}}[G]$. Since

$\tilde{\Phi} \in \mathrm{OD}_{\{x,G\}}$, one has that $\tilde{\Phi} \in \mathrm{HOD}_{\{x,G\}} = \mathrm{HOD}_{\{x\}}[G]$. $\mathrm{HOD}_{\{x\}}[G] \models$ "$\tilde{\Phi} : \mathscr{P}(\kappa) \to [\kappa]^\epsilon \cup \mathscr{P}_B(\kappa)$" by absoluteness. Since $\mathrm{HOD}_{\{x\}}[G] \models \mathsf{AC}$ and boldface GCH holds below κ in $L(\mathbb{R})$, $\mathrm{HOD}_{\{x\}}[G] \models |\mathscr{P}(\lambda)| < \kappa$ for all $\lambda < \kappa$. Thus $\mathrm{HOD}_{\{x\}}[G] \models |\mathscr{P}_B(\kappa)| = |\bigcup_{\lambda<\kappa} \mathscr{P}(\lambda)| = \kappa$. Now one cannot have that $\tilde{\Phi}[A] \subseteq \mathscr{P}_B(\kappa)$ since then $\mathrm{HOD}_{\{x\}}[G] \models \tilde{\Phi} \restriction A : A \to \mathscr{P}_B(\kappa)$ is an injection". This is impossible since $\mathrm{HOD}_{\{x\}}[G] \models |A| = |\mathscr{P}(\kappa)| = 2^\kappa > \kappa = |\mathscr{P}_B(\kappa)|$. This implies that $\tilde{\Phi}[A] \setminus \mathscr{P}_B(\kappa) \neq \emptyset$. Let $\ell \in \Phi[A] \setminus \mathscr{P}_B(\kappa)$ and hence $\ell \in [\kappa]^\epsilon$. There is some $K \in \mathscr{P}(\kappa)^{\mathrm{HOD}_{\{x\}}[G]}$ such that $\tilde{g} \cup K \in A$ and $\Phi(\tilde{g} \cup K) = \ell$. Since $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_{\{x\}}}$ does not add any new functions from ϵ into $\mathrm{HOD}_{\{x\}}$ because $\mathrm{HOD}_{\{x\}} \models$ "$\mathrm{Coll}(\epsilon^+, \epsilon^{++})$ is $< \epsilon^+$ closed, $\ell \in \mathrm{HOD}_{\{x\}}$. Then $\tilde{g} \cup K = \Phi^{-1}(\Phi(g \cup K)) = \Phi^{-1}(\ell) = \tilde{\Phi}^{-1}(\ell) \in \mathrm{HOD}_{\{x\}}$. So $\tilde{g} \in \mathrm{HOD}_{\{x\}}$. Then $g \in \mathrm{HOD}_{\{x\}}$ which is impossible since then $\mathrm{HOD}_{\{x\}} \models$ "$g : \epsilon^+ \to \epsilon^{++}$ is a surjection" by absoluteness. (See the comments after Theorem 6.15 concerning this proof and the conjecture of whether $\mathscr{P}(\kappa)$ has 2-regular cardinality.) ☐

The first step in the proof of Theorem 6.2 for $L(\mathbb{R})$ used the fact that $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$ is a Σ_1-stable ordinal to reflect to counterexamples below $\boldsymbol{\delta}^2_1$ which is the largest Suslin cardinal of $L(\mathbb{R})$ by the scale analysis. The analogous reflection property will be given by Woodin's Σ_1-reflection into Suslin-coSuslin sets under AD^+ (and this reflection principle is essentially equivalent to AD^+)[15].

Definition 6.4. Define $\mathcal{K}$ to be the collection of all transitive set A such that there are Suslin and coSuslin sets $E \subseteq \mathbb{R}^2$ and $F \subseteq \mathbb{R}^2$ with the following properties.

(1) E is an equivalence relation on $\mathbb{R}$. F is a binary relation.
(2) Define $\dot{\in}_F$ on $\mathbb{R}/E$ by $[x]_E \dot{\in}_F [y]_E$ if and only if $F(x,y)$. $(\mathbb{R}/E, \dot{\in}_F)$ is $\mathscr{L}_{\mathrm{set}}$ isomorphic to $(A, \in)$ or in other words, $(A, \in)$ is the Mostowski collapse of $(\mathbb{R}/E, \dot{\in}_F)$.

Let $\mathcal{S} = \bigcup \mathcal{K}$ which will be called the Suslin and coSuslin sets.

Fact 6.5. *(Woodin; [56]) Assume AD^+ and $V = L(\mathscr{P}(\mathbb{R}))$. $(\mathcal{S}, \in) \prec_{\Sigma_1} (V, \in)$, that is, $\mathcal{S}$ is a Σ_1-elementary substructure of the true universe V in the language $\mathscr{L}_{\mathrm{set}}$.*

If $X, Y \in \mathscr{P}(\mathbb{R})$, then X is Wadge reducible to Y (denoted $X \leq_W Y$) if and only if there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ so that for all $r \in \mathbb{R}$, $r \in X$ if and only if $f(r) \in Y$. Under AD, the Wadge lemma ([42] Theorem 2.4) states that for any $X, Y \in \mathscr{P}(\mathbb{R})$, either $X \leq_W Y$ or $Y \leq_W (\mathbb{R} \setminus X)$. Martin and Monk ([42] Theorem 2.9) showed that under AD and $\mathsf{DC}_{\mathbb{R}}$, $\leq_W$ is a wellfounded relation. For each $X \in \mathscr{P}(\mathbb{R})$, $\mathrm{rk}_W(X)$ will denote the Wadge rank of X, that is, the rank of X in $\leq_W$. The length of $\leq_W$ is Θ. For each $\alpha \leq \Theta$, let $\mathcal{W}_\alpha = \{B \in \mathscr{P}(\mathbb{R}) : \mathrm{rk}_W(B) < \alpha\}$ and note that $\mathcal{W}_\Theta = \mathscr{P}(\mathbb{R})$.

A (lightface) pointclass Γ is a good pointclass ([62] Definition 3.1) if and only if Γ is ω-parametrized, closed under recursive substitution, $\exists^\omega$, $\forall^\omega$, $\exists^{\mathbb{R}}$, the category quantifiers[16], and has the scale property.

Next, a very terse sketch will be given to show that for any Suslin and coSuslin set B, there is a lightface analytical hierarchy of good pointclasses containing B. Let κ be the least Suslin cardinal such that B is κ-Suslin. Since B is Suslin and coSuslin, κ is not the largest Suslin cardinal. Let $\tilde{B} \in \mathscr{P}(\mathbb{R})$ be a Suslin set which is not κ-Suslin. Note that $B \leq_W \tilde{B}$ by the Wadge lemma. B is Suslin and coSuslin in $L(\tilde{B}, \mathbb{R})$ (by using the coding lemma to code the trees). In $L(\tilde{B}, \mathbb{R})$, all sets are ordinal definable from $\tilde{B}$ and a real. Let $R \subseteq \mathbb{R} \times \mathbb{R}$ be defined by $R(x,y)$ if and only if $y \notin \mathrm{OD}_{\{\tilde{B},x\}}$. R does not have a uniformization in $L(\tilde{B}, \mathbb{R})$ by the Kechris and Solovay argument as described earlier. Since $\mathsf{AC}^{\mathbb{R}}_{\mathbb{R}}$ (or $\mathsf{Uniformization}$) fails, $\sup(\mathrm{SN}^{L(\tilde{B},\mathbb{R})})$, the supremum of the Suslin cardinals of $L(\tilde{B}, \mathbb{R})$, is below $\Theta^{L(\tilde{B},\mathbb{R})}$. It was shown earlier that $\sup(\mathrm{SN}^{L(\tilde{B},\mathbb{R})})$ is a limit of Suslin cardinals. Let λ be the supremum of the first ω many Suslin cardinals of $L(\tilde{B}, \mathbb{R})$ above κ and note that $\lambda \leq \sup(\mathsf{SN}^{L(\tilde{B},\mathbb{R})}) < \Theta^{L(\tilde{B},\mathbb{R})}$. Since the supremum of a countable set of Suslin cardinals is a Suslin cardinal of $L(\tilde{B}, \mathbb{R})$ if the supremum is below $\Theta^{L(\tilde{B},\mathbb{R})}$, λ is a Suslin cardinal of cofinality ω. Let $S_{<\lambda}$ be the collection of sets which are α-Suslin for some $\alpha < \lambda$ which forms a pointclass closed under Wadge reduction. Let Λ_0 be the countable union of sets in $S_{<\lambda}$. Λ_0 is closed under Wadge reductions, $\exists^{\mathbb{R}}$, $\exists^\omega$, has the scale property, but is not closed under $\forall^\omega$. By the second periodicity theorem ([45] Theorem 6C.3),

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[15] The proof of the main theorem never explicitly uses the ordinal determinacy clause of AD^+ or that any set of reals has ∞-Borels. The only genuine use of AD^+ beyond AD and $\mathsf{DC}_{\mathbb{R}}$ will be at the beginning of the proof by contradiction where the Σ_1-reflection into Suslin-coSuslin (Fact 6.5) is used to find a Suslin and coSuslin counterexample.

[16] The following is an example of a category quantifier: if $B \subseteq \mathbb{R} \times \mathbb{R}$ belongs to Γ, then $B^* = \{x \in \mathbb{R} : B_x \text{ is comeager}\}$ belongs to Γ.

$\Lambda_1 = \forall^{\mathbb{R}}\Lambda_0$ has the scale property, closed under Wadge reduction, $\forall^{\mathbb{R}}$, $\exists^{\omega}$, $\forall^{\omega}$, but is not closed under $\exists^{\mathbb{R}}$. By [45] Lemma 6C.1 , $\Lambda_2 = \exists^{\mathbb{R}}\Lambda_1$ has the scale property, is closed under Wadge reduction, $\exists^{\mathbb{R}}$, $\forall^{\omega}$, $\exists^{\omega}$, but is not closed under $\forall^{\mathbb{R}}$. (See [30] Theorem 3.28 for more details.) Let U be a universal Λ_2 set. Let $\bar{\varphi} = \langle \varphi_n : n \in \omega \rangle$ be a Λ_2-scale on U. Let $\prec^*, \preceq^* \in \Lambda_2$ be the relations witnessing that $\vec{\varphi}$ is a Λ_2-scale. Pick $x_0, x_1, x_2 \in \mathbb{R}$ such that $U_{x_0} = B$, $U_{x_1} = \prec^*$, and $U_{x_2} = \preceq^*$. Let $\Gamma_{-1} = \text{pos}\Sigma^1_1(U, x_0, x_1, x_2)$. Γ_{-1} is ω-parametrized, has the scale property, closed under $\exists^{\mathbb{R}}$, $\forall^{\omega}$, $\exists^{\omega}$, recursive substitution, contains B, and is not closed under $\forall^{\mathbb{R}}$. By recursion, define for $n \in \omega$, $\Gamma_n = \exists^{\mathbb{R}}\forall^{\mathbb{R}}\Gamma_{n-1}$. For $n \in \omega$, Γ_n is closed under $\exists^{\mathbb{R}}$, $\forall^{\omega}$, $\exists^{\omega}$, recursive substitution, has the scale property by [45] Lemma 6C.1 and the second periodicity theorem, contains B, is not closed $\forall^{\mathbb{R}}$, and closed under category quantifiers (which can be seem by the fact that $\Gamma_n = \exists^{\mathbb{R}}\forall^{\mathbb{R}}\Gamma_{n-1}$ and that comeagerness can be expressed by the existence of a winning strategy for a certain player in an unfolded Banach-Mazur game).

Let M be a countable transitive model of ZFC and Σ be an (ω, ω_1)-iteration strategy. The pair (M, Σ) Suslin-coSuslin captures a set A at δ via (T, U) if and only if

(1) $M \models \delta$ is a Woodin cardinal.
(2) $(T, U) \in M$ are absolutely complementing trees for the forcing $\text{Coll}(\omega, \delta)$.
(3) For all $x \in \mathbb{R}$, $x \in A$ if and only if there is a countable iteration tree $\mathcal{T}$ on N according to Σ with last model P and iteration map $i : N \to P$ such that $x \in \pi_1[i(T)]$ (in the real world) where π_1 is the projection onto the first coordinate.

(See [59] Definition 10.1.) The construction of such objects are closely related to Woodin's construction of a coarse Γ-Woodin mouse (see [62] Definition 3.11 or [63] Definition 7.2.3) while controlling the complexity of its iteration strategy which is found in [62] Lemma 3.13.

Fix a Suslin and coSuslin set A. The following sketches the construction of a pair (M, Σ) which Suslin-coSuslin captures A from [62] Theorem 3.13. Let $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be the first five pointclasses starting an analytical hierarchy of good pointclasses containing A as described above which are not closed under $\forall^{\mathbb{R}}$. Let T_4 be a tree projecting onto a complete Γ_4 set. Woodin ([40] Theorem 5.40) showed that there is some $z \in \mathbb{R}$ so that $\text{HOD}^{L[T_4,z]}_{T_4} \models$ "$\omega_2^{L[T,z]}$ is a Woodin cardinal". $V^{\text{HOD}^{L[T_4,z]}_{T_4}}_{\omega_2^{L[T_4,z]}}$ is used to create a coarse Γ_4-Woodin premouse P. Let K_1 denote the universal Γ_1 set. Let (T_1, U_1) be trees projecting to the universal Γ_1 set K_1 and its complement which is constructible from T_4. There is some η and model P such that V^P_η can be used to create a Γ_0-Woodin premouse. Let w be a wellordering of V^P_η in P and let $\pi : M \to L_\alpha(V^P_\eta \cup \{w, T_1, U_1\})$ be elementary for some appropriate α and M countable and transitive. By elementarity, there are $J, \bar{w}, T, U \in M$ such that $\pi(J) = V^P_\eta$, $\pi(\bar{w}) = w$, $\pi(T) = T_1$, $\pi(U) = U_1$ and $M = L_{\bar{\alpha}}(J \cup \{\bar{w}, T, U\})$ for some ordinal $\bar{\alpha}$. Note that M has a definable wellordering of itself using $\bar{w}$ since $L_\alpha(V^P_\eta \cup \{w, T_1, U_1\})$ does using w. M is a Γ_0-Woodin mouse with strategy defined by choosing the unique π-realizable branch. (M, Σ) will Suslin-coSuslin capture the universal Γ_1 set. (See [62] Lemma 3.13 for more details.) Moreover, the strategy Σ is determined by the C_{Γ_1}-operator[17] in the sense that if b is a cofinal branch π-realizable branch on a countable tree $\mathcal{T}$ using extenders from $V^M_{\bar{\eta}}$, then b is the unique cofinal branch such that $C_{\Gamma_1}(V^{\mathcal{M}^{\mathcal{T}}_b}_{\delta(\mathcal{T})}) \subseteq \mathcal{M}^{\mathcal{T}}_b$ (in the notation of [62] Lemma 3.13). Also see [63] Lemma 7.2.7. This will be abbreviated by saying "Σ is the strategy on M guided by C_{Γ_1}". Since the strategy Σ picks π-realizable branches, there is embedding of the last model of any iteration tree according to Σ back into $L_\alpha(V^P_\eta \cup \{w, T_1, U_1\})$ which maps the image of (T, U) under the ultrapower map of the iteration tree back into (T_1, U_1). This shows that (M, Σ) Suslin-CoSuslin captures the universal Γ_1-set.

As noted in [59] Theorem 10.2 and the proof of [59] Theorem 10.3, (M, Σ) can be constructed to have condensation and the Dodd-Jensen property. There seems to be a few ways to do this which will be briefly described: (M, Σ) is a coarse structure and Schlutzenberg has observed that the restriction of Σ to certain nice trees satisfies a form of comparison. The structure M is a weak coarse premouse of Schlutzenberg presentation ([51] Definition 3.1). When Σ is restricted to a certain nice $\text{OD}_{M \cup \{\Sigma\}}$ family of trees, Schlutzenberg shows that (M, Σ) satisfies a form of comparison ([51] Lemma 5.2) and even a form of comparison necessary for

[17] See [62] Definition 1.2. $C_\Gamma(x)$ is the set of $y \in \mathbb{R}$ which is $\Delta(x)$ in some countable ordinals where $\Delta = \Gamma \cap \check{\Gamma}$. See [32] and [33] for more on C_Γ.

genericity iteration ([51] Section 5.2).[18] Let $Z = M\cup\{\Sigma\}$. Note that $Z \in \mathrm{OD}_Z^{L(Z,\mathbb{R})}$ and there is an $\mathrm{OD}_Z^{L(Z,\mathbb{R})}$ wellordering of Z (since $M = J_{\bar{\alpha}}(J\cup\{\bar{w}, T, U\})$ and $\bar{w}$ is a wellordering of J). As noted in the remarks before [59] Theorem 10.2, comparison arguments can be used to form a directed system of countable iterates of M according to Σ. The universal Γ_1-set is ordinal definable from Z since $x \in K_1$ if and only if $L(Z,\mathbb{R})$ satisfies "there is a countable iteration tree $\mathcal{T}$ according to Σ with last model P and iteration map $i : M \to P$ such that $x \in \pi_1[i(T)]$". Since the set A belongs to the ω-parameterized pointclass $\Gamma_0 \subseteq \Gamma_1$, A is OD_Z using the fact that K_1 is OD_Z. As in the remarks before [59] Theorem 10.2, if one let (T_∞, U_∞) be the image of (T, U) in the directed system of countable iterates of M according to Σ (using the comparison of Schlutzenberg [51]), then (T_∞, U_∞) projects onto the universal Γ_1-set K_1 and its complement. Note that $(T_\infty, U_\infty) \in \mathrm{OD}_Z$ and hence belongs to HOD_Z since they are sets of ordinals. Let $Y = M \cup \{T_\infty, U_\infty\}$. $Y \in \mathrm{HOD}_Z$. It is clear that $\mathrm{OD}_Y \subseteq \mathrm{OD}_Z$. The C_{Γ_1} operator can be defined from the universal Γ_1-set K_1 and hence can be defined from (T_∞, U_∞). $\Sigma \in \mathrm{OD}_Y$ since "Σ is the strategy on M guided by C_{Γ_1}". This shows $\mathrm{OD}_Z = \mathrm{OD}_Y$. (Alternatively, Steel ([59] Theorem 10.3) and Sargsyan ([48] Theorem 2.3.8; [49] Theorem 2.25) showed that M_1^Z Suslin-coSuslin captures K_1. M_1^Z has a unique (ω, ω_1)-iteration strategy which is OD_Z and has fine structure. Thus comparison arguments and condensation work as usual. In the above discussion, one could replace (M, Σ) with a cut-off of M_1^Z and its unique iteration strategy.)

Fix the above (M, Σ). A Σ-premouse is a hybrid strategy premouse of the form $J_\alpha^{\vec{E},S}(M)$ which codes fragments of Σ in the sense of [48] Definition 2.2.12 and Definition 2.2.13. As in the proof of boldface GCH sketched in [61] Theorem 2.16, $\mathrm{HOD}_Z^{L(Z,\mathbb{R})}$ is a direct limit of a certain system of Σ-mice with suitable properties generalizing the construction in [64] for $L(\mathbb{R})$. Thus $\mathrm{HOD}_Z^{L(Z,\mathbb{R})} \models \mathsf{GCH}$ and for any $\epsilon < \Theta^{L(Z,\mathbb{R})}$, $\mathrm{cof}^V((\epsilon^+)^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}) = \mathrm{cof}^{L(Z,\mathbb{R})}((\epsilon^+)^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}) = \omega$.

In summary, one has the following by taking $Z = M \cup \{\Sigma\}$ and $Y = M \cup \{T_\infty, U_\infty\}$ from the above discussion.

Lemma 6.6. *Assume* AD *and* $\mathsf{DC}_\mathbb{R}$. *Let* $A \in \mathscr{P}(\mathbb{R})$ *be Suslin and coSuslin. Then there are sets* Z *and* Y *such that the following holds.*

1. *There is an* $\mathrm{OD}_Z^{L(Z,\mathbb{R})}$ *wellordering of* Z.
2. $Y \in \mathrm{HOD}_Z^{L(Z,\mathbb{R})}$ *and* $\mathrm{OD}_Z^{L(Z,\mathbb{R})} = \mathrm{OD}_Y^{L(Z,\mathbb{R})} = \mathrm{OD}_Y^{L(Y,\mathbb{R})}$. *(Thus* Z *is a very good parameter set and* Y *is a very good parameter-witness for* Z.*)*
3. $A \in \mathrm{OD}_Z^{L(Z,\mathbb{R})}$ *and thus* $A \in L(Z,\mathbb{R})$.
4. $\mathrm{HOD}_Z^{L(Z,\mathbb{R})} \models \mathsf{GCH}$.
5. *For all* $\epsilon < \Theta^{L(Z,\mathbb{R})}$, $\mathrm{cof}^{L(Z,\mathbb{R})}((\epsilon^+)^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}) = \omega$.

One can now solve the Bounded Power Set conjecture.

Theorem 6.7. *Assume* AD^+. *For all cardinals* κ *such that* $\omega < \kappa < \Theta$ *and all* $\epsilon < \kappa$, *there is no injection of* $\mathscr{P}_B(\kappa)$ *into* ${}^\epsilon\mathrm{ON}$.

Proof. Suppose the theorem is false. Then there exist an uncountable cardinal $\lambda < \Theta$, $\chi < \lambda$, and an injection $\Psi : \mathscr{P}_B(\lambda) \to {}^\chi\mathrm{ON}$. As observed in the proof of Theorem 6.2, χ must be infinite. Let $\tau_0 : \mathbb{R} \to \chi$ be a surjection. Since $\mathbb{R}$ surjects onto $\lambda < \Theta$, the Moschovakis coding lemma (Fact 2.15) implies there is a surjection of $\mathbb{R}$ onto $\mathscr{P}(\lambda)$ and in particular $\mathscr{P}_B(\lambda)$. Pick a surjection $\tau_1 : \mathbb{R} \to \mathscr{P}_B(\lambda)$. Let $A = \{\Psi(B)(\alpha) : \alpha < \chi \wedge B \in \mathscr{P}_B(\lambda)\}$. Let $\tau_2 : \mathbb{R} \to A$ be defined by $\tau_2(r) = \Psi(\tau_1(r^{[1]}))(\tau_0(r^{[0]}))$. τ_2 is a surjection and thus $\mathrm{ot}(A) < \Theta$. Let $\eta = \mathrm{ot}(A)$. Let $\tau_3 : A \to \eta$ be an order preserving bijection. Let $\Upsilon : \mathscr{P}_B(\kappa) \to {}^\chi\eta$ be defined by $\Upsilon(B)(\alpha) = \tau_3(\Psi(B)(\alpha))$ for all $\alpha < \chi$. Υ is an injection. It has been shown that there exist $\chi < \lambda < \Theta$, $\eta < \Theta$, and injection $\Upsilon : \mathscr{P}_B(\lambda) \to {}^\chi\eta$. Using the Moschovakis coding lemma, such a function Υ can be coded by a set of reals and thus $\Upsilon \in L(\mathscr{P}(\mathbb{R}))$. Also since AD^+ holds, $L(\mathscr{P}(\mathbb{R})) \models \mathsf{AD}^+$. Let ϕ be the $\mathscr{L}_{\mathrm{set}}$ sentences asserting "there exists $\epsilon < \kappa < \Theta$, $\delta < \Theta$, and an injection $\Phi : \mathscr{P}_B(\kappa) \to {}^\epsilon\delta$". Recall if $\beta \leq \Theta$, then $\mathcal{W}_\beta = \{A \in \mathbb{R} : \mathrm{rk}_W(A) < \beta\}$. Let ZF^* be some sufficiently large finite fragment of ZF. Let ψ be the formula asserting "there exist $\beta, \xi \in \mathrm{ON}$ such that $L_\xi(\mathcal{W}_\beta) \models \mathsf{ZF}^*$, AD, and ϕ". It was shown (as witnessed by the function Υ) that $L(\mathscr{P}(\mathbb{R})) \models \phi$. By using reflection on the hierarchy $\langle L_\xi(\mathscr{P}(\mathbb{R})) : \xi \in \mathrm{ON}\rangle$, there is a $\xi \in \mathrm{ON}$ so that $L_\xi(\mathscr{P}(\mathbb{R})) \models \mathsf{ZF}^*$, AD, and ϕ. Letting $\beta = \Theta$, $\mathcal{W}_\beta = \mathcal{W}_\Theta = \mathscr{P}(\mathbb{R})$. Thus

[18] This comparison for coarse structures is also used in [52] to give a proof of the derived model theorem.

$L_\xi(\mathcal{W}_\beta) \models \mathsf{ZF}^*$, AD, and ϕ. This shows that $L(\mathscr{P}(\mathbb{R})) \models \psi$. Since $\mathcal{S} \prec_1 L(\mathscr{P}(\mathbb{R}))$ by Σ_1-reflection into Suslin-coSuslin sets (Fact 6.5), there exist $\beta, \xi \in \mathcal{S}$ so that $L_\xi(\mathcal{W}_\beta) \models \mathsf{ZF}^*$, AD, and ϕ. There is a function $\Phi \in L_\xi(\mathcal{W}_\beta)$, $\epsilon < \kappa < \Theta^{L_\xi(\mathcal{W}_\beta)}$ and $\delta < \Theta^{L_\xi(\mathcal{W}_\beta)}$ so that $L_\xi(\mathcal{W}_\beta) \models$ "$\Phi : \mathscr{P}_B(\kappa) \to {}^\epsilon\delta$ is an injection". By an application of the Moschovakis coding lemma (Fact 2.15) in $L(\mathscr{P}(\mathbb{R}))$ using a prewellordering and pointclass in $L_\xi(\mathcal{W}_\beta)$ as in the proof of Theorem 6.2, $\mathscr{P}_B(\kappa) = \mathscr{P}_B(\kappa)^{L_\xi(\mathcal{W}_\beta)}$ and $\Phi : \mathscr{P}_B(\kappa) \to {}^\epsilon\delta$ is still an injection in the real world by absoluteness. Since $\Phi \in L_\xi(\mathcal{W}_\beta) \subseteq \mathcal{S}$, the transitive closure of $\{\Phi\}$, $\mathrm{tc}(\{\Phi\})$ belong to $\mathcal{K}$ (from Definition 6.4). Thus there are Suslin and coSuslin sets E and F such that $\mathrm{tc}(\{\Phi\})$ is the Mostowski collapse of $(\mathbb{R}/E, \dot{\in}_F)$. Now since $\kappa \in L_\xi(\mathcal{W}_\beta) \subseteq \mathcal{S}$, κ is below a Suslin cardinal. Since the Suslin cardinals are unbounded below its supremum, there is a Suslin cardinal λ such that $\kappa < \lambda$. Thus there is a tree $T \in \mathcal{S}$ so that T is a tree on λ which projects onto the complete S_λ set. Let (E_0, F_0) be Suslin and coSuslin sets so that $\mathrm{tc}(\{T\})$ is the Mostowski collapse of $(\mathbb{R}/E_0, \dot{\in}_{F_0})$. Let $A = \langle E, F, E_0, F_0\rangle$ be a single set of reals coding these four sets of reals in a simple manner so that A is Suslin and coSuslin. Lemma 6.6 supplies set Z and Y with the properties stated in the lemma for A. Since $A \in L(Z,\mathbb{R})$ and $A \in \mathrm{OD}_Z^{L(Z,\mathbb{R})}$, one has that $E, F, E_0, F_0 \in L(Z,\mathbb{R})$ and also $E, F, E_0, F_0 \in \mathrm{OD}_Z^{L(Z,\mathbb{R})}$. $T \in L(Z,\mathbb{R})$ since T is the element of highest rank in the Mostowski collapse of $(\mathbb{R}/E_0, \dot{\in}_{F_0})$. So the universal S_λ set and the tree T witnessing it is λ-Suslin belong to $L(Z,\mathbb{R})$. So $L(Z,\mathbb{R}) \models$ "λ is a Suslin cardinal greater than κ". Note that $\Phi \in \mathrm{OD}_Z^{L(Z,\mathbb{R})}$ since $E_0, F_0 \in \mathrm{OD}_Z^{L(Z,\mathbb{R})}$ and Φ is the element of highest rank in the Mostowski collapse of $(\mathbb{R}/E, \dot{\in}_F)$. Since $\mathrm{HOD}_Z^{L(Z,\mathbb{R})} \models \mathsf{GCH}$ and $\mathrm{cof}^{L(Z;\mathbb{R})}((\epsilon^+)^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}) = \mathrm{cof}^{L(Z;\mathbb{R})}((\epsilon^{++})^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}) = \mathrm{cof}^{L(Z;\mathbb{R})}((\epsilon^{+++})^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}) = \omega$, Lemma 5.6 implies there is a $G \subseteq \mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}$ which belongs to $L(Z,\mathbb{R})$ and is $\mathrm{Coll}(\epsilon^+, \epsilon^{++})$-generic over $\mathrm{HOD}_Z^{L(Z,\mathbb{R})}$. Let $g : (\epsilon^+)^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}} \to (\epsilon^{++})^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}$ be the generic surjection associated to G. Since $\mathrm{cof}^{L(Z,\mathbb{R})}((\epsilon^+)^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}) = \omega$, let $\rho : \omega \to (\epsilon^+)^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}$ be cofinal increasing in $(\epsilon^+)^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}$ and $\rho \in L(Z,\mathbb{R})$. Let $\sigma = \{g \restriction \rho(n) : n \in \omega\}$. Then $G = \langle \sigma \rangle_{\mathrm{Coll}(\epsilon^+,\epsilon^{++})^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}}$ and thus G is a countably generated $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}$-filter. Since $L(Z,\mathbb{R}) = L(Y,\mathbb{R})$ for $Y \in \mathrm{HOD}_Z^{L(Z,\mathbb{R})}$, $\mathrm{HOD}_Z^{L(Z,\mathbb{R})} \models |\mathrm{Coll}(\epsilon^+, \epsilon^{++})| = |\epsilon^{++}|$, and $\epsilon^{++} < \kappa < \lambda$ which is a Suslin cardinal of $L(Z,\mathbb{R})$, Lemma 5.9 implies that $\mathrm{HOD}_Z^{L(Z,\mathbb{R})}[G] = \mathrm{HOD}_{Z\cup\{G\}}^{L(Z,\mathbb{R})}$. Let $\tilde{g} = \{\langle \alpha, g(\alpha)\rangle : \alpha < (\epsilon^+)^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}\}$ and note that $\tilde{g} \in \mathscr{P}_B(\kappa)$ (by Fact 2.18) and $\tilde{g} \in \mathrm{OD}_{\{G\}}^{L(Z,\mathbb{R})}$. Since $\Phi \in \mathrm{OD}_Z^{L(Z,\mathbb{R})}$, one has that $\Phi(\tilde{g}) \in \mathrm{OD}_{Z\cup\{G\}}^{L(Z,\mathbb{R})}$. Hence $\Phi(\tilde{g}) \in \mathrm{HOD}_{Z\cup\{G\}}^{L(Z,\mathbb{R})} = \mathrm{HOD}_Z^{L(Z,\mathbb{R})}[G]$. Since $\Phi(\tilde{g}) \in {}^\epsilon\mathrm{ON}$ and $\mathrm{Coll}(\epsilon^+, \epsilon^{++})^{\mathrm{HOD}_Z^{L(Z,\mathbb{R})}}$ adds no new functions from ϵ into $\mathrm{HOD}_Z^{L(Z,\mathbb{R})}$ since $\mathrm{HOD}_Z^{L(Z,\mathbb{R})} \models$ "$\mathrm{Coll}(\epsilon^+, \epsilon^{++})$ is $< \epsilon^+$-closed", one has that $\Phi(\tilde{g}) \in \mathrm{HOD}_Z^{L(Z,\mathbb{R})}$. Then $\tilde{g} \in \mathrm{OD}_Z^{L(Z,\mathbb{R})}$ since $\tilde{g} = \Phi^{-1}(\Phi(\tilde{g}))$. Thus $\tilde{g} \in \mathrm{HOD}_Z^{L(Z,\mathbb{R})}$ and therefore $g \in \mathrm{HOD}_Z^{L(Z,\mathbb{R})}$ and $G \in \mathrm{HOD}_Z^{L(Z,\mathbb{R})}$. This is impossible since $\mathrm{HOD}_Z^{L(Z,\mathbb{R})} \models$ "$g : \epsilon^+ \to \epsilon^{++}$ is a surjection" by absoluteness. ☐

By adapting the proof of Theorem 6.3 with the proof of Theorem 6.7, one obtains the following result.

Theorem 6.8. *Assume* AD^+. *Let* $\kappa < \Theta$ *be a singular cardinal and let* $\epsilon < \kappa$ *be such that that* $\mathrm{cof}(\epsilon) = \mathrm{cof}(\kappa)$. *Then the following holds.*

(1) $|\mathscr{P}_B(\kappa)| < |[\kappa]^\epsilon \cup \mathscr{P}_B(\kappa)|$.
(2) $|[\kappa]^\epsilon| < |[\kappa]^\epsilon \cup \mathscr{P}_B(\kappa)|$.
(3) $|[\kappa]^\epsilon \cup \mathscr{P}_B(\kappa)| < |\mathscr{P}(\kappa)|$.

Finally, one can prove the ABCD conjecture using the Bounded Power Set conjecture.

Theorem 6.9. *Assume* AD^+. *Let* α, β, γ, *and* δ *be cardinals such that* $\omega \leq \alpha \leq \beta < \Theta$ *and* $\omega \leq \gamma \leq \delta < \Theta$. $|{}^\alpha\beta| \leq |{}^\gamma\delta|$ *if and only if* $\alpha \leq \gamma$ *and* $\beta \leq \delta$.

Proof. ($\Leftarrow$) Suppose $\alpha \leq \gamma$ and $\beta \leq \delta$. Let $f : \alpha \to \beta$ be a function. Define $\Phi(f) : \gamma \to \delta$ by

$$\Phi(f)(\xi) = \begin{cases} f(\xi) & \xi < \alpha \\ 0 & \alpha \leq \xi \end{cases}$$

This defines $\Phi : {}^\alpha\beta \to {}^\gamma\delta$. Suppose $f \neq g$. There is a $\nu < \alpha$ so that $f(\nu) \neq g(\nu)$. Thus $\Phi(f)(\nu) = f(\nu) \neq g(\nu) = \Phi(g)(\nu)$. This shows that Φ is an injection and hence $|{}^\alpha\beta| \leq |{}^\gamma\delta|$.

($\Rightarrow$) Suppose this is false. There is an injection $\Phi : {}^\alpha\beta \to {}^\gamma\delta$ and $\neg(\alpha \leq \gamma \wedge \beta \leq \delta)$. Suppose $\delta < \beta$. Then $|\beta| \leq |{}^\alpha\beta| \leq |{}^\gamma\delta| \leq |{}^\delta\delta| = |\mathscr{P}(\delta)|$ where the second inequality is witnessed by Φ (and one can use ($\Leftarrow$) from above for the third inequality). Since $\delta < \beta$, $\delta^+ \leq \beta$. Hence there is an injection of δ^+ into $\mathscr{P}(\delta)$ which violates boldface GCH at δ (Fact 2.22). Suppose $\gamma < \alpha$. $|\mathscr{P}_B(\alpha)| \leq |\mathscr{P}(\alpha)| \leq |{}^\alpha 2| \leq |{}^\alpha\beta| \leq |{}^\gamma\delta|$. Thus there is an injection of $\mathscr{P}_B(\alpha)$ into ${}^\gamma\mathrm{ON}$ where $\gamma < \alpha$ in violation of Theorem 6.7. This yields a contradiction. $\square$

Remark 6.10. With the ABCD hypothesis below Θ, one can completely classify the cardinality relationship between any pair of infinite cardinal exponentiations below Θ under AD^+: Suppose $\omega \leq \alpha, \beta, \gamma, \delta < \Theta$ are four infinite cardinals below Θ. If $\alpha \leq \beta$, then let $\bar{\alpha} = \alpha$ and $\bar{\beta} = \beta$. If $\beta < \alpha$, then let $\bar{\alpha} = \alpha$ and $\bar{\beta} = \alpha$. Since $\omega \leq \beta < \alpha$, one has that $|{}^\alpha\beta| = |{}^\alpha 2| = |\mathscr{P}(\alpha)| = |{}^\alpha\alpha| = |{}^{\bar{\alpha}}\bar{\beta}|$. Thus one has found cardinals $\omega \leq \bar{\alpha} \leq \bar{\beta} < \Theta$ so that $|{}^\alpha\beta| = |{}^{\bar{\alpha}}\bar{\beta}|$. Similiarly, one can find $\omega \leq \bar{\gamma} \leq \bar{\delta} < \Theta$ so that $|{}^\gamma\delta| = |{}^{\bar{\gamma}}\bar{\delta}|$. Now one can apply the ABCD hypothesis below Θ (Theorem 6.7) to $\omega \leq \bar{\alpha} \leq \bar{\beta} < \Theta$ and $\omega \leq \bar{\gamma} \leq \bar{\delta} < \Theta$ to determine what is the cardinality relationship between $|{}^\alpha\beta| = |{}^{\bar{\alpha}}\bar{\beta}|$ and $|{}^\gamma\delta| = |{}^{\bar{\gamma}}\bar{\delta}|$.

Let $\epsilon \leq \kappa$ be ordinals. Recall that ${}^\epsilon\kappa$ is the set of functions $f : \epsilon \to \kappa$ and $[\kappa]^\epsilon$ is the set of increasing functions $f : \epsilon \to \kappa$. Let $B(\epsilon, \kappa) = \{f \in {}^\epsilon\kappa : \sup(f) < \kappa\}$, which is the set of functions $f : \epsilon \to \kappa$ which are bounded below κ. Next, $B(\epsilon, \kappa)$, $[\kappa]^\epsilon$, and ${}^\epsilon\kappa$ will be distinguished and classified.

Fact 6.11. *Let $\epsilon \leq \kappa$ be limit ordinals.*

(1) $|{}^\epsilon\kappa| = |{}^{|\epsilon|}|\kappa||$.
(2) $|B(\epsilon, \kappa)| = |B(|\epsilon|, \kappa)|$.
(3) If $\epsilon \leq |\kappa| < \kappa$, *then* $|B(\epsilon, \kappa)| = |{}^{|\epsilon|}|\kappa||$.
(4) If $\mathrm{cof}(\epsilon) \neq \mathrm{cof}(\kappa)$, *then* $[\kappa]^\epsilon = B(\epsilon, \kappa)$.
(5) If $\epsilon \leq \mathrm{cof}(\kappa)$, *then* $|[\kappa]^\epsilon| = |{}^\epsilon\kappa|$.
(6) $|\mathscr{P}_B(|\epsilon|)| \leq |[\kappa]^\epsilon| \leq |{}^\epsilon\kappa|$.
(7) If $\mathrm{cof}(\epsilon) = \mathrm{cof}(\kappa)$ *and κ is a limit of indecomposable ordinals, then* $|[\kappa]^\epsilon| = |{}^\epsilon\kappa|$.
(8) Assume for all $\delta < \kappa$, $\neg(|\kappa| \leq |\mathscr{P}(\delta)|)$. If $\mathrm{cof}(\epsilon) = \mathrm{cof}(\kappa)$, then $|B(\epsilon, \kappa)| < |[\kappa]^\epsilon|$.
(9) Assume AD^+. Suppose $\delta < \epsilon \leq \kappa$ and $|\delta| < |\epsilon|$. Then any member of $\{B(\epsilon, \kappa), [\kappa]^\epsilon, {}^\epsilon\kappa\}$ does not inject into any member of $\{B(\delta, \kappa), [\kappa]^\delta, {}^\delta\kappa\}$.

Proof. (1), (2), (3), and (4) are clear.

(5) $[\kappa]^\epsilon \subseteq {}^\epsilon\kappa$. Let $f \in {}^\epsilon\kappa$. $\Phi(f) \in [\kappa]^\epsilon$ will be defined by induction as follows. Suppose $\alpha < \epsilon$ and $\Phi(f) \restriction \alpha \in [\kappa]^\alpha$ has been defined. Since $\alpha < \mathrm{cof}(\kappa)$, $\sup(\Phi(f) \restriction \alpha) < \kappa$. Let $\Phi(\alpha) = \sup(\Phi \restriction \alpha) + f(\alpha)$. This completes the definition of $\Phi(f) \in [\kappa]^\epsilon$. $\Phi : {}^\epsilon\kappa \to [\kappa]^\epsilon$ is an injection. Thus $|{}^\epsilon\kappa| = |[\kappa]^\epsilon|$.

(6) Let k be any element of $[\kappa]^{\epsilon - |\epsilon|}$ such that $\min(k) \geq |\epsilon|$ (if $|\epsilon| = \epsilon$, then $k = \emptyset$). Since $|\epsilon|$ is a cardinal, let A and B be two disjoint subsets of $|\epsilon|$ such that $|A| = |B| = |\epsilon|$. Let $E_A : |\epsilon| \to A$ and $E_B : |\epsilon| \to B$ be the increasing enumeration of A and B, respectively. If $\ell \in \mathscr{P}_B(|\epsilon|)$, let $f_\ell : \mathrm{ot}(\ell) \to \ell$ be the increasing enumeration of ℓ. Define $h_\ell \in [\kappa]^\epsilon$ by

$$h_\ell(\alpha) = \begin{cases} E_A(f_\ell(\alpha)) & \alpha < \mathrm{ot}(\ell) \\ E_B(\sup(E_A \circ f_\ell) + (\alpha - \mathrm{ot}(\ell))) & \mathrm{ot}(\ell) \leq \alpha < |\epsilon| \\ k(\alpha - |\epsilon|) & |\epsilon| \leq \alpha < \epsilon \end{cases}.$$

Since $\ell \in \mathscr{P}_B(|\epsilon|)$, $\sup(f_\ell) < |\epsilon|$. Thus $\sup(E_A \circ f_\ell) < |\epsilon|$. These two observations can be used to show that h_ℓ is indeed an increasing function. Define $\Psi : \mathscr{P}_B(|\epsilon|) \to [\kappa]^\epsilon$ by $\Psi(\ell) = h_\ell$. Ψ is an injection. Thus $|\mathscr{P}_B(|\epsilon|)| \leq |[\kappa]^\epsilon| \leq |{}^\epsilon\kappa|$.

(7) It is clear that $[\kappa]^\epsilon \subseteq {}^\epsilon\kappa$. If κ is regular, then the result follows from (5). Assume $\mathrm{cof}(\kappa) < \kappa$. Let $\delta = \mathrm{cof}(\epsilon) = \mathrm{cof}(\kappa)$. Let $\rho_0 : \delta \to \epsilon$ and $\rho_1 : \delta \to \kappa$ be increasing cofinal maps from δ into ϵ and κ, respectively. One may assume that $\delta < \rho_1(0)$. By the hypothesis, one may assume that the image of ρ_1 consists entirely of indecomposable ordinals (which are closed under the Gödel pairing function, $\langle \cdot, \cdot \rangle$). Let $f \in {}^\epsilon\kappa$. One will define $\Gamma(f) \in [\kappa]^\epsilon$ by recursion as follows. For each $\xi < \delta$, let $B^f_\xi = \{\alpha < \rho_0(\xi) : \sup(\rho_1 \restriction \xi) \leq f(\alpha) < \rho_1(\xi)\}$. For $\xi < \delta$, let $\iota_\xi = \mathrm{ot}(B^f_\xi)$. Let $E^f_\xi : \iota_\xi \to B^f_\xi$ be the increasing enumeration of B^f_ξ. Suppose $\xi < \delta$ and for all $\gamma < \xi$, an increasing function g^f_γ has been defined so that g^f_γ has domain less than δ and $\sup(g^f_\gamma) < \rho_1(\gamma + 1)$ and for all $\gamma_0 < \gamma_1 < \xi$, $g^f_{\gamma_0} \subseteq g^f_{\gamma_1}$. Let $h = \bigcup_{\gamma < \xi} g^f_\gamma$ and note that $\mathrm{dom}(h) = \sup\{\mathrm{dom}(g^f_\gamma) : \gamma < \xi\} = \sup\{\rho_1(\gamma + 1) : \gamma < \xi\} \leq \rho_1(\xi)$. Define g^f_ξ with domain $\mathrm{dom}(h) + \iota^f_\xi$

by recursion as follows. If $\alpha < \mathrm{dom}(h)$, then $g^f_\xi(\alpha) = h(\alpha)$. Suppose $\beta < \iota^f_\xi$ and $g^f_\xi \restriction \mathrm{dom}(h) + \beta$ has been defined for some $\beta < \iota^f_\xi$. Then let $g^f_\xi(\mathrm{dom}(h) + \beta) = \sup(g^f_\xi \restriction \mathrm{dom}(h) + \beta) + \langle E^f_\xi(\beta), f(E^f_\xi(\beta))\rangle$. This completes the construction of g^f_ξ. Since $E^f_\xi(\beta) \in B^f_\xi$, $f(E^f_\xi(\beta)) < \rho_1(\xi)$ for all $\beta < \iota^f_\xi$. Since ρ_1 takes image among ordinals which are closed under the Gödel pairing function, $\langle E^f_\xi(\beta), f(E^f_\xi(\beta))\rangle < \rho_1(\xi)$. Hence $\sup(g^f_\xi) \leq \sup(h) + \rho_1(\xi) \cdot \delta < \rho_1(\xi+1)$ since $\rho_1(\xi+1)$ is indecomposable. This completes the construction of $\langle g^f_\xi : \xi < \delta\rangle$. Let $\Gamma(f) = \bigcup_{\xi<\delta} g^f_\xi$. Note that $\Gamma(f) \in [\kappa]^\epsilon$ and $\Gamma : {}^\epsilon\kappa \to [\kappa]^\epsilon$ is an injection. Thus $|{}^\epsilon\kappa| \leq |[\kappa]^\epsilon|$.

(8) There is no injection of $[\kappa]^\epsilon$ into $B(\epsilon, \kappa)$ since $|B(\epsilon,\kappa)| \leq |\mathscr{P}_B(\kappa)|$ and there is no injection of $[\kappa]^\epsilon$ into $\mathscr{P}_B(\kappa)$ by Theorem 2.2 when $\mathrm{cof}(\epsilon) = \mathrm{cof}(\kappa)$

(9) This follows from (6) and Theorem 6.7. $\square$

One can now show that $|[\omega_3]^{\omega_2}| < |[\omega_3]^{<\omega_3}| \leq |\mathscr{P}(\omega_3)|$ under AD^+.

Theorem 6.12. *Assume* AD^+. *If $\kappa < \Theta$ is a cardinal and $\epsilon < \kappa$, then $|{}^\epsilon\kappa| < |{}^{<\kappa}\kappa|$.*

Proof. Since $\mathscr{P}_B(\kappa)$ injects into ${}^{<\kappa}\kappa$, ${}^{<\kappa}\kappa$ cannot inject into ${}^\epsilon\kappa$ by Theorem 6.7. $\square$

[17] defined a notion of regularity and described the cofinality of some familiar sets in the choiceless determinacy context.

Definition 6.13. ([17]) Let X be a set and Y be a set or class. X has Y-regular cardinality if and only if for all $\Phi : X \to Y$, there is a $y \in Y$ so that $|\Phi^{-1}[\{y\}]| = |X|$.

A set X is said to have locally regular cardinality if and only if for all sets Y with $|Y| < |X|$, X has Y-regular cardinality.

A set X is said to have globally regular cardinality if and only if for all sets Y with $\neg(|X| \leq |Y|)$, X has Y-regular cardinality.

Under AD, $\mathbb{R} \sqcup \omega_1$ does not have 2-regular cardinality. Under AD^+, [17] shows that $\mathbb{R}$ and $\mathbb{R}/E_0$ have globally regular cardinality. Section 7 will show that under AD^+, if $\omega \leq \kappa < \Theta$ and $\mathrm{cof}(\kappa) = \omega$, then ${}^\omega\kappa$ has ON-regular cardinality, $B(\omega,\kappa)$ has n-regular cardinality for all $n \in \omega$, and if $\omega < \kappa$, then $B(\omega, \kappa)$ does not have ω-regular cardinality.

Some information is known about the cofinality of the power set of a strong partition cardinal.

Fact 6.14. *([6]) If $\kappa \to_* (\kappa)^\kappa_2$ (κ is a strong partition cardinal), then $\mathscr{P}(\kappa)$ is* ON*-regular: for all $\delta \in \mathrm{ON}$, $\mathscr{P}(\kappa)$ has δ-regular cardinality.*

([17]) If $\kappa \to_* (\kappa)^\kappa_{<\kappa}$ (κ is a very strong partition cardinal), then $\mathscr{P}(\kappa)$ has ${}^{<\kappa}\mathrm{ON}$-regular cardinality: for all $\delta \in \mathrm{ON}$, $\mathscr{P}(\kappa)$ has ${}^{<\kappa}\delta$-regular cardinality.

The main theorem of this section gives the following new ordinal regularity result. It implies that ${}^{<\omega_\omega}(\omega_{\omega+1})$ does not have ω-regular cardinality which was not previously known.

Theorem 6.15. *Assume* AD^+. *Let $\delta \leq \kappa < \Theta$ be such that δ is a limit ordinal. Then ${}^{<\delta}\kappa$ does not have $\mathrm{cof}(\delta)$-regular cardinality.*

Proof. Let $\rho : \mathrm{cof}(\delta) \to \delta$ be an increasing cofinal function. Define $\Phi : {}^{<\delta}\kappa \to \mathrm{cof}(\delta)$ by $\Phi(f)$ is the least α so that $\mathrm{dom}(f) < \rho(\alpha)$. For each $\alpha < \mathrm{cof}(\delta)$, $\Phi^{-1}[\{\alpha\}] \subseteq {}^{<\rho(\alpha)}\kappa$. Thus $|\Phi^{-1}[\{\alpha\}]| \leq |{}^{<\rho(\alpha)}\kappa| < |{}^{<\delta}\kappa|$ since $\mathscr{P}_B(\delta) \subseteq {}^{<\delta}\kappa$ and $\mathscr{P}_B(\delta)$ cannot inject into ${}^{<\rho(\alpha)}\kappa$ or even ${}^{\rho(\alpha)}\mathrm{ON}$ by Theorem 6.7. $\square$

[17] shows that under AD^+, $\mathscr{P}(\omega_1)$ is regular relative to essentially every set which does not already contain an injective copy of $\mathscr{P}(\omega_1)$ for which one currently have a practical understanding. This gives empirical evidence for the conjecture that $\mathscr{P}(\omega_1)$ has globally regular cardinality under AD^+.

If $\mathscr{P}(\kappa)$ has 2-regular cardinality, then statement (3) in Theorem 6.3 and Theorem 6.8 would follow immediately from the fact that $\mathscr{P}(\kappa)$ does not inject into $\mathscr{P}_B(\kappa)$ and does not inject into $[\kappa]^\epsilon$ when $\mathrm{cof}(\epsilon) = \mathrm{cof}(\kappa) < \kappa$. However, it is not even known if $\mathscr{P}(\omega_2)$ or $\mathscr{P}(\omega_3)$ has 2-regular cardinality. For this reason, the proofs of statement (3) in Theorem 6.3 and Theorem 6.8 require the more complicated argument resembling the proofs of Theorem 6.2 and Theorem 6.7. Since ω_3 is the first singular cardinal of determinacy, the first new instance of Theorem 6.8 (3) is that AD^+ implies $|[\omega_3]^{\omega_2} \cup \mathscr{P}_B(\omega_3)| < |\mathscr{P}(\omega_3)|$.

At a singular cardinal κ, there are two natural combinatorial sets whose cardinality have not yet been distinguished. They are $|{}^{<\kappa}\kappa| = |[\kappa]^{<\kappa}|$ and $|\mathscr{P}(\kappa)|$. The first open case occurs at the first singular cardinal of determinacy. Is $|[\omega_3]^{<\omega_3}| < |\mathscr{P}(\omega_3)|$? Another closely related question is whether $\mathscr{P}(\omega_2)$ injects into ${}^{<\omega_2}\mathrm{ON}$. The main result of the paper can be used to resolve these questions if one assumes a regularity condition on $\mathscr{P}(\kappa)$.

Fact 6.16. *Assume* AD^+. *Let* $\kappa < \Theta$ *and assume* $\mathscr{P}(\kappa)$ *has* κ*-regular cardinality. Then* $|[\kappa]^{<\kappa}| < |\mathscr{P}(\kappa)|$ *and* $\mathscr{P}(\kappa)$ *does not inject into* ${}^{<\kappa}\mathrm{ON}$.

Proof. The second statement implies the first. Suppose $\Phi : \mathscr{P}(\kappa) \to {}^{<\kappa}\mathrm{ON}$ is an injection. Let $\Gamma : \mathscr{P}(\kappa) \to \kappa$ be defined by $\Gamma(f) = |\Phi(f)| = \mathrm{dom}(\Phi(f))$. By the hypothesis that $\mathscr{P}(\kappa)$ is κ-regular, there is an $\epsilon < \kappa$ so that $|\Gamma^{-1}[\{\epsilon\}]| = |\mathscr{P}(\kappa)|$. Let $\Upsilon : \mathscr{P}(\kappa) \to \Gamma^{-1}[\{\epsilon\}]$ be an injection. Then $\Phi \circ \Upsilon : \mathscr{P}(\kappa) \to {}^{\epsilon}\mathrm{ON}$ is an injection. Then $\Phi \circ \Upsilon \restriction \mathscr{P}_B(\kappa) : \mathscr{P}_B(\kappa) \to {}^{\epsilon}\mathrm{ON}$ is an injection which violates Theorem 6.7. ☐

However, Fact 6.16 is applicable for the power set of a strong partition cardinal.

Theorem 6.17. *Assume* AD^+. *Let* $\kappa < \Theta$ *and assume* $\kappa \to_* (\kappa)^\kappa_2$. *Then* $|[\kappa]^{<\kappa}| < |\mathscr{P}(\kappa)|$ *and* $\mathscr{P}(\kappa)$ *does not inject* ${}^{<\kappa}\mathrm{ON}$.

Proof. Fact 6.14 showed that $\kappa \to_* (\kappa)^\kappa_2$ implies that $\mathscr{P}(\kappa)$ has ON-regular cardinality. The result follows from Fact 6.16. ☐

If κ is a strong partition cardinal, then the first statement of Theorem 6.17 can be proved purely combinatorially.

Theorem 6.18. *Assume* κ *satisfies* $\kappa \to_* (\kappa)^\kappa_2$. *Then* $|[\kappa]^{<\kappa}| < |\mathscr{P}(\kappa)|$.

Proof. Suppose $\Phi : \mathscr{P}(\kappa) \to [\kappa]^{<\kappa}$ is an injection. Since $\kappa \to_* (\kappa)^2_2$ implies κ is regular, for all $f \in \mathscr{P}(\kappa)$, $\sup(\Phi(f)) < \kappa$. Let $\Gamma : \mathscr{P}(\kappa) \to \kappa$ be defined by $\Gamma(f) = \sup(f) + 1$. $\kappa \to_* (\kappa)^\kappa_2$ implies $\mathscr{P}(\kappa)$ has ON-regular cardinality by Fact 6.14. Thus there is a $\delta < \kappa$ so that $|\Gamma^{-1}[\{\delta\}]| = |\mathscr{P}(\kappa)|$. Let $\Upsilon : \mathscr{P}(\kappa) \to \Gamma^{-1}[\{\delta\}]$ be an injection. Then $\Phi \circ \Upsilon : \mathscr{P}(\kappa) \to [\delta]^{<\delta}$ is an injection. This is impossible since $\kappa \to_* (\kappa)^2_2$ implies that κ is measurable and hence there no injection of κ into $\mathscr{P}(\delta)$ by Fact 2.3. ☐

If κ is a very strong partition cardinal, then the second statement of Theorem 6.17 can be proved purely combinatorially. (It is open whether every strong partition cardinal is a very strong partition cardinal even under AD or AD^+.)

Theorem 6.19. *Assume* κ *satisfies* $\kappa \to_* (\kappa)^\kappa_{<\kappa}$. *Then* $\mathscr{P}(\kappa)$ *does not inject into* ${}^{<\kappa}\mathrm{ON}$.

Proof. Fact 6.14 shows that $\kappa \to_* (\kappa)^\kappa_{<\kappa}$ implies that $\mathscr{P}(\kappa)$ has ${}^{<\kappa}\mathrm{ON}$-regular cardinality. So no function $\Phi : \mathscr{P}(\kappa) \to {}^{<\kappa}\mathrm{ON}$ can be injective. ☐

Next, one will create a large family of intermediate cardinalities below $\mathscr{P}_B(\kappa)$ when $\kappa < \Theta$ is a cardinal in $L(\mathbb{R})$. The following will review some basic facts about J-constructibility reduction when J is a set of ordinals.

Definition 6.20. Let $J \subseteq \mathrm{ON}$ be a set of ordinals. For $x, y \in \mathbb{R}$, let $x \leq_J y$ if and only if $x \in L[J, y]$ which is the J-constructibility degrees. Define $x \equiv_J y$ if and only if $x \leq_J y$ and $y \leq_J x$ which is the J-constructibility equivalence relation. Let $\mathcal{D}_J = \mathbb{R}/\equiv_J$ be the set of J-degrees or $\equiv_J$-equivalence classes. For $X, Y \in \mathcal{D}_J$, let $X \leq_J Y$ if and only if for any $x \in X$ and $y \in Y$, $x \leq_J y$. For each $X \in \mathcal{D}_J$, $\mathcal{C}_J(X) = \{Y \in \mathcal{D}_J : X \leq_J Y\}$ which is the J-constructibility cone above X. Let μ_J be a filter on $\mathcal{D}_J$ defined by $A \in \mu_J$ if and only if there is an $X \in \mathcal{D}_J$ so that $\mathcal{C}_J(X) \subseteq A$. A perfect tree p on 2 is J-pointed if and only if for all $x \in [p]$, $p \leq_J x$. (Recall that a perfect tree p on 2 is a Turing pointed perfect tree if and only if for all $x \in [p]$, $p \leq_{\mathsf{Turing}} x$, where $\leq_{\mathsf{Turing}}$ refers to Turing reduction. Every Turing pointed perfect tree is an J-pointed perfect tree.) If p is a J-pointed tree, then $\{[x]_{\equiv_J} : x \in [p]\} = \mathcal{C}_J([p]_{\equiv_J})$. A function f with domain $\mathbb{R}$ is J-invariant if and only if for all $x, y \in \mathbb{R}$, if $x \equiv_J y$, then $f(x) = f(y)$. A J-invariant function $f : \mathbb{R} \to \mathrm{ON}$ is everywhere J-increasing if and only if for all $x, y \in \mathbb{R}$, $x \leq_J y$ implies $f(x) \leq f(y)$. A J-invariant function $f : \mathbb{R} \to \mathrm{ON}$ is μ_J-almost everywhere increasing if and only if there is an $a \in \mathbb{R}$ so that for all $a \leq_J x \leq_J y$, $f(x) \leq f(y)$. If $F, G : \mathcal{D}_J \to \mathrm{ON}$, then define $F \sim_{\mu_J} G$ if and only if $\{X \in \mathcal{D}_J : F(X) = G(X)\} \in \mu_J$ and $F <_{\mu_J} G$ if and only if $\{X \in \mathcal{D}_J : F(X) < G(X)\}$. Let $\prod_{\mathcal{D}_J} \mathrm{ON}/\mu_J$ be the ultrapower of ON by μ_J which are the

$\sim_{\mu_J}$-equivalence classes of function $F : \mathcal{D}_J \to \mathrm{ON}$ ordered by the relation induced by $<_{\mu_J}$. If $f, g : \mathbb{R} \to \mathrm{ON}$ are J-invariant functions, then define $f \sim_{\mu_J} g$ if and only if $\{[x]_{\equiv_J} : f(x) = g(x)\} \in \mu_J$ and $f \leq_{\mu_J} g$ if and only if $\{[x]_{\equiv_J} : f(x) < g(x)\} \in \mu_J$.

The following basic facts are proved using the same argument as for the corresponding facts for Turing reducibility.

Fact 6.21. *(Martin) Assume* AD.

(1) For any $B \subseteq \mathbb{R}$, there is a J-pointed perfect tree (even a Turing pointed perfect tree) p so that $[p] \subseteq B$ or $[p] \subseteq \mathbb{R} \setminus B$.
(2) μ_J is a countably complete ultrafilter on $\mathcal{D}_J$.
(3) For all $A \subseteq \mathbb{R}$ such that there is a $x \in \mathbb{R}$ with $\{y \in \mathbb{R} : x \leq_J y\} \subseteq A$, then there is a J-pointed tree such that $[p] \subseteq A$.

Fact 6.22. *Assume* AD *and* $\mathsf{DC}_{\mathbb{R}}$. *For every J-invariant function $f : \mathbb{R} \to \mathrm{ON}$, there is a J-invariant everywhere J-increasing function $g : \mathbb{R} \to \mathrm{ON}$ so that $f \sim_{\mu_J} g$.*

Proof. Let $f : \mathbb{R} \to \mathrm{ON}$ be J-invariant. Let $\bar{f} : \mathcal{D}_J \to \mathrm{ON}$ be defined by $\bar{f}([x]_{\equiv_J}) = f(x)$ which is well defined since f is J-invariant. Let $A = \{X \in \mathcal{D}_J : (\forall Y \geq X)(\bar{f}(X) \leq \bar{f}(Y)\}$ and $B = \{X \in \mathcal{D}_J : (\exists Y \geq X)(\bar{f}(Y) < \bar{f}(X))\}$. Since $A \cap B = \emptyset$, $A \cup B = \mathcal{D}_J$, and μ_J is an ultrafilter by Fact 6.21, exactly one of A or B belongs to μ_J. Suppose $B \in \mu_J$. Let $Z \in \mathcal{D}_J$ be such that $\mathcal{C}_J(Z) \subseteq B$. Pick $z \in Z$. Define $R(x, y)$ if and only if $f(y \oplus z) < f(x \oplus z)$. For any $x \in \mathbb{R}$, $[x \oplus z]_{\equiv_J} \in \mathcal{C}_J(Z) \subseteq B$. So there is a $Y \geq [x \oplus z]$ so that $\bar{f}(Y) < \bar{f}([x \oplus z]_{\equiv_J})$. Pick any $y \in Y$ and note that $y \equiv_J y \oplus z$. Then $f(y \oplus z) = \bar{f}([y \oplus z]_{\equiv_J}) = \bar{f}([y]_{\equiv_J}) = \bar{f}(Y) < \bar{f}([x \oplus z]_{\equiv_J}) = f(x \oplus z)$. Thus $R(x, y)$. By $\mathsf{DC}_{\mathbb{R}}$, there is a sequence $\langle x_n : n \in \omega\rangle$ so that $R(x_n, x_{n+1})$ for all $n \in \omega$. Then $\langle f(x_n \oplus z) : n \in \omega\rangle$ is a descending sequence of ordinals which is a contradiction. Thus it must be the case that $A \in \mu_J$. Let $U \in \mathcal{D}_J$ so that $\mathcal{C}_J(U) \subseteq A$. Let $u \in U$. Define $g(x) = \min\{f(y) : u \oplus x \leq_J y\}$. g is clearly J-invariant. Suppose $x_0 \leq_J x_1$. Then $\{f(y) : u \oplus x_1 \leq_J y\} \subseteq \{f(y) : u \oplus x_0 \leq_J y\}$. Hence $g(x_0) \leq g(x_1)$. Thus g is everywhere J-increasing. Suppose $X \geq U$. Let $x \in X$. Note that $x \oplus u \equiv_J x$ and for all $y \in \mathbb{R}$, if $y \geq x \oplus u$, then $y \geq_J x \geq_J u$. Since $[x]_{\mu_J} = X \in A$, one has that $f(y) = \bar{f}([y]_{\equiv_J}) \geq \bar{f}([x]_{\equiv_J}) = f(x)$ for all $y \geq_J x \oplus u$. Thus $g(x) = \min\{f(y) : y \geq_J x \oplus u\} = f(x)$. This shows that $f \sim_{\mu_J} g$. ☐

Definition 6.23. Let $\mathfrak{gp} : \mathrm{ON} \times \mathrm{ON} \to \mathrm{ON}$ be the Gödel pairing function. If $\delta, \epsilon \in \mathrm{ON}$ and $A \subseteq \mathrm{ON}$, then A is said to code a function from δ into ϵ if and only if $\mathfrak{gp}^{-1}[A]$ is the graph of a function $f : \delta \to \epsilon$. If $f : \delta \to \epsilon$, let $\mathfrak{G}_f = \{\mathfrak{gp}(\alpha, f(\alpha)) : \alpha < \delta\}$. Note that if ϵ is indecomposable, then $\mathfrak{G}_f \subseteq \epsilon$.

Let $J \subseteq \mathrm{ON}$, $\epsilon \in \mathrm{ON}$, $r \in \mathbb{R}$, and $\alpha < \omega_1$. Let $G^{J,\epsilon}_{\alpha,r}$ be the set of all $\mathfrak{G}_f$ such that $f : \delta \to (\epsilon^{+(\alpha+1)})^{L[J,r]}$ is a function and $\delta \leq (\epsilon^{+\alpha})^{L[J,r]}$. Note that $G^{J,\epsilon}_{\alpha,r} \subseteq \mathscr{P}((\epsilon^{+(\alpha+1)})^{L[J,r]}) \subseteq \mathscr{P}_B(\epsilon^+)$ since Fact 2.19 implies that $(\epsilon^{+(\alpha+1)})^{L[J,r]} < \epsilon^+$. Thus $|G^{J,\epsilon}_{\alpha,r}| \leq |\mathscr{P}(\epsilon)|$. Also if $A \subseteq \epsilon$, then the characteristic function of A, $\chi_A : \epsilon \to 2$ belongs to $G^{J,\epsilon}_{\alpha,r}$. This shows that $|\mathscr{P}(\epsilon)| \leq |G^{J,\epsilon}_{\alpha,r}|$. Hence $|G^{J,\epsilon}_{\alpha,r}| = |\mathscr{P}(\epsilon)|$.

Let $h : \mathbb{R} \to \omega_1$ be a J-invariant everywhere J-increasing function. Then let $F^{J,\epsilon}_h = \bigsqcup_{r \in \mathbb{R}} G^{J,\epsilon}_{h(r),r} = \{(r, f) : r \in \mathbb{R} \wedge f \in G^{J,\epsilon}_{h(r),r}\}$. Note that $|F^{J,\epsilon}_h| \leq |\mathscr{P}_B(\epsilon^+)|$.

Fact 6.24. *Assume* AD. *Let J be a set of ordinals, $\epsilon \in \mathrm{ON}$ with $\omega \leq \epsilon$, and $h : \mathbb{R} \to \omega_1$ be a J-invariant everywhere J-increasing function. Assume boldface* GCH *at ϵ holds. Then $\neg(|\epsilon^+| \leq |F^{J,\epsilon}_h|)$ and thus $|\mathscr{P}(\epsilon)| \leq |F^{J,\epsilon}_h| < |\mathscr{P}_B(\epsilon^+)|$.*

Proof. Suppose $\Phi : \epsilon^+ \to F^{J,\epsilon}_h$ is an injection. Let $\pi_1 : \mathbb{R} \times \mathscr{P}(\epsilon^+) \to \mathbb{R}$ be the projection onto the first coordinate. Thinking of $F^{J,\epsilon}_h$ as a a subset of $\mathbb{R} \times \mathscr{P}(\epsilon^+)$, $\pi_1[\Phi[\epsilon^+]]$ is a wellorderable subset of $\mathbb{R}$. Since AD implies boldface GCH at ω (Fact 2.6), $\pi_1[\Phi[\epsilon^+]]$ is countable. By Fact 2.16, $\mathrm{cof}(\epsilon^+) > \omega$. Thus there is an $s \in \mathbb{R}$ so that $|(\pi_1 \circ \Phi)^{-1}[\{s\}]| = |\epsilon^+|$. Then $\Phi : (\pi_1 \circ \Phi)^{-1}[\{s\}] \to \{s\} \times G^{J,\epsilon}_{h(s),s}$ is an injection. As noted above, $|G^{J,\epsilon}_{h(s),s}| = |\mathscr{P}(\epsilon)|$. Thus this implies there is an injection of ϵ^+ into $\mathscr{P}(\epsilon)$ which violates boldface GCH at ϵ. The last statement follows from the fact that ϵ^+ injects into $\mathscr{P}_B(\epsilon^+)$. ☐

When $\mathsf{AC}^{\mathbb{R}}_{\mathbb{R}}$ (or $\mathsf{Uniformization}$) holds, the cardinality of $F^{J,\epsilon}_h$ is simply $|\mathscr{P}(\epsilon)|$.

Fact 6.25. *Assume* $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$. *Let* J *be a set of ordinals,* $\omega \leq \epsilon \in \mathrm{ON}$*, and* $h : \mathbb{R} \to \omega_1$ *be a* J*-invariant everywhere* J*-increasing function. Then* $|F_h^{J,\epsilon}| = |\mathscr{P}(\epsilon)|$.

Proof. As shown above, $|\mathscr{P}(\epsilon)| = |G^{J,\epsilon}_{h(r),r}|$ for any $r \in \mathbb{R}$. Thus $|\mathscr{P}(\epsilon)| \leq |F_h^{J,\epsilon}|$. By Fact 2.19, $(\epsilon^{+(h(r)+1)})^{L[J,r]} < \epsilon^+$ for any $r \in \mathbb{R}$. Thus for each $r \in \mathbb{R}$, there is a bijection $(\epsilon^{+(h(r)+1)})^{L[J,r]}$ onto ϵ. By the Moschovakis coding lemma (Fact 2.15), let $\varpi : \mathbb{R} \to \mathscr{P}(\epsilon^+)$ be a surjection. Define $R \subseteq \mathbb{R}\times\mathbb{R}$ by $R(r,s)$ if and only if $\varpi(s)$ codes the graph of a bijection from $(\epsilon^{+(h(r)+1)})^{L[J,r]}$ to ϵ. Note that $\mathrm{dom}(R) = \mathbb{R}$. By $\mathsf{AC}^{\mathbb{R}}_{\mathbb{R}}$, let $\hat{\Lambda} : \mathbb{R} \to \mathbb{R}$ be a such that $R(r, \hat{\Lambda}(r))$ for all $r \in \mathbb{R}$. Let $\Lambda(r) : (\epsilon^{+(h(r)+1)})^{L[J,r]} \to \epsilon$ be the bijection coded by $\hat{\Lambda}(r)$. Define $\Phi : F_h^{J,\epsilon} \to \mathbb{R}\times\mathscr{P}(\epsilon)$ by $\Phi(r, A) = (r, \Lambda[A])$. Φ is an injection and therefore, $|F_h^{J,\epsilon}| \leq |\mathbb{R}\times\mathscr{P}(\epsilon)| = |\mathscr{P}(\epsilon)|$. ☐

Fact 6.26. *Let* $h_0, h_1 : \mathbb{R} \to \omega_1$ *be two* J*-invariant everywhere* J*-increasing functions such that* $h_0 \leq_{\mu_J} h_1$*. Then* $|F_{h_0}^{J,\epsilon}| \leq |F_{h_1}^{J,\epsilon}|$*. Thus if* $h_0 \sim_{\mu_j} h_1$*, then* $|F_{h_0}^{J,\epsilon}| = |F_{h_1}^{J,\epsilon}|$.

Proof. Let $B = \{x \in \mathbb{R} : h_0(x) \leq h_1(x)\}$. Since $h_0 \leq_{\mu_J} h_1$, there is a $u \in \mathbb{R}$ so that $\{x \in \mathbb{R} : u \leq_J x\} \subseteq B$. By Fact 6.21, there is a J-pointed perfect tree p so that $[p] \subseteq B$. An $s \in p$ is called a split node of p if and only if $s\hat{\ }0 \in p$ and $s\hat{\ }1 \in p$. For $n \in \omega$, s is called an n-split node of p if and only if there are exactly n-many proper initial segment of s which are split nodes of p. Let $\mathrm{split}_n(p)$ be the collection of n-split nodes of p. Let $\Xi_p : {}^{<\omega}2 \to \bigcup_{n\in\omega} \mathrm{split}_n(p)$ be defined as follows: $\Xi_p(\emptyset)$ is the shortest split node node of p. If $\Xi_p(s)$ has been defined and $i \in 2$, then let $\Xi_p(s\hat{\ }i)$ be the shortest split node of p which extends $\Xi_p(s)\hat{\ }i$. Let $\Upsilon_p : {}^{\omega}2 \to [p]$ be defined by $\Upsilon_p(f) = \bigcup_{n\in\omega} \Xi_p(f \upharpoonright n)$. Since p is J-pointed and $\Upsilon_p(f) \in [p]$, $p \leq_J \Upsilon_p(f)$. Since f can be recovered from $\Upsilon_p(f)$ and p by a simple computable manner, $f \leq_J \Upsilon_p(f)$.

Define a function Φ on $F_{h_0}^{J,\epsilon}$ by $\Phi(r, A) = \Phi(\Upsilon_p(r), A)$. Φ is clearly an injective function. Let $(r, A) \in F_{h_0}^{J,\epsilon}$. This means that $A \in G^{J,\epsilon}_{h(r),r}$. Thus $A = \mathfrak{G}_f$ where $f : \delta \to (\epsilon^{+(h_0(r)+1)})^{L[J,r]}$ and $\delta \leq (\epsilon^{+h_0(r)})^{L[J,r]}$. Since h_0 is everywhere J-increasing, $r \leq_J \Upsilon_p(r)$, and $\Upsilon_p(r) \in [p] \subseteq B$, one has that $h_0(r) \leq h_0(\Upsilon_p(r)) \leq h_1(\Upsilon_p(r))$. Since $r \leq_J \Upsilon_p(r)$, $L[J,r] \subseteq L[J, \Upsilon_p(r)]$. Thus $\delta \leq (\epsilon^{+h_0(r)})^{L[J,r]} \leq (\epsilon^{+h_0(r)})^{L[J,\Upsilon_p(r)]} \leq (\epsilon^{+h_1(r)})^{L[J,\Upsilon_p(r)]}$. Similiarly, $(\epsilon^{+(h_0(r)+1)})^{L[J,r]} \leq (\epsilon^{+(h_1(r)+1)})^{L[J,\Upsilon_p(r)]}$. Thus $A = \mathfrak{G}_f \in G^{J,\epsilon}_{h_1(\Upsilon_p(r)),\Upsilon_p(r)}$. So $(\Upsilon_p(r), A) \in F_{h_1}^{J,\epsilon}$. $\Phi : F_{h_0}^{J,\epsilon} \to F_{h_1}^{J,\epsilon}$ is an injection and hence $|F_{h_0}^{J,\epsilon}| \leq |F_{h_1}^{J,\epsilon}|$. ☐

In contrast to Fact 6.25, if $V = L(\mathbb{R})$ and J is a set of ordinal such that $L[J] = \mathrm{HOD}$, then $F_h^{J,\epsilon}$ will represent a cardinality intermediate between $\mathscr{P}(\epsilon)$ and $\mathscr{P}_B(\epsilon^+)$. By Fact 4.20, $\mathrm{HOD}^{L(\mathbb{R})} = L[\vec{\mathbb{B}}^{\omega;\emptyset}]$ so $\vec{\mathbb{B}}^{\omega;\emptyset}$ is an example of such a set of ordinals J.

Theorem 6.27. *Assume* AD *and* $V = L(\mathbb{R})$*. Let* J *be a set of ordinals so that* $\mathrm{HOD} = L[J]$*,* $\omega \leq \epsilon < \Theta$*, and* $h_0, h_1 : \mathbb{R} \to \omega_1$ *be two* J*-invariant everywhere* J*-increasing functions such that* $h_0 <_{\mu_J} h_1$*, then* $|F_{h_0}^{J,\epsilon}| < |F_{h_1}^{J,\epsilon}|$.

Proof. By Fact 6.26, $|F_{h_0}^{J,\epsilon}| \leq |F_{h_1}^{J,\epsilon}|$. Suppose $\Phi : F_{h_1}^{J,\epsilon} \to F_{h_0}^{J,\epsilon}$ is an injection. Since $V = L(\mathbb{R})$, there is an $\tilde{x} \in \mathbb{R}$ so that Φ is $\mathrm{OD}_{\{\tilde{x}\}}$. Since $h_0 \leq_{\mu_J} h_1$, there exists an $x \in \mathbb{R}$ so that $h_0(x) < h_1(x)$ and $\tilde{x} \leq_J x$. So $\tilde{x} \in \mathrm{HOD}_{\{\tilde{x}\}} = \mathrm{HOD}[\tilde{x}] = L[J][\tilde{x}] = L[J,\tilde{x}] \subseteq L[J,x] = L[J][x] = \mathrm{HOD}[x] = \mathrm{HOD}_{\{x\}}$ by Fact 4.19, the hypothesis on J, and $\tilde{x} \leq_J x$. Thus $\tilde{x} \in \mathrm{OD}_{\{x\}}$. Hence $h_0(x) < h_x(x)$ and $\Phi \in \mathrm{OD}_{\{x\}}$. Consider the forcing $\mathrm{Coll}(\epsilon^{+h_1(x)}, \epsilon^{+(h_1(x)+1)})^{\mathrm{HOD}_{\{x\}}}$. By Lemma 5.3, Lemma 5.5, and Fact 6.1 much like in the proof of Theorem 6.2, there is a $G \subseteq \mathrm{Coll}(\epsilon^{+h_1(x)}, \epsilon^{+(h_1(x)+1)})^{\mathrm{HOD}_{\{x\}}}$ which is generic over $\mathrm{HOD}_{\{x\}}$. Let $g : (\epsilon^{+h_1(x)})^{\mathrm{HOD}_{\{x\}}} \to (\epsilon^{+(h_1(x)+1)})^{\mathrm{HOD}_{\{x\}}}$ be the generic surjection. Since $\mathrm{HOD}_{\{x\}} = \mathrm{HOD}[x] = L[J,x]$, $g : (\epsilon^{+h_1(x)})^{L[J,x]} \to (\epsilon^{+(h_1(x)+1)})^{L[J,x]}$. Thus $\mathfrak{G}_g \in G^{J,\epsilon}_{h_1(x),x}$ (using the notation of Definition 6.23). Hence $(x, \mathfrak{G}_g) \in F_{h_1}^{J,\epsilon}$. Then $\Phi(x, \mathfrak{G}_g)$ is $\mathrm{OD}_{\{x,G\}}$. In $\mathrm{HOD}_{\{x\}}$, this forcing is in bijection with $(\epsilon^{+(h_1(x)+1)})^{\mathrm{HOD}_{\{x\}}}$. By Fact 5.7 of Neeman and Woodin, there is a unique supercompact measure on $\mathscr{P}_{\omega_1}((\epsilon^{+(h_1(x)+1)})^{\mathrm{HOD}_{\{x\}}})$. (If ϵ is below a Suslin cardinal or equivalently below $\boldsymbol{\delta}^2_1$, such supercompact measure exists by Harrington-Kechris [24] and Woodin [66].) By Lemma 5.9, $\Phi(x, \mathfrak{G}_g) \in \mathrm{HOD}_{\{x,G\}} = \mathrm{HOD}_{\{x\}}[G]$. Since $\Phi : F_{h_1}^{J,\epsilon} \to F_{h_0}^{J,\epsilon}$, $\Phi(x, \mathfrak{G}_g) \in F_{h_0}^{J,\epsilon} \cap \mathrm{HOD}_{\{x\}}[G]$. Thus there is some $r \in \mathbb{R}$ and some f such that $f : \tilde{\delta} \to (\epsilon^{+(h_0(r)+1)})^{L[J,r]}$, $\tilde{\delta} \leq (\epsilon^{+h_0(r)})^{L[J,r]}$, and $\Phi(x, \mathfrak{G}_g) = (r, \mathfrak{G}_f)$. Since $(r, \mathfrak{G}_f) = \Phi(x, \mathfrak{G}_g) \in \mathrm{HOD}_{\{x\}}[G]$, one has that $r \in \mathrm{HOD}_{\{x\}}[G]$ and $f \in \mathrm{HOD}_{\{x\}}[G]$ (since f is the function whose graph is $\mathfrak{gp}^{-1}[\mathfrak{G}_g]$ where $\mathfrak{gp}$ is the Gödel pairing function). Since $\mathrm{HOD}_{\{x\}} \models$ "$\mathrm{Coll}(\epsilon^{+h_1(x)}, \epsilon^{+(h_1(x)+1)})$ is $< \epsilon^{+h_1(x)}$-closed", no new functions from a $\delta < (\epsilon^{+h_1(x)})^{\mathrm{HOD}_{\{x\}}}$ into the ordinals are added by this forcing. No new reals are added by this forcing. So $r \in \mathrm{HOD}_{\{x\}} = L[J,x]$ and thus $r \leq_J x$. Since h_0 is everywhere J-increasing, $h_0(r) \leq h_0(x) < h_1(x)$. So $\tilde{\delta} \leq (\epsilon^{+h_0(r)})^{L[J,r]} \leq (\epsilon^{+h_0(r)})^{L[J,x]} \leq (\epsilon^{+h_0(x)})^{L[J,x]} < (\epsilon^{+(h_0(x)+1)})^{L[J,x]} \leq (\epsilon^{+h_1(x)})^{L[J,x]} = (\epsilon^{+h_1(x)})^{\mathrm{HOD}_{\{x\}}}$. So f is a function from $\tilde{\delta} < (\epsilon^{+h_1(x)})^{\mathrm{HOD}_{\{x\}}}$ into the ordinals. Again since no new functions from a $\delta < (\epsilon^{+h_1(x)})^{\mathrm{HOD}_{\{x\}}}$ into the ordinals are added by this forcing, $f \in \mathrm{HOD}_{\{x\}}$. So $\Phi(x, \mathfrak{G}_g) = (r, \mathfrak{G}_f) \in \mathrm{HOD}_{\{x\}}$. So $(x, \mathfrak{G}_g) = \Phi^{-1}(\Phi(x, \mathfrak{G}_g))$ is $\mathrm{OD}_{\{x\}}$ and so $(x, \mathfrak{G}_g) \in \mathrm{HOD}_{\{x\}}$. In particular, $\mathfrak{G}_g \in \mathrm{HOD}_{\{x\}}$. Then $g \in \mathrm{HOD}_{\{x\}}$ since g is the function whose graph is $\mathfrak{gp}^{-1}(\mathfrak{G}_g)$. By absoluteness, $\mathrm{HOD}_{\{x\}} \models$ "$g : \epsilon^{+h_1(x)} \to \epsilon^{+(h_1(x)+1)}$ is a surjection". Contradiction. ☐

If $X \in \mathcal{D}_J$, defined $L[J,X]$ to be $L[J,x]$ for any $x \in X$. Note that this is well defined independent of the choice of $x \in X$. (However, the standard global constructibility ordering of $L[J,x]$ does depend on x.) Since $L(\mathbb{R}) \models \mathsf{DC}$, $\prod_{\mathcal{D}_J} \omega_1/\mu_J$ is wellfounded.

Fact 6.28. *(Woodin) Assume* AD. *Let J be a set of ordinals.* $\prod_{X \in \mathcal{D}_J} \omega_1^{L[J,X]}/\mu_J = \omega_1$.

(Woodin) Assume $V = L(\mathbb{R})$ and J is a set of ordinals such that $\mathrm{HOD} = L[J]$, *then* $\prod_{X \in \mathcal{D}_J} \omega_2^{L[J,X]} = \Theta$.

Proof. The first statement is [42] Theorem 9.6.

Now suppose $V = L(\mathbb{R})$ and J is a set of ordinals such that $\mathrm{HOD} = L[J]$. Let $A \in \mathscr{P}(\mathbb{R})$. A is $\mathrm{OD}_{\{x\}}$ for some $x \in \mathbb{R}$. For any $y \in \mathbb{R}$ such that $x \leq_J y$, $x \in L[J,y] = L[J][y] = \mathrm{HOD}[y] = \mathrm{HOD}_{\{y\}}$ by Fact 4.19. So for any y such that $x \leq_J y$, A is $\mathrm{OD}_{\{y\}}$. Then for all $y \geq_J x$, $A \cap \mathrm{HOD}_{\{y\}}$ is $\mathrm{OD}_{\{y\}}$ and hence $A \cap \mathrm{HOD}_{\{y\}} \in \mathrm{HOD}_{\{y\}}$. For all $y \geq_J x$, $A \cap L[J,y] \in L[J,y]$ since $L[J,y] = \mathrm{HOD}_{\{y\}}$. This verifies that J is a strongly maximal set of ordinals in the terminology of [42]. The result now follow from [42] Corollary 9.26 and Corollary 9.27. ☐

If $h : \mathbb{R} \to \mathrm{ON}$ is a J-invariant function, let $\bar{h} : \mathcal{D}_J \to \mathrm{ON}$ be defined by $\bar{h}([x]_{\equiv_J}) = h(x)$ and note this is well defined by J-invariance. For each $\alpha \in \prod_{\mathcal{D}_J} \omega_1/\mu_J$, there is a J-invariant everywhere J-increasing function $h : \mathbb{R} \to \omega_1$ so that $[\bar{h}]_{\sim_{\mu_J}} = \alpha$ by Fact 6.22. Thus for each $\alpha \in \prod_{\mathcal{D}_J} \omega_1/\mu_J$, let $\mathfrak{F}^{J,\epsilon}_\alpha = |F^{J,\epsilon}_h|$ for any J-invariant everywhere J-increasing function so that $[\bar{h}]_{\sim_{\mu_J}} = \alpha$. $\mathfrak{F}^{J,\epsilon}_\alpha$ is independent of the choice of such h by Fact 6.26. Assume $V = L(\mathbb{R})$ and let J be a set of ordinals such that $L[J] = \mathrm{HOD}$. By Fact 6.24 and Theorem 6.27, the map Σ on $\prod_{\mathcal{D}_J} \omega_1/\mu_J$ defined by $\Sigma(\alpha) = \mathfrak{F}^{J,\epsilon}_\alpha$ is an order preserving injection of $\prod_{\mathcal{D}_J} \omega_1/\mu$ with the ultrapower ordering (which is isomorphic to an ordinal larger than Θ by Fact 6.28) into the cardinalities between $\mathscr{P}(\epsilon)$ and $\mathscr{P}_B(\epsilon^+)$ under the cardinality injection ordering.

7. ω-Sequences of Ordinals

The ABCD conjecture merely distinguished the cardinality of different cardinal exponentiations. This section will focus on deeper questions concerning cardinality and combinatorics for ω-sequences of ordinals which correspond to the smallest nonwellorderable cardinal exponentiation. The techniques introduced here seem very flexible for resolving many questions concerning ω-sequences of ordinals including primeness, Jónssonness, and basis for linear orderings which will be addressed in forthcoming work.

First, one will consider the question of whether ${}^\omega\kappa$ injects into $\mathscr{P}(\delta) \times \mathrm{ON}$ when $\delta < \kappa$. Recall that if $\delta < \kappa$, then Fact 2.10 says that $\kappa \to_* (\kappa)^\omega_\delta$ and more directly the δ^+-completeness of μ^ω_κ imply that ${}^\omega\kappa$ does not inject into $\mathscr{P}(\delta) \times \mathrm{ON}$. However, ω_3 is a singular cardinal and hence does not possess any partition properties. Another approach to the question of whether ${}^\omega\kappa$ injects into $\mathscr{P}(\delta) \times \mathrm{ON}$ is through a combinatorial property called primeness investigated under determinacy in [16].

Definition 7.1. Let A be a set. A is said to be prime if and only if for all sets B and C, if $|A| \leq |B \times C|$, then $|A| \leq |B|$ or $|A| \leq |C|$ (that is, if A injects into $B \times C$, then A already injects into B or C).

Fact 7.2. *(Chan-Jackson-Trang; [16]) Assume* $\mathsf{AC}^{\mathbb{R}}_\omega$ *and all subsets of* $\mathbb{R}$ *have the Baire property.* $\mathbb{R}$ *and* $\mathbb{R}/E_0$ *are prime.*

Fact 7.3. *(Chan-Jackson-Trang; [16]) Assume* $\kappa \to_* (\kappa)^{\omega+\omega}_2$. ${}^\omega\kappa$ *is prime.*

Fact 7.4. *(Chan-Jackson-Trang; [16]) Assume* AD. *For all* $n \in \omega$, ${}^\omega(\omega_n)$ *is prime.* ${}^\omega(\omega_\omega)$ *is prime.*

Fact 7.5. *Assume boldface* GCH *holds below* κ, ${}^\omega\kappa$ *is not wellorderable*[19], *and* ${}^\omega\kappa$ *is prime. Then* ${}^\omega\kappa$ *does not inject into* $\mathscr{P}(\delta) \times \mathrm{ON}$ *for all* $\delta < \kappa$.

Proof. Suppose $\Phi : {}^\omega\kappa \to \mathscr{P}(\delta) \times \mathrm{ON}$ is an injection. By replacement, there is a $\lambda \in \mathrm{ON}$ so that $\Phi : {}^\omega\kappa \to \mathscr{P}(\delta) \times \lambda$. By primeness, ${}^\omega\kappa$ injects into $\mathscr{P}(\delta)$ or λ. Since $\delta^+ \leq \kappa$, $|\kappa| \leq |{}^\omega\kappa|$, and boldface GCH holds at δ, ${}^\omega\kappa$ cannot inject into $\mathscr{P}(\delta)$. Since ${}^\omega\kappa$ is not wellorderable, ${}^\omega\kappa$ cannot inject into λ. Contradiction. □

Fact 7.6. *Assume* AD. *For all* $1 \leq n < \omega$, ${}^\omega(\omega_n)$ *does not inject into* $\mathscr{P}(\omega_m) \times \mathrm{ON}$ *for all* $m < n$. ${}^\omega(\omega_\omega)$ *does not inject into* $\mathscr{P}(\omega_n) \times \mathrm{ON}$ *for all* $n < \omega$.

Proof. This follows from Fact 7.4 and Fact 7.5. □

It will be shown next using the methods from the previous sections that for all $\kappa < \Theta$, ${}^\omega\kappa$ does not inject into $\mathscr{P}(\delta) \times \mathrm{ON}$ for all $\delta < \kappa$. The basic template for this argument is Theorem 3.5.

Definition 7.7. An measure sequence $\mathfrak{u}$ consists of $\vec{\mu}^{\mathfrak{u}} = \langle \mu_n^{\mathfrak{u}} : n \in \omega\rangle$, $\vec{\kappa}^{\mathfrak{u}} = \langle \kappa_n^{\mathfrak{u}} : n \in \omega\rangle$, and $\lambda^{\mathfrak{u}}$ such that for all $n \in \omega$, $\omega < \kappa_n^{\mathfrak{u}}$, $\vec{\kappa}^{\mathfrak{u}}$ is a strictly increasing sequence, $\mu_n^{\mathfrak{u}}$ is a $\kappa_n^{\mathfrak{u}}$-complete uniform normal measure on $\kappa_n^{\mathfrak{u}}$, and $\lambda^{\mathfrak{u}} = \sup \vec{\kappa}^{\mathfrak{u}} = \sup\{\kappa_n^{\mathfrak{u}} : n \in \omega\}$.

Definition 7.8. Let $\mathfrak{u}$ be a measure sequence. A $\mathfrak{u}$-Namba condition is a function $p \in \prod_{n\in\omega} \kappa_n^{\mathfrak{u}} \cup \mu_n^{\mathfrak{u}}$ with the following properties.

- There is an $m \in \omega$ so that for all $i < m$, $p(i) \in \kappa_i^{\mathfrak{u}}$ and for all $i \geq m$, $p(i) \in \mu_i^{\mathfrak{u}}$. The unique such m called the stem-length of p and is denoted $\mathsf{slength}(p)$. (In other words, if $i < \mathsf{slength}(p)$, then $p(i)$ is an ordinal below $\kappa_i^{\mathfrak{u}}$. If $\mathsf{slength}(p) \leq i$, then $p(i)$ is a subset of $\kappa_i^{\mathfrak{u}}$ which belongs to the measure $\mu_i^{\mathfrak{u}}$.)
- $p \restriction \mathsf{slength}(p)$ is called the stem of p and is denoted $\mathsf{stem}(p)$.

Let $\mathbb{N}_{\mathfrak{u}}$ denote the collection of all $\mathfrak{u}$-Namba conditions. Define $\leq_{\mathbb{N}_{\mathfrak{u}}}$ on $\mathbb{N}_{\mathfrak{u}}$ as follows. If $p, q \in \mathbb{N}_{\mathfrak{u}}$, then $p \leq_{\mathbb{N}_{\mathfrak{u}}} q$ if and only if the following holds.

- $\mathsf{slength}(p) \geq \mathsf{slength}(q)$.
- $\mathsf{stem}(q) = \mathsf{stem}(p) \restriction \mathsf{slength}(q)$.
- For all $\mathsf{slength}(q) \leq i < \mathsf{slength}(p)$, $p(i) \in q(i)$.
- For all $\mathsf{slength}(p) \leq i < \omega$, $p(i) \subseteq q(i)$.

Let $1_{\mathbb{N}_{\mathfrak{u}}}$ be the function defined by $1_{\mathbb{N}_{\mathfrak{u}}}(n) = \kappa_n^{\mathfrak{u}}$ for all $n \in \omega$. $\mathbb{N}_{\mathfrak{u}} = (\mathbb{N}_{\mathfrak{u}}, \leq_{\mathbb{N}_{\mathfrak{u}}}, 1_{\mathbb{N}_{\mathfrak{u}}})$ is $\mathfrak{u}$-Namba forcing.

For $k \in \omega$ and $p, q \in \mathbb{N}_{\mathfrak{u}}$, let $p \leq_{\mathfrak{u}}^k q$ if and only if $\mathsf{stem}(p) = \mathsf{stem}(q)$ and for all $i < k$, $p(i) = q(i)$. A sequence $\langle p_k : k \in \omega\rangle$ such that $p_{k+1} \leq_{\mathfrak{u}}^k p_k$ for all $k \in \omega$ is a $\mathfrak{u}$-fusion sequence. The fusion of the $\mathfrak{u}$-fusion sequence $\langle p_k : k \in \omega\rangle$ is p_ω defined by $p_\omega(i) = p_{i+1}(i)$. It is clear that p_ω is a $\mathfrak{u}$-Namba condition and $\mathsf{stem}(p_\omega) = \mathsf{stem}(p_k)$ for any $k \in \omega$. Note that for all $i \in \omega$, $p_\omega \leq_{\mathfrak{u}}^i p_i$.

If $s \in {}^{<\omega}(\lambda^{\mathfrak{u}})$ and $p \in \mathbb{N}_{\mathfrak{u}}$, then say that $s \in^{\mathfrak{u}} p$ if and only if for all $i < \mathsf{slength}(p)$, $s(i) = p(i)$ and for all $\mathsf{slength}(p) \leq i < |s|$, $s(i) \in p(i)$. Define the body of p, denoted $[p]$, to be the set of $f \in {}^\omega(\lambda^{\mathfrak{u}})$ such that for all $n \in \omega$, $f \restriction n \in^{\mathfrak{u}} p$. If $p \in \mathbb{N}_{\mathfrak{u}}$ and $s \in^{\mathfrak{u}} p$, then let $(p)_s \in \mathbb{N}_{\mathfrak{u}}$ be defined by

$$(p)_s(i) = \begin{cases} s(i) & i < |s| \\ p(i) & |s| \leq i \end{cases}.$$

Note that if $|s| \leq \mathsf{slength}(p)$, then $\mathsf{stem}((p)_s) = s$ and $(p)_s = p$, and if $|s| > \mathsf{slength}(p)$, then $\mathsf{stem}((p)_s) = s$.

Let F is a $\mathbb{N}_{\mathfrak{u}}$-filter and $p, q \in G$. There is an $r \in F$ so that $r \leq_{\mathbb{N}^{\mathfrak{u}}} p$ and $r \leq_{\mathbb{N}^{\mathfrak{u}}} q$. Thus for all $i < \min\{\mathsf{slength}(p), \mathsf{slength}(q)\}$, $p(i) = r(i) = q(i)$. Let $f_F = \bigcup\{\mathsf{stem}(p) : p \in F\}$. So there is an $\alpha \leq \omega$ so that $f_F : \alpha \to \lambda^{\mathfrak{u}}$. Let $\dot{f}_{\dot{G}}$ be the canonical $\mathbb{N}_{\mathfrak{u}}$-name for the ω-sequence added by $\dot{G}$, which is the name for the $\mathbb{N}_{\mathfrak{u}}$-generic filter.

Remark 7.9. Namba forcing typically refers to forcings whose conditions are trees with some properties. $\mathbb{N}_{\mathfrak{u}}$ more closely resembles some form of Prikry forcing along ω many measures on an increasing ω-sequence of measurable cardinals. $\mathbb{N}_{\mathfrak{u}}$ is essentially equivalent to diagonal Prikry forcing. $\mathbb{N}_{\mathfrak{u}}$ is an instance of a more general Namba forcing developed by Cox and Krueger in [23]. Since here $\mathfrak{u}$ is an increasing sequence of measurable cardinals along with its sequence of measures, one has simplified the presentation by removing

[19] If $\kappa > \omega$, then boldface GCH below κ implies boldface GCH at ω. This implies that $\mathbb{R}$ is not wellorderable. Since $|\mathbb{R}| \leq |{}^\omega\kappa|$, ${}^\omega\kappa$ is not wellorderable.

the trees. This Prikry-style presentation seems notationally more practical for handling higher dimensional Namba forcing which will be used in forthcoming work.

In [23], a $\mathfrak{u}$-Namba-tree is a $T \subseteq {}^{<\omega}(\lambda^{\mathfrak{u}})$ so that for all $s \in T$, $\mathrm{succ}_p(s) = \{\alpha \in \kappa^{\mathfrak{u}}_{|s|} : s\hat{\ }\langle\alpha\rangle \in T\} \in \mu^{\mathfrak{u}}_{|s|}$. Let $\bar{\mathbb{N}}_{\mathfrak{u}}$ be the Cox-Krueger $\mathfrak{u}$-Namba forcing consisting of $\mathfrak{u}$-Namba trees ordered by the subset relation. If $p \in \mathbb{N}_{\mathfrak{u}}$ is a $\mathfrak{u}$-Namba condition, then $T_p = \{s \in {}^{<\omega}(\lambda^{\mathfrak{u}}) : s \in^{\mathfrak{u}} p\}$ is a $\mathfrak{u}$-Namba tree. The map $\Phi : \mathbb{N}_{\mathfrak{u}} \to \bar{\mathbb{N}}_{\mathfrak{u}}$ defined by $\Phi(p) = T_p$ is a dense embedding of forcing posets. Lemma 7.11 and Fact 7.13 corresponds to [23] Lemma 6.4 and [23] Proposition 6.14, respectively.

Fact 7.10. *Let $M \models \mathsf{ZF}$ be an inner model and $\mathfrak{u} \in M$ such that $M \models$ "$\mathfrak{u}$ is a measure sequence". Suppose $G \subseteq \mathbb{N}_{\mathfrak{u}}$ is $\mathbb{N}_{\mathfrak{u}}$-generic over M. Then $f_G : \omega \to \lambda^{\mathfrak{u}}$ is cofinal through $\lambda^{\mathfrak{u}}$ and $f_G \notin M$.*

Proof. For each $n \in \omega$, let $D_n = \{p \in \mathbb{N}_{\mathfrak{u}} : \mathsf{slength}(p) > n\}$. D_n is dense in $\mathbb{N}_{\mathfrak{u}}$. If $G \cap D_n \neq \emptyset$ for all $n \in \omega$, then $\mathrm{dom}(f_G) = \mathrm{dom}(\bigcup\{\mathsf{stem}(p) : p \in G\}) = \omega$.

For each $n \in \omega$, let $E_n = \{p \in \mathbb{N}_{\mathfrak{u}} : \mathsf{slength}(p) > n \wedge (\forall i \geq \mathsf{slength}(p))(p(i) \subseteq \kappa^{\mathfrak{u}}_i \setminus \kappa^{\mathfrak{u}}_n)\}$. Let $q \in \mathbb{N}_{\mathfrak{u}}$. Let $k = \max\{n, \mathsf{slength}(q)\}$. Pick $s \in^{\mathfrak{u}} q$ with $|s| = k$. Define $p \in \mathbb{N}_{\mathfrak{u}}$ as follows:

$$p(i) = \begin{cases} s(i) & i < k \\ q(i) \setminus \kappa^{\mathfrak{u}}_n & k \leq i \end{cases}$$

Note that $p \in E_n$ and $p \leq_{\mathbb{N}_{\mathfrak{u}}} q$. Thus E_n is dense. Suppose $p \in G \cap E_n$. For all $i \geq \mathsf{slength}(p)$ and all $s \in^{\mathfrak{u}} p$, $s(i) \geq \kappa_n$. Thus $f_G(i) \geq \kappa_n$ for all $i \geq \mathsf{slength}(p)$. Thus if $G \cap E_n \neq \emptyset$ for all $n \in \omega$, then $\sup(f_G) = \sup\{\kappa^{\mathfrak{u}}_n : n \in \omega\} = \lambda^{\mathfrak{u}}$.

For each $h \in {}^{\omega}(\lambda^{\mathfrak{u}})$, let $F_h = \{p \in \mathbb{N}_{\mathfrak{u}} : (\exists n < \mathsf{slength}(p))(\mathsf{stem}(p)(n) \neq h(n))\}$. Suppose $q \in \mathbb{N}_{\mathfrak{u}}$. Since $q(\mathsf{slength}(q)) \in \mu^{\mathfrak{u}}_{\mathsf{slength}(q)}$, there exists $\alpha \in q(\mathsf{slength}(q))$ such that $h(\mathsf{slength}(q)) \neq \alpha$. Let $p \in \mathbb{N}_{\mathfrak{u}}$ be defined by $p(i) = q(i)$ for $i < \mathsf{slength}(q)$, $p(\mathsf{slength}(q)) = \alpha$, and $p(i) = q(i) \setminus (\alpha + 1)$ for $i > \mathsf{slength}(q)$. Then $p \leq_{\mathbb{N}_{\mathfrak{u}}} q$ and $q \in F_h$. This shows that F_h is a dense subset of $\mathbb{N}_{\mathfrak{u}}$. If $G \cap F_h$, then $f_G \neq h$. Thus if $G \cap F_h$ for all $h \in ({}^{\omega}(\lambda^{\mathfrak{u}}))^M$, then $f_G \neq h$ for any $h \in ({}^{\omega}(\lambda^{\mathfrak{u}}))^M$. ☐

Lemma 7.11. *Assume AC. Let $\mathfrak{u}$ be a measure sequence, $D \subseteq \mathbb{N}_{\mathfrak{u}}$ be a dense open subset of $\mathbb{N}_{\mathfrak{u}}$, and $p \in \mathbb{N}_{\mathfrak{u}}$. Then there is a $q \leq_{\mathbb{N}_{\mathfrak{u}}} p$ and a $k \in \omega$ so that $\mathsf{slength}(p) \leq k < \omega$, $\mathsf{slength}(q) = \mathsf{slength}(p)$, and for all $s \in^{\mathfrak{u}} q$ with $|s| = k$, $(q)_s \in D$.*

Proof. Say that $p \in \mathbb{N}_{\mathfrak{u}}$ is D-good if and only if there is a $q \leq_{\mathbb{N}_{\mathfrak{u}}} p$ and $\mathsf{slength}(p) \leq k < \omega$ so that $\mathsf{slength}(q) = \mathsf{slength}(p)$ and for all $s \in^{\mathfrak{u}} q$ with $|s| = k$, $(q)_s \in D$. Let $\mathsf{bad}_D(p) = \{\alpha \in \kappa^{\mathfrak{u}}_{\mathsf{slength}(p)} : (p)_{\mathsf{stem}(p)\hat{\ }\langle\alpha\rangle}$ is not D-good$\}$.

Claim: If p is not D-good, then $\mathsf{bad}_D(p) \in \mu^{\mathfrak{u}}_{\mathsf{slength}(p)}$.

To see the claim: Let $m = \mathsf{slength}(p)$. Suppose $\mathsf{bad}_D(p) \notin \mu^{\mathfrak{u}}_m$. Then $p(m) \setminus \mathsf{bad}_D(p) \in \mu^{\mathfrak{u}}_m$. For each $\alpha \in p(m) \setminus \mathsf{bad}_D(p)$, $(p)_{\mathsf{stem}(p)\hat{\ }\langle\alpha\rangle}$ is D-good. Let k_α be the least k with $m + 1 \leq k < \omega$ so that there exists some $r \leq_{\mathbb{N}_{\mathfrak{u}}} (p)_{\mathsf{stem}(p)\hat{\ }\langle\alpha\rangle}$ with the property that $\mathsf{slength}(r) = m + 1$ and for all $s \in^{\mathfrak{u}} r$ with $|s| = k$, $(r)_s \in D$. Since $\omega < \kappa_m$ and $\mu^{\mathfrak{u}}_m$ is $\kappa^{\mathfrak{u}}_m$-complete, there is a $\bar{k}$ so that $A = \{\alpha \in p(m) \setminus \mathsf{bad}_D(p) : k_\alpha = \bar{k}\} \in \mu^{\mathfrak{u}}_m$. Since AC holds, fix a wellordering $\prec$ of $\mathbb{N}_{\mathfrak{u}}$. For each $\alpha \in A$, let r_α be the $\prec$-least $r \in \mathbb{N}_{\mathfrak{u}}$ so that $r \leq_{\mathbb{N}_{\mathfrak{u}}} (p)_{\mathsf{stem}(p)\hat{\ }\langle\alpha\rangle}$ with $\mathsf{slength}(r) = m+1$ and for all $s \in^{\mathfrak{u}} r$, if $|s| = \bar{k}$, then $(r)_s \in D$. If $i < m$, let $q(i) = p(i)$. Let $q(m) = A$. For all $m < i$, let $q(i) = \bigcap\{r_\alpha(i) : \alpha \in A\}$. Note that for all $m < i$, since $|A| = \kappa^{\mathfrak{u}}_m < \kappa^{\mathfrak{u}}_i$ and $\mu^{\mathfrak{u}}_i$ is $\kappa^{\mathfrak{u}}_i$-complete, $q(i) \in \mu^{\mathfrak{u}}_i$. Thus $q \in \mathbb{N}_{\mathfrak{u}}$ and $\mathsf{slength}(q) = m$. Let $s \in^{\mathfrak{u}} q$ with $|s| = \bar{k}$. Since $q(m) = A$, $s(m) \in A$. $(q)_s \leq_{\mathbb{N}_{\mathfrak{u}}} (r_{s(m)})_s$. So $(q)_s \in D$ since D is dense open and $(r_{s(m)})_s \in D$ by the choice of $r_{s(m)}$. This shows that $\bar{k}$ and q witness that p is D-good. Contradiction. This completes the proof of the claim.

The fact is equivalent to the assertion that every $p \in \mathbb{N}_{\mathfrak{u}}$ is D-good. Suppose $p \in \mathbb{N}_{\mathfrak{u}}$ is not D-good. Let $m = \mathsf{slength}(p)$. For $j \leq m$, let $p_j = p$. Suppose $m \leq j$ and p_j has been defined so that for all $s \in^{\mathfrak{u}} p_j$ with $|s| = j$, $(p_j)_s$ is not D-good and for all $i < j$, $p_{i+1} \leq^i_{\mathfrak{u}} p_i$. (This is true for all $j \leq m$, since $p_j = p$.) Thus for all $s \in^{\mathfrak{u}} p_j$ with $|s| = j$, $\mathsf{bad}_D((p_j)_s) \in \mu^{\mathfrak{u}}_j$. If $i \neq j$, let $p_{j+1}(i) = p_j(i)$. Let $p_{j+1}(j) = \bigcap\{\mathsf{bad}_D((p_j)_s) : s \in^{\mathfrak{u}} p_i \wedge |s| = j\}$. Note that $|s| = j$ and $s \in^{u} p_j$ imply that $s(i) \in \kappa^{\mathfrak{u}}_i$ for each $i < j$. Thus $|\{s \in^{\mathfrak{u}} p_j : |s| = j\}| \leq \kappa^{\mathfrak{u}}_{j-1}$. Since $\mu^{\mathfrak{u}}_j$ is $\kappa^{\mathfrak{u}}_j$-complete, $p_{j+1}(j) \in \mu^{\mathfrak{u}}_j$. This shows that $p_{j+1} \in \mathbb{N}_{\mathfrak{u}}$. Since $p_{j+1} \upharpoonright j = p_j \upharpoonright j$, $p_{j+1} \leq^j_{\mathfrak{u}} p_j$. Suppose $s \in^{\mathfrak{u}} p_{j+1}$ with $|s| = j + 1$. $s \upharpoonright j \in^{u} p_j$ and $s(j) \in p_{j+1}(j) \subseteq \mathsf{bad}_D((p_j)_{s \upharpoonright j})$. Thus $(p_{j+1})_s$ is not D-good. This completes the construction of the $\mathfrak{u}$-fusion sequence $\langle p_n : n \in \omega\rangle$. Let p_ω be the fusion of $\langle p_n : n \in \omega\rangle$ defined by $p_\omega(i) = p_{i+1}(i)$. Since D is dense, there is a $q \leq_{\mathbb{N}_\mathfrak{u}} p_\omega$ with $q \in D$. Let $m^* = \mathsf{slength}(q)$ and $s^* = \mathsf{stem}(q)$. Since $p_\omega \leq_\mathfrak{u}^{m^*} p_{m^*}$, $s^* \in^\mathfrak{u} p_{m^*}$, and $q \leq_{\mathbb{N}_\mathfrak{u}} (p_\omega)_{s^*} = (p_{m^*})_{s^*}$, one has that q and $m^* = |s^*| = \mathsf{slength}((p_{m^*})_{s^*})$ witness that $(p_{m^*})_{s^*}$ is D-good. However, by construction of p_{m^*}, $(p_{m^*})_s$ is not D-good for any $s \in^\mathfrak{u} p_{m^*}$ with $|s| = m^*$. Contradiction. This completes the proof. □

Lemma 7.12. *Assume* AC. *Let* $\mathfrak{u}$ *be a measure sequence,* $p \in \mathbb{N}_\mathfrak{u}$*, and* φ *be a sentence in the forcing language of* $\mathbb{N}_\mathfrak{u}$*. Then exactly one of the following holds.*

1. *There is a* $q \leq_{\mathbb{N}_\mathfrak{u}} p$ *with* $\mathsf{stem}(p) = \mathsf{stem}(q)$ *and* $q \Vdash_{\mathbb{N}_\mathfrak{u}} \varphi$.
2. *There is a* $q \leq_{\mathbb{N}_\mathfrak{u}} p$ *with* $\mathsf{stem}(p) = \mathsf{stem}(q)$ *and* $q \Vdash_{\mathbb{N}_\mathfrak{u}} \neg\varphi$.

Proof. Let $D = \{p \in \mathbb{N}_\mathfrak{u} : p \Vdash_\mathbb{P} \varphi \vee p \Vdash_\mathbb{P} \neg\varphi\}$ which is a dense open subset of $\mathbb{N}_\mathfrak{u}$. By Lemma 7.11, there is a $k \in \omega$ with $\mathsf{slength}(p) \leq k < \omega$ and a $q \leq_{\mathbb{N}_\mathfrak{u}} p$ with $\mathsf{slength}(p) = \mathsf{slength}(q)$ so that for all $s \in^\mathfrak{u} q$ with $|s| = k$, $(q)_s \in D$. Let $\bar{k}$ be the least such k. Fix a $\bar{q} \leq_{\mathbb{N}_\mathfrak{u}} p$ so that $\mathsf{slength}(p) = \mathsf{slength}(\bar{q})$ and for all $s \in^\mathfrak{u} \bar{q}$ with $|s| = \bar{k}$, $(\bar{q})_s \in D$. The claim is that $\bar{k} = \mathsf{slength}(p) = \mathsf{slength}(\bar{q})$. Suppose otherwise that $\bar{k} > \mathsf{slength}(\bar{q})$. Since $\mathsf{slength}(\bar{q}) < \bar{k}$, $q(\bar{k}-1) \in \mu^\mathfrak{u}_{\bar{k}-1}$. For each $s \in^\mathfrak{u} \bar{q}$ with $|s| = \bar{k}-1$ and $\alpha \in q(\bar{k}-1)$, one has that $(\bar{q})_{s\hat{\ }\langle\alpha\rangle} \Vdash_{\mathbb{N}_\mathfrak{u}} \varphi$ or $(\bar{q})_{s\hat{\ }\langle\alpha\rangle} \Vdash_{\mathbb{N}_\mathfrak{u}} \neg\varphi$ by the property of $\bar{k}$ and $\bar{q}$. Let $A^0_s = \{\alpha \in \bar{q}(\bar{k}-1) : (q)_{s\hat{\ }\langle\alpha\rangle} \Vdash_{\mathbb{N}_\mathfrak{u}} \neg\varphi\}$ and $A^i_s = \{\alpha \in \bar{q}(\bar{k}-1) : (q)_{s\hat{\ }\langle\alpha\rangle} \Vdash_{\mathbb{N}_\mathfrak{u}} \varphi\}$. Since $A^0_s \cap A^1_s = \emptyset$ and $A^0_s \cup A^1_s = \bar{q}(\bar{k}-1) \in \mu^\mathfrak{u}_{\bar{k}-1}$, there is a unique $i_s \in 2$ so that $A^{i_s}_s \in \mu^\mathfrak{u}_{\bar{k}-1}$. Define $\underline{q}$ as follows: For $i < \bar{k}-1$, $\underline{q}(i) = \bar{q}(i)$. Let $\underline{q}(\bar{k}-1) = \bigcap\{A^{i_s}_s : s \in^\mathfrak{u} \bar{q} \wedge |s| = \bar{k}-1\}$. For $\bar{k}-1 < i < \omega$, let $\underline{q}(i) = \bar{q}(i)$. Since $\mu^\mathfrak{u}_{\bar{k}-1}$ is $\kappa^\mathfrak{u}_{\bar{k}-1}$-complete and $|\{s \in^\mathfrak{u} \bar{q} : |s| = \bar{k}-1\}| < \kappa^\mathfrak{u}_{\bar{k}-1}$, $\underline{q}(\bar{k}-1) \in \mu^\mathfrak{u}_{\bar{k}-1}$. Thus $\underline{q} \in \mathbb{N}_\mathfrak{u}$ and $\mathsf{slength}(\underline{q}) = \mathsf{slength}(p)$. Let $s \in^\mathfrak{u} \underline{q}$ with $|s| = \bar{k}-1$. Without loss of generality, suppose $i_s = 1$. One seeks to show that $(q)_s \Vdash_{\mathbb{N}_\mathfrak{u}} \varphi$. Suppose not. Then one can find an $r \leq_{\mathbb{N}_\mathfrak{u}} (\underline{q})_s$ so that $\mathsf{slength}(r) \geq \bar{k}$ and $r \Vdash_{\mathbb{N}_\mathfrak{u}} \neg\varphi$. However $r(\bar{k}-1) \in \underline{q}(\bar{k}-1) \subseteq A^1_s$ and $r \leq_{\mathbb{N}_\mathfrak{u}} (\underline{q})_{s\hat{\ }\langle r(\bar{k}-1)\rangle}$. So $(\underline{q})_{s\hat{\ }\langle r(\bar{k}-1)\rangle} \Vdash_{\mathbb{N}_\mathfrak{u}} \varphi$. Thus $r \Vdash_\mathbb{N} \varphi$. Contradiction. This means that $(\underline{q})_s \in D$. Since $s \in^\mathfrak{u} \underline{q}$ with $|s| = \bar{k}-1$ was arbitrary, this violates the minimality of $\bar{k}$. This shows that $\bar{k} = \mathsf{slength}(\bar{q})$. The only $s \in^\mathfrak{u} \bar{q}$ with $|s| = \mathsf{slength}(\bar{q})$ is $\mathsf{stem}(\bar{q})$. This means $\bar{q} = (\bar{q})_{\mathsf{stem}(q)} \in D$. Thus $\bar{q} \Vdash_{\mathbb{N}_\mathfrak{u}} \varphi$ or $\bar{q} \Vdash_{\mathbb{N}_\mathfrak{u}} \neg\varphi$. $\bar{q}$ is the desired extension of q.

This shows one of the two cases must occur. Suppose both occurred. Let $q_0, q_1 \in \mathbb{N}_\mathfrak{u}$ be such that $q_0 \leq_{\mathbb{N}_\mathfrak{u}} p$, $q_1 \leq_{\mathbb{N}_\mathfrak{u}} p$, $\mathsf{stem}(p) = \mathsf{stem}(q_0) = \mathsf{stem}(q_1)$, $q_0 \Vdash_{\mathbb{N}_\mathfrak{u}} \neg\varphi$, and $q_1 \Vdash_{\mathbb{N}_\mathfrak{u}} \varphi$. Let q_2 be defined by $q_2(i) = p(i)$ for all $i < \mathsf{slength}(p)$ and $q_2(i) = q_0(i) \cap q_1(i)$ for all $\mathsf{slength}(p) \leq i < \omega$. Then $q_2 \in \mathbb{N}_\mathfrak{u}$, $q_2 \leq_{\mathbb{N}_\mathfrak{u}} q_0$, and $q_2 \leq_{\mathbb{N}_\mathfrak{u}} q_0$. Thus $q_2 \Vdash_{\mathbb{N}_\mathfrak{u}} \varphi$ and $q_2 \Vdash_{\mathbb{N}_\mathfrak{u}} \neg\varphi$. Contradiction. □

Fact 7.13. *Assume* AC. *Let* $\mathfrak{u}$ *be a measure sequence. For any* $\delta < \lambda^\mathfrak{u}$, $1_{\mathbb{N}_\mathfrak{u}} \Vdash_{\mathbb{N}_\mathfrak{u}} \mathscr{P}(\check{\delta}) = \check{\mathscr{P}}(\delta)$. *(Thus* $\mathbb{N}_\mathfrak{u}$ *does not add any new bounded subsets of* $\lambda^\mathfrak{u}$*.)*

Proof. Using AC, let $\prec$ be a wellordering of $\mathbb{N}_\mathfrak{u}$. Let $\delta < \lambda^\mathfrak{u}$. Let $\tilde{p} \in \mathbb{N}_\mathfrak{u}$ and $\dot{A}$ be a $\mathbb{N}_\mathfrak{u}$-name such that $\tilde{p} \Vdash_{\mathbb{N}_\mathfrak{u}} \dot{A} \subseteq \check{\delta}$. Let $D_{\tilde{p},\dot{A}}$ be the set of $p \in \mathbb{N}_\mathfrak{u}$ such that the disjunction of the following holds.

1. p is incompatible with $\tilde{p}$.
2. $p \leq_{\mathbb{N}_\mathfrak{u}} \tilde{p}$ and there exists a $B \in \mathscr{P}(\delta)$ so that $p \Vdash_{\mathbb{N}_\mathfrak{u}} \dot{A} = \check{B}$.

One will show that $D_{\tilde{p},\dot{A}}$ is dense. Let $\bar{p} \in \mathbb{N}_\mathfrak{u}$ be arbitrary. If $\bar{p}$ is incompatible with $\tilde{p}$, then $\bar{p} \in D_{\tilde{p},\dot{A}}$. Now suppose $\bar{p}$ is compatible with $\tilde{p}$. Let $\hat{p}$ be such that $\hat{p} \leq_{\mathbb{N}_\mathfrak{u}} \bar{p}$ and $\hat{p} \leq_{\mathbb{N}_\mathfrak{u}} \tilde{p}$. Let $\bar{n}$ be the least n such that $\mathsf{slength}(\hat{p}) \leq n$ and $\delta < \kappa^\mathfrak{u}_n$. Fix $\eta < \delta$. By Lemma 7.12, there is a $q \leq_\mathfrak{u} \hat{p}$ with $\mathsf{slength}(q) = \bar{n}$ and an $i_\eta \in 2$ so that $q \Vdash_{\mathbb{N}_\mathfrak{u}} \dot{A}(\check{\eta}) = \check{i}_\eta$ (here $\dot{A}$ is regarded as a name for the characteristic function of a subset of δ). Let q_η be the $\prec$-least $q \in \mathbb{N}_\mathfrak{u}$ so that $\mathsf{slength}(q) = \bar{n}$, $q \leq_{\mathbb{N}_\mathfrak{u}} \hat{p}$, and $q \Vdash_{\mathbb{N}_\mathfrak{u}} \dot{A}(\check{\eta}) = \dot{i}_\eta$. Let $B = \{\eta < \delta : i_\eta = 1\}$. Define $\underline{q}$ by

$$\underline{q}(i) = \begin{cases} \hat{p}(i) & i < \bar{n} \\ \bigcap\{q_\eta(i) : \eta < \delta\} & \bar{n} \leq i \end{cases}.$$

By choice of $\bar{n}$, $\delta < \kappa^\mathfrak{u}_{\bar{n}}$. For all $i \geq \bar{n}$, $\kappa^\mathfrak{u}_i \geq \kappa^\mathfrak{u}_{\bar{n}}$ and $\mu^\mathfrak{u}_i$ is $\kappa^\mathfrak{u}_i$-complete. Therefore, $\underline{q}(i) \in \mu^\mathfrak{u}_i$ for all $\bar{n} \leq i$. This shows that $\underline{q} \in \mathbb{N}_\mathfrak{u}$, $\mathsf{stem}(\underline{q}) = \bar{n}$, and $\underline{q} \leq_{\mathbb{N}_\mathfrak{u}} q_\eta$ for each $\eta < \delta$. Since $q_\eta \Vdash_{\mathbb{N}_\mathfrak{u}} \dot{A}(\check{\eta}) = i_\eta = \check{B}(\check{\eta})$, one has that $\underline{q} \Vdash_{\mathbb{N}_\mathfrak{u}} A(\check{\eta}) = \check{B}(\check{\eta})$ for each $\eta < \delta$. Thus $\underline{q} \leq_{\mathbb{N}_\mathfrak{u}} \tilde{p}$ and $\underline{q} \Vdash_{\mathbb{N}_\mathfrak{u}} \dot{A} = \check{B}$. Thus $\underline{q} \leq_{\mathbb{N}_\mathfrak{u}} \bar{p}$ and $\underline{q} \in D_{\tilde{p},\dot{A}}$. Since $\bar{p}$ was arbitrary, this shows that $D_{\tilde{p},\dot{A}}$ is dense. Since $\tilde{p}$ and $\dot{A}$ were arbitrary with the above properties, this shows that $1_{\mathbb{N}_\mathfrak{u}} \Vdash_{\mathbb{N}_\mathfrak{u}} \mathscr{P}(\check{\delta}) = \check{\mathscr{P}}(\delta)$. □

The next lemma will provide a setting in which there is an inner model M of ZFC and a $(\mathbb{N}_\mathfrak{u})^M$-generic filter over M in the real world. Moreover, it will be shown that there are many such generic filters corresponding to each element of the body of a $\mathbb{N}_\mathfrak{u}$ condition (belonging to the real world but external to M). For later purposes, it will be stated to make the uniformity explicit.

Lemma 7.14. *Let M be an inner model of ZFC. Suppose the following holds.*

(1) $\mathfrak{u} \in M$ and $M \models$ "$\mathfrak{u}$ is a measure sequence, $2^{\lambda^\mathfrak{u}} = (\lambda^\mathfrak{u})^+$, and $2^{(\lambda^\mathfrak{u})^+} = (\lambda^\mathfrak{u})^{++}$".
(2) $\prec$ is a wellordering of all sets in M of rank less than $\lambda^\mathfrak{u} + 3$.
(3) $\tau_1 : \omega \to ((\lambda^\mathfrak{u})^+)^M$ is an increasing cofinal function.
(4) $\tau_2 : \omega \to ((\lambda^\mathfrak{u})^{++})^M$ is an increasing cofinal function.
(5) $p \in (\mathbb{N}_\mathfrak{u})^M$.

Then there is a $q \in \mathbb{N}_\mathfrak{u}$ (but $q \notin M$) such that $q \leq_{\mathbb{N}_\mathfrak{u}} p$, $\mathsf{slength}(p) = \mathsf{slength}(q)$, and for all $g \in [q]$, $G_g = \{p \in (\mathbb{N}_\mathfrak{u})^M : g \in [p]\}$ is a $(\mathbb{N}_\mathfrak{u})^M$-generic filter over M. q is produced uniformly from $\mathfrak{u}$, $\prec$, τ_1, τ_2, and p.

Proof. Since $M \models$ "$\kappa_n^\mathfrak{u}$ is measurable (and hence strong limit)" for all $n \in \omega$, $M \models$ "$\mathscr{P}(\kappa_n^\mathfrak{u}) < \kappa_{n+1}^\mathfrak{u}$". Thus $M \models$ "$|\mathbb{N}_\mathfrak{u}| \leq |\mathscr{P}(\lambda^\mathfrak{u})| = 2^{\lambda^\mathfrak{u}} = (\lambda^\mathfrak{u})^+$" by the hypothesis. Hence $M \models$ "$|\mathscr{P}(\mathbb{N}_\mathfrak{u})| \leq 2^{(\lambda^\mathfrak{u})^+} = (\lambda^\mathfrak{u})^{++}$" by the hypothesis. Let $\tau_0 : \omega \to \lambda^\mathfrak{u}$ be defined by $\tau_0(n) = \kappa_n^\mathfrak{u}$ (which is defined uniformly from $\mathfrak{u}$). By Lemma 5.4, there is a M-amenable surjection $\bar{h} : \lambda^\mathfrak{u} \to ((\lambda^\mathfrak{u})^{++})^M$ which is produced uniformly from τ_0, τ_1, τ_2, and $\prec$. Let $\mathcal{D}$ be the collection of dense open subsets of $(\mathbb{N}_\mathfrak{u})^M$. Since $M \models$ "$|\mathcal{D}| \leq |\mathscr{P}(\mathbb{N}_\mathfrak{u})| \leq (\lambda^\mathfrak{u})^{++}$, let $\hat{h} : (\lambda^\mathfrak{u})^{++} \to \mathcal{D}$ be the $\prec$-least bijection in M between $(\lambda^\mathfrak{u})^{++}$ and $\mathcal{D}$. Then $h : \lambda^\mathfrak{u} \to \mathcal{D}$ defined by $h = \hat{h} \circ \bar{h}$ is an M-amenable surjection produced uniformly from $\mathfrak{u}$, $\prec$, τ_1, and τ_2. Fix a $p \in (\mathbb{N}_\mathfrak{u})^M$. One will build an external (to M) $\mathfrak{u}$-fusion sequence $\langle p_n : n \in \omega\rangle$ with the following properties.

(1) $p_n \in (\mathbb{N}_\mathfrak{u})^M$ (that is, $p_n \in M$) .
(2) $p_0 = p_1 = p$.
(3) For all $1 \leq n < \omega$, $s \in^\mathfrak{u} p_{n+1}$ with $|s| = n$, for all $\eta < \kappa_{n-1}^\mathfrak{u}$, there is an $m \geq n$ so that for all $t \in^\mathfrak{u} (p_{n+1})_s$ with $|t| = m$, $(p_{n+1})_t \in h(\eta)$.

To construct this $\mathfrak{u}$-fusion sequence: Let $p_0 = p_1 = p$. Now suppose $1 \leq n < \omega$ and p_n has been defined. For each $s \in^\mathfrak{u} p_n$ with $|s| = n$ and $\eta < \kappa_{n-1}^\mathfrak{u}$, there is some $q \in (\mathbb{N}_\mathfrak{u})^M$ and some $n \leq m < \omega$ so that $q \leq_{\mathbb{N}_\mathfrak{u}} (p_n)_s$, $\mathsf{stem}(q) = \mathsf{stem}((p_n)_s) = s$, and for all $t \in^\mathfrak{u} q$ with $|t| = m$, $(q)_t \in h(\eta)$ by Lemma 7.11 applied in M. Let $q_s^{n,\eta} \in (\mathbb{N}_\mathfrak{u})^M$ be the $\prec$-least such q and let $m_s^{n,\eta}$ be the least such m with the above property for $q_s^{n,\eta}$. This defines $\langle q_s^{n,\eta} : s \in^\mathfrak{u} p_n \wedge |s| = n \wedge \eta < \kappa_{n-1}^\mathfrak{u}\rangle$ and this sequence belongs to M because $h \restriction \kappa_{n-1}^\mathfrak{u} \in M$ since h is M-amenable. Define p_{n+1} by

$$p_{n+1}(i) = \begin{cases} p_{n+1}(i) = p_n(i) & i < n \\ p_n(i) = \bigcap\{q_s^{n,\eta}(i) : s \in^\mathfrak{u} p_n \wedge |s| = n \wedge \eta < \kappa_{n-1}^\mathfrak{u}\} & n \leq i \end{cases}.$$

Note that for $n \leq i$, $p_{n+1}(i) \in \mu_i$ since $|\{q_s^{n,\eta} : s \in^\mathfrak{u} p_n \wedge |s| = n \wedge \eta < \kappa_{n-1}^\mathfrak{u}\}| = \kappa_{n-1}^\mathfrak{u}$, $\mu_i^\mathfrak{u}$ is $\kappa_i^\mathfrak{u}$-complete, and $\kappa_{n-1}^\mathfrak{u} < \kappa_i$. Thus $p_{n+1} \in (\mathbb{N}_\mathfrak{u})^M$ and $p_{n+1} \leq_\mathfrak{u}^n p_n$. Suppose $s \in^\mathfrak{u} p_{n+1}$, $|s| = n$, and $\eta < \kappa_{n-1}^\mathfrak{u}$. So $s \in^\mathfrak{u} p_n$. By construction, $(p_{n+1})_s \leq_{\mathbb{N}_\mathfrak{u}} q_s^{n,\eta}$. Let $t \in^\mathfrak{u} (p_{n+1})_s$ with $|t| = m_s^{n,\eta}$. Then $t \in^\mathfrak{u} q_s^{n,\eta}$ and $(q_s^{n,\eta})_t \in h(\eta)$ be the properties of $q_s^{n,\eta}$. Since $h(\eta)$ is dense open and $(p_{n+1})_t \leq_{\mathbb{N}_\mathfrak{u}} q_t^{n,\eta}$, $(p_{n+1})_t \in h(\eta)$. This shows p_{n+1} has property (3). This completes the construction of the $\mathfrak{u}$-fusion sequence $\langle p_n : n \in \omega\rangle$ with the desired properties above. (However this sequence does not belong to M.) Let p_ω be the fusion of $\langle p_n : n \in \omega\rangle$ defined by $p_\omega(i) = p_{i+1}(i)$ which belongs to $\mathbb{N}_\mathfrak{u} \setminus (\mathbb{N}_\mathfrak{u})^M$ (is a $\mathfrak{u}$-Namba condition but not a member of M). Note that p_ω is defined uniformly from $\mathfrak{u}$, p, h, and $\prec$ and thus uniformly from $\mathfrak{u}$, p, $\prec$, τ_1, and τ_2. Let $g \in [p]$. $G_g = \{p \in (\mathbb{N}_\mathfrak{u})^M : g \in [p]\}$ is a $(\mathbb{N}_\mathfrak{u})^M$-filter. Let $D \in M$ be a dense open subset of $(\mathbb{N}_\mathfrak{u})^M$. Since $h : \lambda^\mathfrak{u} \to \mathcal{D}$ is a surjection, there is an $\eta < \lambda^\mathfrak{u}$ so that $h(\eta) = D$. Let $1 \leq n < \omega$ be such that $\eta < \kappa_{n-1}^\mathfrak{u}$. Since $g \restriction n \in^\mathfrak{u} p_\omega$, $g \restriction n \in^\mathfrak{u} p_{n+1}$. By the property of p_{n+1}, there is an $m \geq n$ so that for all $t \in^\mathfrak{u} (p_{n+1})_{g\restriction n}$ with $|t| = m$, $(p_{n+1})_t \in h(\eta)$. Thus $(p_{n+1})_{g\restriction m} \in h(\eta) = D$. Since $p_\omega \leq_{\mathbb{N}_\mathfrak{u}} p_{n+1}$, $g \restriction k \in (p_{n+1})_{g\restriction m}$ for all $k \in \omega$. Thus $g \in [(p_{n+1})_{g\restriction m}]$. By definition of G_g, $(p_{n+1})_{g\restriction m} \in G_g$. Thus $(p_{n+1})_{g\restriction m} \in G_g \cap D$. It has been shown that G_g is $(\mathbb{N}_\mathfrak{u})^M$-generic over M. ☐

In particular, one has the following less uniform version.

Lemma 7.15. *Assume* AD. *Let* $M \models \mathsf{ZFC}$ *be an inner model. Let* $\mathfrak{u} \in M$ *be such that* $\lambda^{\mathfrak{u}} < \Theta$, $M \models$ *"*$\mathfrak{u}$ *is a measure sequence,* $2^{\lambda^{\mathfrak{u}}} = (\lambda^{\mathfrak{u}})^+$, $2^{(\lambda^{\mathfrak{u}})^+} = (\lambda^{\mathfrak{u}})^{++}$*", and* $\mathrm{cof}(((\lambda^{\mathfrak{u}})^+)^M) = \mathrm{cof}(((\lambda^{\mathfrak{u}})^{++})^M) = \omega$. *For all* $p \in (\mathbb{N}_{\mathfrak{u}})^M$*, there is a* $q \in \mathbb{N}_{\mathfrak{u}}$ *such that* $q \leq_{\mathbb{N}_{\mathfrak{u}}} p$, $\mathsf{slength}(p) = \mathsf{slength}(q)$*, and for all* $g \in [q]$, G_g *is* $(\mathbb{N}_{\mathfrak{u}})^M$*-generic filter over* M.

The following result shows there is an abundance of measurable cardinals of HOD below the true cardinals greater than ω_1. (ω_1 is the first measurable cardinal of HOD.)

Fact 7.16. *(Jackson, Ketchersid, Schlutzenberg, Woodin;* [27] *Lemma 2.7*[20]*) Assume* AD *and* $V = L(\mathbb{R})$. *If* $x \in \mathbb{R}$ *and* κ *is a cardinal such that* $\omega_1 < \kappa < \Theta$*, then* $\mathrm{HOD}_x \models$ *"there are* κ*-many measurable cardinals below* κ*".*

Let κ be a cardinal. Recall $B(\omega,\kappa) = \{f \in {}^{\omega}\kappa : \sup(f) < \kappa\}$. If $\mathrm{cof}(\kappa) > \omega$, then $B(\omega,\kappa) = {}^{\omega}\kappa$. If $\mathrm{cof}(\kappa) = \omega$, then $|B(\omega,\kappa)| < |{}^{\omega}\kappa|$ by Fact 6.11.

Theorem 7.17. *Assume* AD. $L(\mathbb{R}) \models$ *"For all cardinals* κ *with* $2 \leq \kappa < \Theta$ *and all* $\chi < \kappa$, $B(\omega,\kappa)$ *does not inject into* $\mathscr{P}(\chi) \times \mathrm{ON}$*". In particular,* $L(\mathbb{R}) \models$ *"For all cardinals* κ *with* $2 \leq \kappa < \Theta$ *and all* $\chi < \kappa$, ${}^{\omega}\kappa$ *does not inject into* $\mathscr{P}(\chi) \times \mathrm{ON}$*".*

Proof. By going into the inner model $L(\mathbb{R})$, one may assume $V = L(\mathbb{R})$. Suppose the result fails. There are $2 \leq \kappa < \Theta$, $\chi < \kappa$, and an injection $\Phi : B(\omega,\kappa) \to \mathscr{P}(\chi) \times \mathrm{ON}$. Since $V = L(\mathbb{R})$, there is an $x \in \mathbb{R}$ so that Φ is $\mathrm{OD}_{\{x\}}$. First, $2 \leq \kappa \leq \omega$ is not possible. This is because $|B(\omega,\kappa)| = |\mathbb{R}|$ and is hence not wellorderable. However, $\mathscr{P}(\chi) \times \mathrm{ON}$ is wellorderable for all $\chi < \kappa \leq \omega$. Thus Φ cannot be an injection. Next, $\kappa = \omega_1$ cannot be possible. This follows from Fact 2.10 or Theorem 3.5.[21] Thus $\kappa > \omega_1$. By Fact 7.16, $\mathrm{HOD}_{\{x\}} \models$ "there are κ-many measurable cardinals below κ". Let $\vec{\kappa} = \langle \kappa_n : n \in \omega \rangle$ be the first ω-many measurable cardinals between χ and κ. Let $\lambda = \sup\{\kappa_n : n \in \omega\}$ and note that $\chi < \kappa_0 < \lambda < \kappa$ since $\kappa > \omega$. Let $\vec{\mu} = \langle \mu_n : n \in \omega \rangle \in \mathrm{HOD}_{\{x\}}$ be such that for each $n \in \omega$, μ_n is a κ_n-complete uniform normal measure on κ_n. Let $\mathfrak{u}$ be the measure sequence so that $\kappa_n^{\mathfrak{u}} = \kappa_n$, $\mu_n^{\mathfrak{u}} = \mu_n$, and $\lambda^{\mathfrak{u}} = \lambda$. Let $\mathbb{N}_{\mathfrak{u}}$ denote the collection of all $\mathfrak{u}$-Namba conditions and $(\mathbb{N}_{\mathfrak{u}})^{\mathrm{HOD}_{\{x\}}}$ be the $\mathfrak{u}$-Namba conditions that belong to $\mathrm{HOD}_{\{x\}}$. By Fact 6.1, the inner model $\mathrm{HOD}_{\{x\}}$ satisfies the hypothesis of Lemma 7.14 or Lemma 7.15. Thus by applying Lemma 7.14 or even the less uniform Lemma 7.15, there is a $p \leq_{\mathbb{N}_{\mathfrak{u}}} 1_{\mathbb{N}}$ such that for any $g \in [p]$, G_g is $(\mathbb{N}_{\mathfrak{u}})^{\mathrm{HOD}_{\{x\}}}$-generic over $\mathrm{HOD}_{\{x\}}$. Fix a $g \in [p]$. Note that $g = f_{G_g}$. Since $\lambda < \Theta$ and $g \in {}^{\omega}\lambda$, $\mathrm{HOD}_{\{x\}}[g] = \mathrm{HOD}_{\{x,g\}}$ by Fact 4.19[22] Moreover, since $\lambda < \kappa$, $g \in B(\omega,\kappa)$. Since $\Phi \in \mathrm{OD}_{\{x\}}$, $\Phi(g) \in \mathrm{OD}_{\{x,g\}} \cap (\mathscr{P}(\chi) \times \mathrm{ON})$. Thus $\Phi(g) \in \mathrm{HOD}_{\{x,g\}} \cap (\mathscr{P}(\chi) \times \mathrm{ON}) = \mathrm{HOD}_{\{x\}}[g] \cap (\mathscr{P}(\chi) \times \mathrm{ON}) \subseteq \mathrm{HOD}_{\{x\}}[G_g] \cap (\mathscr{P}(\chi) \times \mathrm{ON})$. Since $(\mathbb{N}_{\vec{\mu}})^{\mathrm{HOD}_{\{x\}}}$ does not add any new bounded subsets of λ by Fact 7.13 and $\chi < \lambda$, $\mathrm{HOD}_{\{x\}}[G_g] \cap (\mathscr{P}(\chi) \times \mathrm{ON}) = \mathrm{HOD}_{\{x\}} \cap (\mathscr{P}(\chi) \times \mathrm{ON})$. Thus $\Phi(g) \in \mathrm{HOD}_{\{x\}}$. Since $\Phi(g) \in \mathrm{OD}_{\{x\}}$ and $\Phi^{-1} \in \mathrm{OD}_{\{x\}}$, one has that $g = \Phi^{-1}(\Phi(g)) \in \mathrm{OD}_{\{x\}}$. Thus $g \in \mathrm{HOD}_{\{x\}}$. This contradicts Fact 7.10 which asserts that $g = f_{G_g} \notin \mathrm{HOD}_{\{x\}}$. □

Corollary 7.18. *Assume* AD. $L(\mathbb{R}) \models$ *"For all cardinals* κ *with* $\omega \leq \kappa < \Theta$*, all* $\chi < \kappa$*, all* $\omega \leq \epsilon \leq \kappa$, $\nu \leq \chi$*, and* $\zeta \in \mathrm{ON}$, $\neg(|[\kappa]^{\epsilon}| \leq |[\chi]^{\nu} \times \zeta|)$*".*

Theorem 7.19. *Assume* AD^+. *For all cardinals* κ *with* $2 \leq \kappa < \Theta$ *and all* $\chi < \kappa$, $B(\omega,\kappa)$ *does not inject into* $\mathscr{P}(\chi) \times \mathrm{ON}$. *In particular, for all cardinals* κ *with* $2 \leq \kappa < \Theta$ *and all* $\chi < \kappa$, ${}^{\omega}\kappa$ *does not inject into* $\mathscr{P}(\chi) \times \mathrm{ON}$.

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[20]As a corollary of Fact 7.16, every uncountable cardinal in HOD_x is either measurable or a limit of measurable cardinals of HOD_x. [27] showed that this corollary is sufficient for showing every cardinal is Jónsson in $L(\mathbb{R}) \models \mathsf{AD}$. However, since $B(\omega,\kappa) \neq {}^{\omega}\kappa$, for the main result of this section concerning $B(\omega,\kappa)$, one needs the stronger form due to Schlutzenberg in the cofinality ω case.

[21]The case $\kappa = \omega_1$ needs to be considered separately. No ordinal countable in the real world can be measurable in $\mathrm{HOD}_{\{x\}}$ by [60] Lemma 8.25. Thus there is no countable ordinal λ which is a limit of measurable cardinals in $\mathrm{HOD}_{\{x\}}$. The generalized Namba forcings considered in this section cannot be used. However, as in the proof of Theorem 3.5, one can use the classical Namba forcing defined in $\mathrm{HOD}_{\{x\}}$ of trees on $(\omega_2)^{\mathrm{HOD}_{\{x\}}}$. Note the existence of generic filters for this Namba forcing over $\mathrm{HOD}_{\{x\}}$ is given by Fact 2.14 since this forcing is countable in the real world.

[22]The generic object of $(\mathbb{N}_{\vec{\mu}})^{\mathrm{HOD}_{\{x\}}}$ is an ω-sequence through $\lambda < \Theta$. Although the generic filter existence problem is more difficult here than for Theorem 6.2, the capturing problem has a much simpler solution using Fact 4.19.

Proof. The proof of this in relation to the proof of Theorem 7.17 is analogous to the proof of Theorem 6.7 in relation to the proof of Theorem 6.2. An initial Σ_1-reflection was not used in the proof of Theorem 7.17. Here Σ_1-reflection into the Suslin-coSuslin sets (Fact 6.5) is still needed to bring a counterexample into a nice model of the form $L(Z,\mathbb{R})$ as in the proof of Theorem 6.7 so that $\mathrm{HOD}_Z^{L(Z,\mathbb{R})}$ possesses a direct system HOD-analysis. With such a HOD-analysis, $\mathrm{HOD}_Z^{L(Z,\mathbb{R})}$ will satisfy the analog of Fact 7.16. The proof then proceeds much like the proof of Theorem 7.17. ☐

Corollary 7.20. *Assume* AD^+*. For all cardinals* κ *with* $\omega \leq \kappa < \Theta$*, all* $\chi < \kappa$*, all* $\omega \leq \epsilon \leq \kappa$*,* $\nu \leq \chi$*, and* $\zeta \in \mathrm{ON}$*,* $\neg(|[\kappa]^\epsilon| \leq |[\chi]^\nu \times \zeta|)$*.*

Lemma 7.14 shows that for suitable inner models M, there is a $p \in \mathbb{N}_\mathfrak{u}$ (which does not belong to M) so that every $g \in [p]$ induces a $(\mathbb{N}_\mathfrak{u})^M$-generic filter over M. The next result computes the cardinality of $[p]$, the body of p.

Fact 7.21. *Let* $M \models \mathsf{ZFC}$ *be an inner model. Let* $\mathfrak{u} \in M$ *be such that* $M \models$ *"*$\mathfrak{u}$ *is a measure sequence"'. Let* $p \in \mathbb{N}_\mathfrak{u}$ *(which may not belong to* M*). Then* $|[p]| = |{}^\omega(\lambda^\mathfrak{u})|$*.*

Proof. It is clear that $[p] \subseteq {}^\omega(\lambda^\mathfrak{u})$ and thus $|[p]| \leq |{}^\omega(\lambda^\mathfrak{u})|$. For each $f \in {}^\omega(\lambda^\mathfrak{u})$, define $\iota_f : \omega \to \omega$ by induction as follows: Let $\iota_f(0)$ be the least $n \in \omega$ so that $\mathsf{slength}(p) \leq n$ and $f(0) < \kappa_n^\mathfrak{u}$. Suppose $\iota_f(i)$ has been defined. Let $\iota_f(i+1)$ be the least n greater than $\iota_f(i)$ so that $f(i+1) < \kappa_n^\mathfrak{u}$. For each $n \in \omega$ with $\mathsf{slength}(p) \leq n < \omega$, let $\mathsf{enum}_{p(n)} : \kappa_n^\mathfrak{u} \to p(n)$ be the increasing enumeration of $p(n)$. Define $\Phi(f) \in [p]$ by

$$\Phi(f)(n) = \begin{cases} \mathsf{enum}_{p(n)}(0) & n \notin \iota_f[\omega] \\ \mathsf{enum}_{p(n)}(1 + f(k)) & n \in \iota_f[\omega] \wedge n = \iota_f(k) \end{cases}.$$

$\Phi : {}^\omega(\lambda^\mathfrak{u}) \to [p]$ is an injection and thus $|{}^\omega(\lambda^\mathfrak{u})| \leq |[p]|$. Thus $|{}^\omega(\lambda^\mathfrak{u})| = |[p]|$. ☐

Fact 7.22. *Suppose* κ *is a cardinal,* $1 \leq \epsilon < \kappa$*, and* κ *does not inject into* $\mathscr{P}(\delta)$ *for any* $\delta < \kappa$*. Then* $B(\epsilon,\kappa)$ *does not have* $\mathrm{cof}(\kappa)$*-regular cardinality.*

Proof. Let $\rho : \mathrm{cof}(\kappa) \to \kappa$ be an increasing cofinal sequence. By definition of $B(\epsilon,\kappa)$, $\sup(f) < \kappa$ for all $f \in B(\epsilon,\kappa)$. Let $\Phi : B(\epsilon,\kappa) \to \mathrm{cof}(\kappa)$ be defined by $\Phi(f)$ is the least α so that $\sup(f) < \rho(\alpha)$. For each $\alpha < \mathrm{cof}(\kappa)$, $|\Phi^{-1}[\{\alpha\}]| < |B(\epsilon,\kappa)|$. Since otherwise, κ would inject into $\mathscr{P}(\max\{\epsilon,\rho(\alpha)\})$ since $|\kappa| \leq |B(\epsilon,\kappa)| = |\Phi^{-1}[\{\alpha\}]| \leq |{}^\epsilon\rho(\alpha)| \leq |\mathscr{P}(\max\{\epsilon,\rho(\alpha)\})|$ which violates the hypothesis since $\max\{\epsilon,\rho(\alpha)\} < \kappa$. Φ witnesses that $B(\epsilon,\kappa)$ does not have $\mathrm{cof}(\kappa)$-regular cardinality. ☐

Corollary 7.23. *Assume* κ *is a cardinal,* $1 \leq \epsilon < \kappa$*,* κ *does not inject into* $\mathscr{P}(\delta)$ *for any* $\delta < \kappa$*, and* $\mathrm{cof}(\kappa) > \epsilon$*. Then* ${}^\epsilon\kappa$ *does not have* $\mathrm{cof}(\kappa)$*-regular cardinality. In particular, if* $\mathrm{cof}(\kappa) > \omega$*,* ${}^\omega\kappa$ *does not have* $\mathrm{cof}(\kappa)$*-regular cardinality.*

Proof. If $\mathrm{cof}(\kappa) > \epsilon$, then ${}^\epsilon\kappa = B(\epsilon,\kappa)$. The result follows from Fact 7.22. ☐

Under AD^+, $B(\omega,\kappa)$ does not have ω-regular cardinality if $\kappa < \Theta$ and $\mathrm{cof}(\kappa) = \omega$. However, under AD^+, it will be shown that if $\kappa < \Theta$ and $\mathrm{cof}(\kappa) = \omega$, then ${}^\omega\kappa$ has a much greater degree of ordinal regularity.

Lemma 7.24. *Let* $M \models \mathsf{ZFC}$ *be an inner model. Let* $\mathfrak{u} \in M$ *be such that* $\lambda^\mathfrak{u} < \Theta$*,* $M \models$ *"*$\mathfrak{u}$ *is a measure sequence,* $2^{\lambda^\mathfrak{u}} = (\lambda^\mathfrak{u})^+$*,* $2^{(\lambda^\mathfrak{u})^+} = (\lambda^\mathfrak{u})^{++}$*", and* $\mathrm{cof}(((\lambda^\mathfrak{u})^+)^M) = \mathrm{cof}(((\lambda^\mathfrak{u})^{++})^M) = \omega$*. Let* $R \subseteq {}^\omega(\lambda^\mathfrak{u}) \times \mathrm{ON}$ *be such that there is an* ∞*-Borel code* (S,φ) *for* R *with* $S \in M$*. Let* $p \in (\mathbb{N}_\mathfrak{u})^M$ *be such that* $[p] \subseteq \mathrm{dom}(R)$ *(here* $[p]$ *is computed in the real world).*

(1) *There is a* $\gamma \in \mathrm{ON}$ *and a* $q \in (\mathbb{N}_\mathfrak{u})^M$ *with* $q \leq_{\mathbb{N}_\mathfrak{u}} p$ *so that* $M \models q \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S},\dot{f}_{\dot{G}}] \models \varphi(\check{S},\dot{f}_{\dot{G}},\check{\gamma})$*.*
(2) *There is a unique* $\gamma \in \mathrm{ON}$ *so that there is a* $q \in (\mathbb{N}_\mathfrak{u})^M$ *with* $q \leq_{\mathbb{N}_\mathfrak{u}} p$*,* $\mathsf{slength}(q) = \mathsf{slength}(p)$*, and* $M \models q \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S},\dot{f}_{\dot{G}}] \models \varphi(\check{S},\dot{f}_{\dot{G}},\check{\gamma})$*.*

Proof. To see statement (1): The first claim is that there exists $q_0 \in (\mathbb{N}_\mathfrak{u})^M$ with $q_0 \leq_{\mathbb{N}_\mathfrak{u}} p$ and $M \models q_0 \Vdash_{\mathbb{N}_\mathfrak{u}} (\exists\beta \in \mathrm{ON})(L[\check{S},\dot{f}_{\dot{G}}] \models \varphi(\check{S},\dot{f}_{\dot{G}},\beta))$.

To see this claim, suppose not. Then there is an $r \in (\mathbb{N}_\mathfrak{u})^M$ with $r \leq_{\mathbb{N}_\mathfrak{u}} p$ so that $r \Vdash_{\mathbb{N}_\mathfrak{u}} (\forall\beta \in \mathrm{ON})(L[\check{S},\dot{f}_{\dot{G}}] \models \neg\varphi(\check{S},\dot{f}_{\dot{G}},\beta))$. Applying Lemma 7.14 to the inner model M and the condition $r \in (\mathbb{N}_\mathfrak{u})^M$ to obtain a condition $\bar{r} \in \mathbb{N}_\mathfrak{u}$ (but not in M) so that $\bar{r} \leq_{\mathbb{N}_\mathfrak{u}} r$ and for all $g \in [\bar{r}]$, G_g is $(\mathbb{N}_\mathfrak{u})^M$-generic over

M.[23] Pick a $g \in [\bar{r}]$. Since $g \in [\bar{r}] \subseteq [r] \subseteq [p] \subseteq \mathrm{dom}(R)$, there is a $\gamma \in \mathrm{ON}$ so that $R(g,\gamma)$. However, since $g \in [\bar{r}] \subseteq [r]$ implies $r \in G_g$ and $\dot{f}_{\dot{G}}[G_g] = g$, one has by the forcing theorem that $M[G_g] \models (\forall \beta \in \mathrm{ON}) L[S,g] \models \neg\varphi(S,g,\gamma)$. Thus $M[G] \models L[S,g] \models \neg\varphi(S,g,\gamma)$. By absoluteness, $L[S,g] \models \neg\varphi(S,g,\gamma)$. Since (S,φ) is the ∞-Borel code for R, $\neg R(g,\gamma)$. Contradiction.

Fix q_0 as in the claim above. There is $q_1 \in (\mathbb{N}_\mathfrak{u})^M$ and $\gamma \in \mathrm{ON}$ so that $q_1 \leq_{\mathbb{N}_\mathfrak{u}} q_0$ and $M \models q_1 \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\gamma})$. This establishes the first statement. Now to prove the second statement. Work in M. Let D be the set of $q \in \mathbb{N}_\mathfrak{u}$ so that one of the following holds:

(1) q is incompatible with p.
(2) There exists a $\beta \in \mathrm{ON}$ such that $q \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\beta})$.

By the first statement just proved above, D is a dense open subset of $\mathbb{N}_\mathfrak{u}$. By lemma 7.11, there is a $\hat{p} \leq_{\mathbb{N}_\mathfrak{u}} p$ with $\mathsf{slength}(p) = \mathsf{slength}(\hat{p})$ and a $\mathsf{slength}(p) \leq k < \omega$ so that for all $s \in^\mathfrak{u} \hat{p}$ with $|s| = k$, $(\hat{p})_s \in D$. Since $\hat{p} \leq_{\mathbb{N}_\mathfrak{u}} p$, the first condition above cannot hold. Thus for each $s \in^\mathfrak{u} \hat{p}$ with $|s| = k$, there is a $\beta_s \in \mathrm{ON}$ so that $(\hat{p})_s \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\beta}_s)$. By Lemma 7.12, for each $s \in^\mathfrak{u} \hat{p}$ with $|s| = k$, exactly one of the following holds:

(a) There is a $q \leq_{\mathbb{N}_\mathfrak{u}} \hat{p}$ with $\mathsf{slength}(q) = \mathsf{slength}(\hat{p}) = \mathsf{slength}(p)$ and $q \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\beta}_s)$.
(b) There is a $q \leq_{\mathbb{N}_\mathfrak{u}} \hat{p}$ with $\mathsf{slength}(q) = \mathsf{slength}(\hat{p}) = \mathsf{slength}(p)$ and $q \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \neg\varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\beta}_s)$.

The claim is that there is an $s \in^\mathfrak{u} \hat{p}$ with $|s| = k$ so that Case (a) holds for s. Since $M \models \mathsf{AC}$, let $\prec$ be a wellordering of $\mathbb{N}_\mathfrak{u}$ in M. Suppose for all $s \in^\mathfrak{u} \hat{p}$, Case (b) holds. Let q_s be the $\prec$-least such $q \in \mathbb{N}_\mathfrak{u}$ which witnesses Case (b) for s. Define $\bar{q}$ by $\hat{p}(i)$ for all $i < k$ and $\bar{q}(i) = \bigcap\{q_s(i) : s \in^\mathfrak{u} \hat{p} \wedge |s| = k\}$ if $k \leq i$. Since $|\{s \in^\mathfrak{u} \hat{p} : |s| = k\}| < |\kappa^\mathfrak{u}_k|$, $\mu^\mathfrak{u}_i$ is $\kappa^\mathfrak{u}_i$-complete, and $\kappa^\mathfrak{u}_k \leq \kappa^\mathfrak{u}_i$ when $k \leq i$, one has that $\bar{q} \in \mathbb{N}_\mathfrak{u}$ and $\bar{q} \leq_{\mathbb{N}_\mathfrak{u}} q_s$ for all $s \in^\mathfrak{u} \hat{p}$ with $|s| = k$.

Now return to the real world. By Lemma 7.15, there is a $\tilde{q} \in \mathbb{N}_\mathfrak{u}$ with $\tilde{q} \leq_{\mathbb{N}_\mathfrak{u}} \bar{q}$ so that $\mathsf{slength}(\tilde{q}) = \mathsf{slength}(\bar{q}) = \mathsf{slength}(p)$ and for all $g \in [\tilde{q}]$, G_g is $(\mathbb{N}_\mathfrak{u})^M$-generic over M. Pick any $g \in [\tilde{q}]$. For any $s \in^\mathfrak{u} \hat{p}$ with $|s| = k$, $g \in [\tilde{q}] \subseteq [\bar{q}] \subseteq [q_s]$ and thus $q_s \in G_g$. For each $s \in^\mathfrak{u} \hat{p}$ with $|s| = k$, $M \models q_s \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \neg\varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\beta}_s)$. In particular, since $g \upharpoonright k \in^\mathfrak{u} \hat{p}$ and $|g \upharpoonright k| = k$, $M \models q_{g\upharpoonright k} \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \neg\varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\beta}_{g\upharpoonright k})$. Also $g \in [\tilde{q}] \subseteq [\bar{q}] \subseteq [\hat{p}]$. In particular, $g \in [(\hat{p})_{g\upharpoonright k}]$ and thus $(\hat{p})_{g\upharpoonright k} \in G_g$. Since $g \upharpoonright k \in^\mathfrak{u} \hat{p}$, $|g \upharpoonright k| = k$, and by the definition of $\beta_{g\upharpoonright k}$, $M \models (\hat{p})_{g\upharpoonright k} \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\beta}_{g\upharpoonright k})$. It has been shown that $M \models$ "$q_{g\upharpoonright k} \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \neg\varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\beta}_{g\upharpoonright k})$", $M \models$ "$(\hat{p})_{g\upharpoonright k} \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\beta}_{g\upharpoonright k})$", and $q_{g\upharpoonright k}, (\hat{p})_{g\upharpoonright k} \in G$. Since G is a filter, there is an $r \in (\mathbb{N}_\mathfrak{u})^M$ with $r \leq_{\mathbb{N}_\mathfrak{u}} q_{g\upharpoonright k}$ and $r \leq_{\mathbb{N}_\mathfrak{u}} (\hat{p})_{g\upharpoonright k}$. Then $r \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\beta}_{g\upharpoonright k})$ and $r \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \neg\varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\beta}_{g\upharpoonright k})$. Contradiction. This establishes the claim. By the claim, there is an $s \in^\mathfrak{u} \hat{p}$ with $|s| = k$ and a $\underline{q} \leq_{\mathbb{N}_\mathfrak{u}} \hat{p}$ with $\mathsf{slength}(\underline{q}) = \mathsf{slength}(p)$ so that $M \models \underline{q} \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\beta}_s)$. This $\underline{q}$ and ordinal β_s verify the second statement. The uniqueness of γ follows from the fact that if $q_0, q_1 \in (\mathbb{N}_\mathfrak{u})^M$ with $\mathsf{slength}(q_0) = \mathsf{slength}(q_1)$, then q_0 and q_1 are $\leq_{\mathbb{N}_\mathfrak{u}}$-compatible via the condition q_2 defined by $q_2(i) = q_0(i) = q_1(i)$ when $i < \mathsf{slength}(q_0) = \mathsf{slength}(q_1)$ and $q_2(i) = q_0(i) \cap q_1(i)$ when $\mathsf{slength}(q_0) = \mathsf{slength}(q_1) \leq i < \omega$. □

Lemma 7.24 statement (1) is sufficient for the following results concerning regularity; however, it is prudent to have the stronger statement (2) which does not change the stem for other constructions. The proof of statement (2) uses statement (1). The proof of statement (1) has an essential use of the ∞-Borel code.

Lemma 7.24 asserts the existence of an internal object $q \in M$ with the desired property and thus uniformity is not relevant. Uniformity in the next lemma will be important.

Lemma 7.25. *Let M be an inner model of* ZFC*. Suppose the following holds.*

(a) $\mathfrak{u} \in M$ and $M \models$ "$\mathfrak{u}$ is a measure sequence, $2^{\lambda^\mathfrak{u}} = (\lambda^\mathfrak{u})^+$, and $2^{(\lambda^\mathfrak{u})^+} = (\lambda^\mathfrak{u})^{++}$".
(b) $\prec$ is a wellordering of all sets in M of rank less than $\lambda^\mathfrak{u} + 3$.
(c) $\tau_1 : \omega \to ((\lambda^\mathfrak{u})^+)^M$ is an increasing cofinal function.
(d) $\tau_2 : \omega \to ((\lambda^\mathfrak{u})^{++})^M$ is an increasing cofinal function.
(e) $p \in (\mathbb{N}_\mathfrak{u})^M$.

[23] When applying Lemma 7.14, choose any τ_1 and τ_2 witnessing $\mathrm{cof}(((\lambda^\mathfrak{u})^+)^M) = \omega$ and $\mathrm{cof}(((\lambda^\mathfrak{u})^{++})^M) = \omega$. Since $M \models \mathsf{AC}$, let $\prec$ be any internal wellordering of the element of M of rank less than $\lambda^\mathfrak{u} + 3$. Uniformity is not relevant in this lemma.

(f) (S,φ) *is an* ∞*-Borel code with* $S \in M$*. If* $R \subseteq {}^\omega(\lambda^\mathfrak{u}) \times \mathrm{ON}$ *is the* ∞*-Borel subset of* ${}^\omega(\lambda^\mathfrak{u}) \times \mathrm{ON}$ *defined by* (S,φ)*, then* $[p] \subseteq \mathrm{dom}(R)$*.*

Then there is a $\gamma \in \mathrm{ON}$ *and* $q \in \mathbb{N}_\mathfrak{u}$ *(but* $q \notin M$*) with* $q \leq_{\mathbb{N}_\mathfrak{u}} p$*,* $\mathsf{slength}(q) = \mathsf{slength}(p)$*, and for all* $g \in [q]$*,* $R(g,\gamma)$*. Also* q *is produced uniformly from* $\mathfrak{u}$*,* $\prec$*,* τ_1*,* τ_2*,* p*, and* (S,φ)*.*

Proof. By Lemma 7.24, there is a unique $\gamma \in \mathrm{ON}$ such that there is a $q' \in (\mathbb{N}_\mathfrak{u})^M$ with $q' \leq_{\mathbb{N}_\mathfrak{u}} p$, $\mathsf{slength}(q') = \mathsf{slength}(p)$, and $M \models q' \Vdash_{\mathbb{N}_\mathfrak{u}} L[\check{S}, \dot{f}_{\dot{G}}] \models \varphi(\check{S}, \dot{f}_{\dot{G}}, \check{\gamma})$. Let $\bar{q}$ be the $\prec$-least such q' witnessing this property for γ. Note that γ and $\bar{q}$ are obtained uniformly from $\mathfrak{u}$, $\prec$, and (S,φ). By Lemma 7.14, there is a $q \leq_{\mathbb{N}} \bar{q}$ with $q \leq_{\mathbb{N}_\mathfrak{u}} p$, $\mathsf{slength}(q) = \mathsf{slength}(\bar{q})$, and for all $g \in [q]$, G_g is $(\mathbb{N}_\mathfrak{u})^M$-generic over M and this q is produced uniformly from $\mathfrak{u}$, $\prec$, τ_1, and τ_2, and $\bar{q}$. Thus q is produced uniformly from $\mathfrak{u}$, $\prec$, τ_1, τ_2, p, and (S,φ). Pick any $g \in [q]$. Since $g \in [q] \subseteq [\bar{q}]$, $\bar{q} \in G_g$. By the forcing theorem, $M[G_g] \models L[S,g] \models \varphi(S,g,\gamma)$. By absoluteness, $L[S,g] \models \varphi(S,g,\gamma)$. Since (S,φ) is the ∞-Borel code for R, $R(g,\gamma)$. Since $g \in [q]$ was arbitrary, the result follows. ☐

Jackson, Trang, and the author ([17]) can show that ${}^\omega(\omega_\omega)$ has ω_ω-regular cardinality under AD using Martin's ultrapower analysis for ω_ω and that ${}^\omega(\omega_\omega)$ has ON-regular cardinality under AD and $\mathsf{DC}_\mathbb{R}$ using Steel's result concerning Suslin bounded prewellordering. The next theorem shows that under AD^+, for all $\kappa < \Theta$ with $\mathrm{cof}(\kappa) = \omega$, ${}^\omega\kappa$ has ON-regular cardinality.

Theorem 7.26. *Assume* AD *and* $V = L(\mathbb{R})$*. Let* $\omega \leq \kappa < \Theta$ *be a cardinal with* $\mathrm{cof}(\kappa) = \omega$*. Then* ${}^\omega\kappa$ *has* ON*-regular cardinality.*

Proof. First, consider $\kappa = \omega$. $|{}^\omega\omega| = |\mathbb{R}|$. Fact 2.5 states that wellordered unions of meager sets are meager. So for any function $\Phi : \mathbb{R} \to \mathrm{ON}$, there is some $\beta \in \mathrm{ON}$ so that $\Phi^{-1}[\{\beta\}]$ is nonmeager. Then $|\Phi^{-1}[\{\beta\}]| = |\mathbb{R}|$.

Now suppose $\omega < \kappa < \Theta$ and $\mathrm{cof}(\kappa) = \omega$. Since $\mathsf{AC}_\omega^\mathbb{R}$ implies ω_1 is regular and $\mathrm{cof}(\kappa) = \omega$, one must have that $\kappa > \omega_1$. Let $\rho : \omega \to \kappa$ be an increasing cofinal sequence. Suppose $\Phi : {}^\omega\kappa \to \mathrm{ON}$ is a function. Since $\kappa < \Theta$, there is a surjection $\pi : \mathbb{R} \to \kappa$. Define $\varsigma : \mathbb{R} \to {}^\omega\kappa$ by $\varsigma(x)(n) = \pi(x^{[n]})$ (where recall that $\langle x^{[n]} : n \in \omega\rangle$ is the recursive function which codes an ω-sequence in $\mathbb{R}$ as an element of $\mathbb{R}$). ς is a surjection. So $\mathbb{R}$ surjects onto $\Phi[{}^\omega\kappa]$ and hence $\mathrm{ot}(\Phi[{}^\omega\kappa]) < \Theta$. By composing Φ with the Mostowski collapse of $\Phi[{}^\omega\kappa]$ if necessary, one will assume without loss of generality that $\Phi : {}^\omega\kappa \to \zeta$ where $\zeta < \Theta$. Since $V = L(\mathbb{R})$, there is an $x \in \mathbb{R}$ so that Φ and ρ are $\mathrm{OD}_{\{x\}}$. Thus $\rho \in \mathrm{HOD}_{\{x\}}$ and $\mathrm{HOD}_{\{x\}} \models \mathrm{cof}(\kappa) = \omega$. Let $R = \{(f,\alpha) \in {}^\omega\kappa \times \zeta : \Phi(f) = \alpha\}$. Note that $\mathrm{dom}(R) = {}^\omega\kappa$ since Φ is a function on ${}^\omega\kappa$. R is $\mathrm{OD}_{\{x\}}$ and essentially a subset of ${}^\omega(\max\{\kappa,\zeta\})$. By Fact 4.21, R has an $\mathrm{OD}_{\{x\}}$ ∞-Borel code (S,φ). Since $\mathrm{HOD}_{\{x\}} \models$ "$\mathrm{cof}(\kappa) = \omega$", Fact 7.16 gives an increasing sequence $\langle \kappa_n : n \in \omega\rangle$ and $\langle \mu_n : n \in \omega\rangle$ which both belong to $\mathrm{HOD}_{\{x\}}$ so that $\sup\{\kappa_n : n \in \omega\} = \kappa$ and $\mathrm{HOD}_{\{x\}} \models$ "μ_n is a normal measure on κ_n" for each $n \in \omega$. Let $\mathfrak{u} \in \mathrm{HOD}_{\{x\}}$ be the associated measure sequence and $\dot{f}_{\dot{G}}$ be the $(\mathbb{N}_\mathfrak{u})^{\mathrm{HOD}_{\{x\}}}$-name for the generic ω-sequence. Since the hypothesis of Lemma 7.25 holds for $\mathrm{HOD}_{\{x\}}$ and $1_{\mathbb{N}_\mathfrak{u}}$ by Fact 6.1, there is a $\gamma < \zeta$ and $q \in \mathbb{N}_\mathfrak{u}$ so that for all $g \in [q]$, $R(g,\gamma)$. By definition of R, $\Phi(g) = \gamma$ for all $g \in [q]$. It has been shown that $[q] \subseteq \Phi^{-1}[\{\gamma\}]$. Since $|[q]| = |{}^\omega\kappa|$ by Fact 7.21, $|\Phi^{-1}[\{\gamma\}]| = |{}^\omega\kappa|$. Since Φ was arbitrary, this shows that ${}^\omega\kappa$ has ON-regular cardinality. ☐

If $\mathrm{cof}(\kappa) = \omega$ and $\omega < \kappa$, Fact 7.22 shows that $B(\omega,\kappa)$ does not have ω-regular cardinality. Under AD, Jackson, Trang, and the author ([17]) can show that $B(\omega,\omega_\omega)$ has n-regular cardinal for all $n \in \omega$ using the Martin's ultrapower analysis for each ω_m when $m < \omega$. This will be generalized next to show under determinacy assumption that $B(\omega,\kappa)$ has n-regular cardinality for all $n \in \omega$. (Note that under AD, there are sets which do not have 2-regular cardinality such as $\mathbb{R} \sqcup \omega_1$.)

Fact 7.27. *Let* $M \models \mathsf{ZFC}$ *be an inner model. Let* $\delta \in \mathrm{ON}$ *and* $\langle \mathfrak{u}_\alpha : \alpha \in \delta\rangle$ *be a sequence so that* $M \models$ *"*$\mathfrak{u}_\alpha$ *is a measure sequence" for each* $\alpha \in \delta$ *and* $\langle \lambda^{\mathfrak{u}_\alpha} : \alpha \in \delta\rangle$ *is a strictly increasing sequence. Let* $\kappa = \sup\{\lambda^{\mathfrak{u}_\alpha} : \alpha \in \delta\}$*. Let* $\langle p_\alpha : \alpha \in \delta\rangle$ *be such that* $p_\alpha \in \mathbb{N}_{\mathfrak{u}_\alpha}$ *(but not necessarily in* M*) for each* $\alpha \in \delta$*. Then* $|\bigcup_{\alpha\in\delta}[p_\delta]| = |B(\omega,\kappa)|$*.*

Proof. Since $\bigcup_{\alpha\in\delta}[p_\alpha] \subseteq B(\omega,\kappa)$, it is clear that $|\bigcup_{\alpha\in\delta}[p_\alpha]| \leq |B(\omega,\kappa)|$. If $f \in B(\omega,\kappa)$, let $\alpha_f \in \delta$ be the least $\alpha \in \delta$ so that $\sup(f) < \lambda_\alpha$. Let ι_f be the least $n \in \omega$ so that $\mathsf{slength}(p_{\alpha_f}) \leq n$ and $f(0) < \kappa_n^{\alpha_f}$. Suppose $\iota_f(k)$ has been defined. Let $\iota_f(k+1)$ be the least n greater than $\iota_f(k)$ so that $f(k+1) < \kappa_n^{\mathfrak{u}_{\alpha_f}}$. For each

$\alpha \in \delta$ and $k \in \omega$ with $\mathsf{slength}(p_\alpha) \leq k < \omega$, let $\mathsf{enum}_{p_\alpha(k)} : \kappa_k^{\mathfrak{u}_\alpha} \to p_\alpha(k)$ be the increasing enumeration of $p_\alpha(k)$. Define $\Phi : B(\omega,\kappa) \to \bigcup_{\alpha\in\delta}[p_\alpha]$ by

$$\Phi(f)(n) = \begin{cases} \mathsf{enum}_{p_{\alpha_f}(n)}(0) & n \notin \iota_f[\omega] \\ \mathsf{enum}_{p_{\alpha_f}(n)}(1 + f(k)) & n \in \iota_f[\omega] \wedge n = \iota_f(k) \end{cases}.$$

Φ is an injection and therefore $|B(\omega,\kappa)| \leq |\bigcup_{\alpha\in\delta}[p_\alpha]|$. Thus $|B(\omega,\kappa)| = |\bigcup_{\alpha\in\delta}[p_\alpha]|$. ☐

Theorem 7.28. *Assume* AD *and* $V = L(\mathbb{R})$. *Let* $\omega \leq \kappa < \Theta$ *be a cardinal with* $\mathrm{cof}(\kappa) = \omega$. *For all* $n < \omega$, $B(\omega,\kappa)$ *has* n*-regular cardinality. If* $\omega < \kappa$, *then* $B(\omega,\kappa)$ *does not have* ω*-regular cardinality.*

Proof. First, suppose $\kappa = \omega$. Note that $|B(\omega,\omega)| = |\mathbb{R}|$. As noted in Theorem 7.26, $\mathbb{R}$ even has ON-regular cardinality.

Next, suppose that $\omega < \kappa < \Theta$ and $\mathrm{cof}(\kappa) = \omega$. Since $\mathsf{AC}_\omega^\mathbb{R}$ implies $\mathrm{cof}(\omega_1) = \omega_1$, one must have $\omega_1 < \kappa$. Let $\rho : \omega \to \kappa$ be an increasing cofinal function. Let $j \in \omega$. Let $\Phi : B(\omega,\kappa) \to j$ be a function. Since $V = L(\mathbb{R})$, there is an $x \in \mathbb{R}$ so that ρ and Φ are $\mathrm{OD}_{\{x\}}$. Thus $\rho \in \mathrm{HOD}_{\{x\}}$ and $\mathrm{HOD}_{\{x\}} \models \mathrm{cof}(\kappa) = \omega$. Let $R = \{(f,i) \in B(\omega,\kappa) \times j : \Phi(f) = i\}$. Note that R may be regarded as a subset of ${}^\omega\kappa$. Also $\mathrm{dom}(R) = B(\omega,\kappa)$ since Φ is a function on $B(\omega,\kappa)$. By Fact 4.21, R has an $\mathrm{OD}_{\{x\}}$ ∞-Borel code (S,φ). By Fact 7.16, $\mathrm{HOD}_{\{x\}} \models$ "there are κ-many measurable cardinals below κ". For each $m,n \in \omega$, let κ_n^m be the $(\omega \cdot \rho(m) + n)^{\text{th}}$ measurable cardinal of $\mathrm{HOD}_{\{x\}}$ and let μ_n^m be the least normal measure on κ_n^m according to the canonical global wellordering of $\mathrm{HOD}_{\{x\}}$. For each $m \in \omega$, let $\lambda_m = \sup\{\kappa_n^m : n \in \omega\}$. For each $m \in \omega$, let $\mathfrak{u}_m$ be the measure sequence so that $\kappa_n^{\mathfrak{u}_m} = \kappa_n^m$, $\mu_n^{\mathfrak{u}_m} = \mu_n^m$, and $\lambda^{\mathfrak{u}_m} = \lambda_m$. Since $\rho \in \mathrm{HOD}_{\{x\}}$, one has that $\mathfrak{u}_m \in \mathrm{HOD}_{\{x\}}$ for each $m \in \omega$. Note that $\sup\{\lambda_m : m \in \omega\} = \kappa$. For each $m \in \omega$, let $R_m = R \cap ({}^\omega(\lambda_m) \times j)$. (S,φ) essentially remains an ∞-Borel code for R_m. The hypothesis of Lemma 7.25 holds for $\mathrm{HOD}_{\{x\}}$ by Fact 6.1. For each $m \in \omega$, by applying Lemma 7.25 to $\mathrm{HOD}_{\{x\}}$, $(\mathbb{N}_{\mathfrak{u}_m})^{\mathrm{HOD}_{\{x\}}}$, and R_m with ∞-Borel code (S,φ), one has a $q \in \mathbb{N}_{\mathfrak{u}_m}$ and $i \in j$ so that for all $g \in [q]$, $R_m(g,i)$ or equivalently $R(g,i)$. Let i_m be the least such $i < j$ so that there exists a q with the above property for m. Let $S \subseteq \omega \times \mathscr{P}(\kappa)$ be defined by $S(m,q)$ if and only if $q \in \mathbb{N}_{\mathfrak{u}_m}$ and for all $g \in [q]$, $R_m(q,i_m)$. By the discussion above, $\mathrm{dom}(S) = \omega$. By the Moschovakis coding lemma and $\mathsf{AC}_\omega^\mathbb{R}$ in the real world,[24], there is a sequence $\langle q_m : m \in \omega\rangle$ so that for all $m \in \omega$, $S(m,q_m)$. There is an $\bar{i} \in j$ so that $A = \{m \in \omega : i_m = \bar{i}\}$ is infinite. Let $Z = \bigcup_{m\in A}[q_m]$. Suppose $g \in Z$. There is a unique $m \in A$ so that $g \in [q_m]$. By the property q_m and since $i_m = \bar{i}$, $R(g,\bar{i})$. By definition of R, $\Phi(g) = \bar{i}$. This shows that $Z \subseteq \Phi^{-1}[\{\bar{i}\}]$. Since $|Z| = |\bigcup_{m\in A}[p_m]| = |B(\omega,\kappa)|$ by Fact 7.27 (applied to the sequence $\langle \mathfrak{u}_m : m \in A\rangle$), one has that $|\Phi^{-1}[\{\bar{i}\}]| = |B(\omega,\kappa)|$. Since Φ was arbitrary, this shows that $B(\omega,\kappa)$ has j-regular cardinality. Since $j \in \omega$ was arbitrary, the theorem follows. ☐

The following is obtained by making similar modifications as in Theorem 6.7 and Theorem 7.19 to Theorem 7.26 and Theorem 7.28.

Theorem 7.29. *Assume* AD^+. *Suppose* $\omega \leq \kappa < \Theta$ *with* $\mathrm{cof}(\kappa) = \omega$. ${}^\omega\kappa$ *has* ON*-regular cardinality.* $B(\omega,\kappa)$ *has* n*-regular cardinality for each* $n \in \omega$. *If* $\omega < \kappa$, *then* $B(\omega,\kappa)$ *does not have* ω*-regular cardinality.*

The results above have been shown for ${}^\omega\kappa$ and $B(\omega,\kappa)$ when $\kappa < \Theta$ with $\mathrm{cof}(\kappa) = \omega$. When $\mathrm{cof}(\kappa) > \omega$, ${}^\omega\kappa = B(\omega,\kappa)$. Fact 7.22 shows that $B(\omega,\kappa)$ does not have $\mathrm{cof}(\kappa)$-regular cardinality. Under AD, using $\omega_1 \to_* (\omega_1)^\omega_\omega$ or the countably completeness of $\mu^\omega_{\omega_1}$, one has that ${}^\omega\omega_1$ has ω-regular cardinality. Under AD, using $\omega_2 \to_* (\omega_1)^\omega_{\omega_1}$ or the ω_2-completeness of the $\mu^\omega_{\omega_2}$, one has that ${}^\omega\omega_2$ has ω_1-regular cardinality. Jackson, Trang, and the author can show that for all $2 \leq n < \omega$, ${}^\omega\omega_m$ has ω_1-regular cardinality. The conjecture is that for all cardinal κ, $B(\omega,\kappa)$ has δ-regular cardinality for all $\delta < \mathrm{cof}(\kappa)$. The methods developed here for $B(\omega,\kappa)$ when $\mathrm{cof}(\kappa) = \omega$ can be extended to ${}^\omega\kappa$ for all cardinal $\kappa < \Theta$ if one assumes an additional hypothesis. The following is the version of this conjecture for $L(\mathbb{R})$.

Definition 7.30. The Uniform Cofinality of HOD Successor and Double Successor Hypothesis for $L(\mathbb{R})$ is the following assertion:

[24]The use of the coding lemma and $\mathsf{AC}_\omega^\mathbb{R}$ in the real world is necessary. S does not belong to $\mathrm{HOD}_{\{x\}}$ and hence one cannot make the choice of $\langle q_m : m \in \omega\rangle$ using the wellordering in $\mathrm{HOD}_{\{x\}}$. Moreover, a q_m such that $S(m,q_m)$ cannot belong to $(\mathbb{N}_{\mathfrak{u}_m})^{\mathrm{HOD}_{\{x\}}}$.

Assume AD and $V = L(\mathbb{R})$. Let $x \in \mathbb{R}$ and $\omega_1 < \kappa < \Theta$ be a cardinal such that $\mathrm{HOD}_{\{x\}} \models \mathrm{cof}(\kappa) = \mathrm{cof}(\kappa)^{L(\mathbb{R})}$. There are functions $h : \mathrm{cof}(\kappa) \to \kappa$, $H : \mathrm{cof}(\kappa) \times \omega \to \kappa$, $S_1 : \mathrm{cof}(\kappa) \times \omega \to \kappa$, and $S_2 : \mathrm{cof}(\kappa) \times \omega \to \kappa$ so that for all $\alpha \in \mathrm{cof}(\kappa)$ and $n \in \omega$, the following holds.

(1) $H(\alpha, n) < H(\alpha, n+1)$, $S_1(\alpha, n) < S_1(\alpha, n+1)$, $S_2(\alpha, n) < S_2(\alpha, n+1)$.
(2) $\langle H(\alpha, n) : n \in \omega\rangle \in \mathrm{HOD}_{\{x\}}$.
(3) $\mathrm{HOD}_{\{x\}} \models H(\alpha, n)$ is a measurable cardinal.
(4) $h(\alpha) = \sup\{H(\alpha, n) : n \in \omega\}$.
(5) h is cofinal in κ.
(6) $(h(\alpha)^+)^{\mathrm{HOD}_{\{x\}}} = \sup\{S_1(\alpha, n) : n \in \omega\}$.
(7) $(h(\alpha)^{++})^{\mathrm{HOD}_{\{x\}}} = \sup\{S_2(\alpha, n) : n \in \omega\}$.

Remark 7.31. Recall that Steel [60] Lemma 8.25 shows that successor cardinals of HOD have cofinality ω in the real world. The above conjecture asserts one can uniformly find a sequence of witnesses to many successor cardinals of HOD having cofinality ω.

The following outline the usefulness of the conjecture. Let $\alpha < \mathrm{cof}(\kappa)$ and $n \in \omega$. Let $\tau_0^\alpha(n) = H(\alpha, n)$. Let μ_n^α be the least normal measure on $H(\alpha, n)$ according to the canonical global wellordering of $\mathrm{HOD}_{\{x\}}$. Let $\tau_1^\alpha(n) = S_1(\alpha, n)$. Let $\tau_2^\alpha(n) = S_2(\alpha, n)$. Let $\mathfrak{u}_\alpha$ be the measure sequence such that $\kappa^{\mathfrak{u}_\alpha} = H(\alpha, n)$, $\mu_n^{\mathfrak{u}_\alpha} = \mu_n^\alpha$, and $\lambda^{\mathfrak{u}_\alpha} = h(\alpha)$. Since for each α, $\langle H(\alpha, n) : n \in \omega\rangle \in \mathrm{HOD}_{\{x\}}$, one has $\mathfrak{u}_\alpha \in \mathrm{HOD}_{\{x\}}$ which is important since one needs $\mathrm{HOD}_{\{x\}}$ to be able to define $\mathbb{N}_{\mathfrak{u}_\alpha}$. In the real world, one has the following two sequences $\langle \tau_i^\alpha : \alpha < \mathrm{cof}(\kappa) \wedge i < 3\rangle$ and $\langle \mathfrak{u}_\alpha : \alpha < \mathrm{cof}(\kappa)\rangle$. By the uniformity asserted in Lemma 5.4, one has a sequence $\langle e_\alpha : \alpha < \mathrm{cof}(\kappa)\rangle$ so that $e_\alpha : h(\alpha) \to (h(\alpha)^{++})^{\mathrm{HOD}_{\{x\}}}$ is a $\mathrm{HOD}_{\{x\}}$-amenable surjection for each $\alpha < \mathrm{cof}(\kappa)$. Similarly, one can apply Lemma 7.15 and Lemma 7.24 uniformly.

Note also that the conjecture is stated with the domain of h being $\mathrm{cof}(\kappa)$. This will be sufficient for the following application. Moreover, for higher dimensional problem such as primeness and Jónssonness addressed in forthcoming work, the measurability of $\mathrm{cof}(\kappa)$ in $\mathrm{HOD}_{\{x\}}$ will be needed.

The belief is that the conjecture will be proved using the directed system analysis of $\mathrm{HOD}_{\{x\}}^{L(\mathbb{R})}$ using certain iterable mice. Theorem 7.34 will provide some evidence for this conjecture by establishing that it holds for those $\kappa < \Theta$ with $\mathrm{cof}(\kappa) = \omega_1$ assuming externally that $M_\omega^\sharp$ exists. For those set Z which appeared in the discussion before Lemma 6.6, $\mathrm{HOD}_Z^{L(Z,\mathbb{R})}$ possesses a similar directed system analysis. Hence one can also formulate the conjecture for such models of the form $L(Z, \mathbb{R})$ and successor and double successor in $\mathrm{HOD}_Z^{L(Z,\mathbb{R})}$. Granting such a conjecture, one can extend as usual the results from $L(\mathbb{R})$ to AD^+.

Next, one will verify the extent of regularity for $B(\omega, \kappa)$ in $L(\mathbb{R}) \models \mathsf{AD}$ assuming the uniform cofinality of HOD successor and double successor hypothesis in $L(\mathbb{R})$.

Theorem 7.32. *Assume* AD *and* $V = L(\mathbb{R})$*, and the uniform cofinality of* HOD *successor and double successor hypothesis in* $L(\mathbb{R})$*. Let* $\omega \leq \kappa < \Theta$ *be a cardinal. Then* $B(\omega, \kappa)$ *has* δ*-regular cardinality for all* $\delta < \mathrm{cof}(\kappa)$*.*

Proof. When $\mathrm{cof}(\kappa) = \omega$, Theorem 7.29 already showed that $B(\omega, \kappa)$ has n-regular cardinality for all $n \in \omega$.

Suppose $\kappa = \omega_1$. Let $\Phi : [\omega_1]_*^{\omega_1} \to \omega$. Since $\omega_1 \to_* (\omega_1)_\omega^\omega$, $\mu_{\omega_1}^\omega$ is a countably complete measure. Thus there is a club $C \subseteq \omega_1$ and an $m \in \omega$ so that for all $f \in [C]_*^\omega$, $\Phi(f) = m$. Thus $|\Phi^{-1}[\{m\}]| = |{}^\omega\omega_1|$. ${}^\omega\omega_1$ has ω-regular cardinality.

Assume $\omega_1 < \kappa$ and $\mathrm{cof}(\kappa) > \omega$. Let $\delta < \mathrm{cof}(\kappa)$ and $\Phi : B(\omega, \kappa) \to \delta$. Let $\rho : \mathrm{cof}(\kappa) \to \kappa$ be an increasing cofinal map. Since $V = L(\mathbb{R})$, there is an $x \in \mathbb{R}$ so that Φ and ρ are $\mathrm{OD}_{\{x\}}$. Thus $\rho \in \mathrm{HOD}_{\{x\}}$ and $\mathrm{HOD}_{\{x\}} \models$ "$\mathrm{cof}(\kappa) = \mathrm{cof}(\kappa)^{L(\mathbb{R})}$". Let $R \subseteq B(\omega, \kappa) \times \delta$ be defined by $R(f, \beta)$ if and only $\Phi(f) = \beta$. R is $\mathrm{OD}_{\{x\}}$ and thus there is an ∞-Borel code $(S, \varphi) \in \mathrm{HOD}_{\{x\}}$ for R by Fact 4.21. By the uniform cofinality of HOD successor and double successor hypothesis, let h, H, S_1, and S_2 be as in Definition 7.30 for $\mathrm{HOD}_{\{x\}}$ and κ. For each $\alpha < \mathrm{cof}(\kappa)$ and $n \in \omega$, let μ_n^α be the least normal measure on $H(\alpha, n)$ according to the canonical global wellordering of $\mathrm{HOD}_{\{x\}}$. Let $\mu_\alpha \in \mathrm{HOD}_{\{x\}}$ be the measure sequence defined by $\kappa_n^{\mathfrak{u}_\alpha} = H(\alpha, n)$, $\mu_n^{\mathfrak{u}_\alpha} = \mu_n^\alpha$, and $\lambda^{\mathfrak{u}_\alpha} = h(\alpha)$. Since for each $\alpha < \mathrm{cof}(\kappa)$, $\langle H(\alpha, n) : n \in \omega\rangle \in \mathrm{HOD}_{\{x\}}$, $\mathfrak{u}_\alpha \in \mathrm{HOD}_{\{x\}}$. For each $\alpha < \mathrm{cof}(\kappa)$, let $\tau_1^\alpha : \omega \to (h(\alpha)^+)^{\mathrm{HOD}_{\{x\}}}$ be defined by $\tau_1^\alpha(n) = S_1(\alpha, n)$. For each $\alpha < \mathrm{cof}(\kappa)$, let $\tau_2^\alpha : \omega \to (h(\alpha)^+)^{\mathrm{HOD}_{\{x\}}}$ be defined by $\tau_2^\alpha(n) = S_2(\alpha, n)$. Note that $\langle \tau_i^\alpha : \alpha < \mathrm{cof}(\kappa) \wedge i \in \{1, 2\}\}$ is a sequence that exists in the real world $L(\mathbb{R})$ but does not belong to $\mathrm{HOD}_{\{x\}}$. Let $\prec$ be the global wellordering of $\mathrm{HOD}_{\{x\}}$. By applying Lemma 7.25 to $\mathfrak{u}_\alpha$, $\prec$, τ_1^α, τ_2^α, $1_{\mathbb{N}_{\mathfrak{u}_\alpha}}$, and (S,φ) uniformly for each α, one has sequences $\langle \gamma_\alpha : \alpha < \mathrm{cof}(\kappa)\rangle$ and $\langle q_\alpha : \alpha < \mathrm{cof}(\kappa)\rangle$ so that for all $\alpha < \mathrm{cof}(\kappa)$ and $g \in [q_\alpha]$, $R(g,\gamma_\alpha)$ holds. By the definition of R, for all $\alpha < \mathrm{cof}(\kappa)$ and $g \in [q_\alpha]$, $\Phi(g) = \gamma_\alpha$. Since $\langle \gamma_\alpha : \alpha < \mathrm{cof}(\kappa)\rangle$ is a sequence below $\delta < \mathrm{cof}(\kappa)$ and $\mathrm{cof}(\kappa)$ is regular, there is a $\bar\gamma$ so that the set $A = \{\alpha < \mathrm{cof}(\kappa) : \gamma_\alpha = \bar\gamma\}$ has cardinality $\mathrm{cof}(\kappa)$. Let $Z = \bigcup_{\alpha\in A}[q_\alpha]$. Let $g \in Z$. There is a unique $\alpha \in A$ so that $g \in [q_\alpha]$. By properties of q_α and since $\gamma_\alpha = \bar\gamma$, one has that $\Phi(g) = \bar\gamma$. Thus $Z \subseteq \Phi^{-1}[\{\bar\gamma\}]$. Since $|Z| = |B(\omega,\kappa)|$ by Fact 7.27, one has that $|\Phi^{-1}[\{\bar\gamma\}]| = |B(\omega,\kappa)|$. Since $\Phi : B(\omega,\kappa) \to \delta$ was arbitrary, this shows that $B(\omega,\kappa)$ has δ-regular cardinality. Since $\delta < \mathrm{cof}(\kappa)$ was arbitrary, this completes the proof. □

Theorem 7.33. *Assume AD^+ and uniform cofinality of $\mathrm{HOD}_Z^{L(Z,\mathbb{R})}$ successor and double successor conjecture for $L(Z,\mathbb{R})$ when Z has the property from the discussion before Lemma 6.6. Let $\omega \leq \kappa < \Theta$ be a cardinal. Then $B(\omega,\kappa)$ has δ-regular cardinality for all $\delta < \mathrm{cof}(\kappa)$.*

Next, one will provide a sketch of the argument that if $M_\omega^\sharp$ exists, then the uniform cofinality of HOD successor and double successor hypothesis hold for $L(\mathbb{R})$ below $\boldsymbol{\delta}_1^2$ at cofinality ω_1.

Theorem 7.34. *Assume $M_\omega^\sharp$ exists. Then $L(\mathbb{R}) \models$ "if $\kappa < \boldsymbol{\delta}_1^2$ is a cardinal and $\mathrm{cof}(\kappa) = \omega_1$, then the uniform cofinality of HOD successor and double successor hypothesis holds at κ".*

Proof. First one will review the relevant theory and notation of the internal directed system from [60] Section 8. By [60] Lemma 8.3, a countable, properly small, and $\Game^{\mathbb{R}}\boldsymbol{\Pi}_1^1$-iterable mouse M has a unique iteration strategy for a certain game $G^*(M,\omega_1,\omega_1)$ in $L(\mathbb{R})$ which is uniformly Σ_1 definable in $L(\mathbb{R})$ with M as a parameter. See [60] Definition 8.4 for the definition of correct iteration for countable, properly small, and $\Game^{\mathbb{R}}\boldsymbol{\Pi}_1^1$-iterable mouse. [60] Lemma 8.5 states that the notion of correct iteration is Σ_1-definable over $L(\mathbb{R})$. Let $\mathcal{F}$ be the set of countable, properly small, $\Game^{\mathbb{R}}\boldsymbol{\Pi}_1^1$-iterable, and full mouse. The ordering on $\mathcal{F}$ is $\prec^*$ defined by $M \prec^* N$ if and only if M iterates correctly to a cutpoint of N (see [60] Definition 8.7), and in this case, $\pi_{M,N}$ will refer to the unique iteration map of M to a cutpoint of N. $\mathcal{F}$, $\prec^*$, and the map $\langle (M,N,\pi_{M,N}) : M,N \in \mathcal{F} \wedge M \prec^* N\rangle$ are ordinal definable in $L(\mathbb{R})$. Let F^+ consists of the correct iterates of M_ω by δ_0-bounded trees. For $M,N \in F^+$, define $M \prec^+ N$ if and only if M iterates correctly to N by a δ_0-bounded tree. (See [60] Definition 8.10, 8.11, and 8.12.) $(\mathcal{F}^+,\prec^+)$ is external to $L(\mathbb{R})$ in the sense that it is not even definable in $L(\mathbb{R})$. However, using $(F^+,\prec^+)$, [60] Lemma 8.14 implies that the internal directed system $\langle M,N,\pi_{M,N} : M,N \in \mathcal{F} \wedge M \prec^* N\rangle$ is countably directed with wellfounded direct limit denoted M_∞. Steel ([60] Theorem 8.20) showed that $\mathrm{HOD}^{L(\mathbb{R})} \cap V_{\boldsymbol{\delta}_1^2} = M_\infty \cap V_{\boldsymbol{\delta}_1^2}$. (The discussion above can be relativized for $\mathrm{HOD}^{L(\mathbb{R})}_{\{x\}}$ where $x \in \mathbb{R}$.)

Now work in $L(\mathbb{R})$. Let $\omega_1 < \kappa < \boldsymbol{\delta}_1^2$ with $\mathrm{cof}(\kappa) = \omega_1$. Since every cofinal map of ω_1 into κ is $\mathrm{OD}_{\{x\}}$ for some $x \in \mathbb{R}$, there is an $x \in \mathbb{R}$ so that $\mathrm{HOD}_{\{x\}} \models \mathrm{cof}(\kappa) = \omega_1^{L(\mathbb{R})}$. Without loss of generality, suppose $\mathrm{HOD} \models \mathrm{cof}(\kappa) = \omega_1^{L(\mathbb{R})}$. Since $\omega_1 < \kappa < \Theta$, there are κ many measurable cardinal of HOD below κ by Fact 7.16. Let $\rho \in \mathrm{HOD} \cap V_{\boldsymbol{\delta}_1^2}$ be such that $\mathrm{HOD} \cap V_{\boldsymbol{\delta}_1^2} \models$ "$\rho : \omega_1^{L(\mathbb{R})} \to \kappa$ is increasing cofinal and for all $\alpha < \omega_1^{L(\mathbb{R})}$, $\rho(\alpha)$ is a measurable cardinal". Since $\mathrm{HOD} \cap V_{\boldsymbol{\delta}_1^2} = M_\infty \cap V_{\boldsymbol{\delta}_1^2}$, there is an $N_0 \in \mathcal{F}$ so that the following holds.

(1) Let $j_{0,\infty} : N_0 \to M_\infty$ denote the direct system map of N_0 into M_∞.
(2) Let $\bar\delta$ be the least measurable cardinal of N_0. It can be shown that $j_{0,\infty}(\bar\delta) = \omega_1^{L(\mathbb{R})}$.
(3) There is a $\bar\kappa \in N_0$ so that $j_{0,\infty}(\bar\kappa) = \kappa$.
(4) There is a $\bar\rho \in N_0$ so that $j_{0,\infty}(\bar\rho) = \rho$. Thus $N_0 \models$ "$\bar\rho : \bar\delta \to \bar\kappa$ is cofinal increasing and for all $\alpha < \bar\delta$, $\bar\rho(\alpha)$ is measurable".
(5) Let $\bar\mu \in N_0$ be such that $N_0 \models$ "$\bar\mu$ is a measure on $\bar\delta$" (on the extender sequence of N_0).
(6) By Fact 7.16, $\mathrm{HOD} \cap V_{\boldsymbol{\delta}_1^2} \models$ "there are κ many measurable cardinals below κ. By the elementarity of $j_{0,\infty}$, $N_0 \models$ "there are $\bar\kappa$ many measurable cardinals below $\bar\kappa$". Let $i_{\bar\mu} : N_0 \to \mathrm{Ult}(N_0,\bar\mu)$ be the ultrapower map. By elementarity, $\mathrm{Ult}(N_0,\bar\mu) \models$ "there are $i_{\bar\mu}(\bar\kappa)$-many measurable cardinals below $i_{\bar\mu}(\bar\kappa)$". Since $\mathrm{cof}(\bar\kappa) = \bar\delta$, $i_{\bar\mu}(\bar\kappa) > \bar\kappa$. Since $\bar\mu \in N_0$, let $\iota \in \mathrm{Ult}(N_0,\bar\mu) \subseteq N_0$ be such that $\mathrm{Ult}(N_0,\bar\mu) \models$ "$\iota : \omega \to i_{\bar\mu}(\bar\kappa)$ enumerates the first ω-many measurable cardinals above $\bar\kappa$". Note this property of ι is first order expressible in N_0 since $\bar\mu \in N_0$.

(7) Let $\lambda = \sup(\iota)$ and note that $\lambda < i_{\bar{\mu}}(\kappa)$. Let $\lambda^* = (\lambda^+)^{\mathrm{Ult}(N_0,\bar{\mu})}$ and let $\lambda^{**} = (\lambda^{++})^{\mathrm{Ult}(N_0,\bar{\mu})}$. λ^* and λ^{**} are the ordinals which become the successor and double successor of λ in the ultrapower of N_0 by $\bar{\mu}$. Note that these facts about $\bar{\mu}$, λ, λ^*, and λ^{**} are first order expressible in N_0.

(8) Since N_0 is countable in $L(\mathbb{R})$, let $\ell_1 : \omega \to \lambda^*$ and $\ell_2 : \omega \to \lambda^{**}$ be increasing cofinal functions through λ^* and λ^{**}, respectively, which belong to $L(\mathbb{R})$.

For each $\alpha < \omega_1$, let $j_{0,\alpha} : N_0 \to N_\alpha$ be the α^{th}-length linear iteration of N_0 by the image of $\bar{\mu}$. Note that $N_\alpha \in \mathcal{F}$ for all $\alpha < \beta$. Let $j_{\alpha,\infty} : N_\alpha \to M_\infty$ be the direct limit map of N_α into M_∞. Note that for all $\alpha < \omega_1$, $j_{\alpha,\infty}$ belongs to $L(\mathbb{R})$ (and this is an important reason to use the internal direct system $\mathcal{F}$ rather than the external direct system $\mathcal{F}^+$). For $\alpha < \beta < \omega$, let $j_{\alpha,\beta} : N_\alpha \to N_\beta$ denote $(\beta - \alpha)^{\text{th}}$-length linear iteration of N_α by the image of $j_{0,\alpha}(\bar{\mu})$. Furthermore, since these are linear iterations, one can check that $j_{\alpha,\beta} = \pi_{N_\alpha,N_\beta}$ in the notation above; that is, $j_{\alpha,\beta}$ is the unique map of the correct iteration witnessing $N_\alpha \prec^* N_\beta$ according to the unique iteration strategy for the appropriate game. Thus for all $\alpha < \beta < \gamma$, $j_{\alpha,\gamma} = j_{\beta,\gamma} \circ j_{\alpha,\beta}$ and $j_{\alpha,\infty} = j_{\beta,\infty} \circ j_{\alpha,\beta}$. Define the following objects:

(a) For $\alpha < \omega_1$ and $n \in \omega$, let $H(\alpha, n) = j_{\alpha+1,\infty}(j_{0,\alpha}(\iota))(n)$.

(b) For $\alpha < \omega_1$, let $h(\alpha) = \sup\{H(\alpha, n) : n \in \omega\} = \sup(j_{\alpha+1,\infty}(j_{0,\alpha}(\iota))) = j_{\alpha+1,\infty}(\sup(j_{0,\alpha}(\iota))) = j_{\alpha+1,\infty}(j_{0,\alpha}(\lambda))$.

(c) For $\alpha < \omega_1$ and $n \in \omega$, let $S_1(\alpha, n) = j_{\alpha+1,\infty}(j_{0,\alpha}(\ell_1(n)))$.

(d) For $\alpha < \omega_1$ and $n \in \omega$, let $S_n(\alpha, n) = j_{\alpha+1,\infty}(j_{0,\alpha}(\ell_2(n)))$.

Note that $j_{0,\alpha}(\iota)$ enumerates the next ω-many measurable above $\sup(j_{0,\alpha}(\bar{\rho})) = j_{0,\alpha}(\bar{\kappa})$ in $\mathrm{Ult}(N_\alpha, j_{0,\alpha}(\bar{\mu})) = N_{\alpha+1}$ by (6) above. Note that $\sup(j_{0,\alpha+1}(\bar{\rho})) = j_{0,\alpha+1}(\bar{\kappa}) > j_{0,\alpha}(\bar{\kappa}) = \sup(j_{0,\alpha}(\bar{\rho}))$. By (4), $N_{\alpha+1}$ sees that $j_{0,\alpha+1}(\bar{\kappa}) = \sup(j_{0,\alpha+1}(\bar{\rho}))$ is a limit of measurable cardinals. So $\sup(j_{0,\alpha}(\iota)) = j_{0,\alpha}(\lambda) < j_{0,\alpha+1}(\bar{\kappa})$ since the supremum of the first ω-many measurable cardinals of $N_{\alpha+1}$ above $j_{0,\alpha}(\bar{\kappa})$ is less than $j_{0,\alpha+1}(\bar{\kappa})$ which is above $j_{0,\alpha}(\bar{\kappa})$ and is a limit of $j_{0,\alpha}(\bar{\delta})$ many measurable cardinals of $N_{\alpha+1}$. Similarly, $j_{0,\alpha}(\lambda^*) < j_{0,\alpha}(\lambda^{**}) < j_{0,\alpha+1}(\bar{\kappa})$ since these are the successor and double successor of $j_{0,\alpha}(\lambda)$ in $N_{\alpha+1}$. Then

$$H(\alpha, n) = j_{\alpha+1,\infty}(j_{0,\alpha}(\ell))(n) < h(\alpha) = j_{\alpha+1,\infty}(\sup(j_{0,\alpha}(\iota))) = j_{\alpha+1,\infty}(j_{0,\alpha}(\lambda))$$

$$< j_{\alpha+1,\infty}(j_{0,\alpha}(\lambda^*)) < j_{\alpha+1,\infty}(j_{0,\alpha}(\lambda^{**})) < j_{\alpha+1,\infty}(j_{0,\alpha+1}(\bar{\kappa})) = \kappa.$$

This shows that that $H : \omega_1 \times \omega_1 \to \kappa$ and $h : \omega_1 \to \kappa$. Also $S_1(\alpha, n) = j_{\alpha+1,\infty}(j_{0,\alpha}(\ell_1(n))) < j_{\alpha+1,\infty}(j_{0,\alpha}(\lambda^*)) < \kappa$. Similarly, $S_2(\alpha, n) < \kappa$. This shows that $S_1 : \omega_1 \times \omega \to \kappa$ and $S_2 : \omega_1 \times \omega \to \kappa$.

Since ι, ℓ_1, and ℓ_2 are increasing functions, Definition 7.30 property (1) holds by the elementary $j_{0,\alpha}$ and $j_{\alpha+1,\infty}$. Since $j_{0,\alpha}(\iota)$ enumerates the next ω-many measurable cardinals above $j_{0,\alpha}(\bar{\kappa})$ in $\mathrm{Ult}(N_\alpha, j_{0,\alpha}(\bar{\mu})) = N_{\alpha+1}$, $j_{\alpha+1,\infty}(j_{0,\alpha}(\iota)) \in M_\infty$ enumerates the next ω-many measurable cardinals above $j_{\alpha+1,\infty}(j_{0,\alpha}(\bar{\kappa}))$ in $M_\infty \cap V_{\boldsymbol{\delta}^2_1} = \mathrm{HOD} \cap V_{\boldsymbol{\delta}^2_1}$. Thus $\langle H(\alpha, n) : n \in \omega\rangle = j_{\alpha+1,\infty}(j_{0,\alpha}(\iota))$ belongs to HOD which establishes Definition 7.30 property (2) and it enumerates the next ω-many measurable cardinal in HOD above $j_{\alpha+1,\infty}(j_{0,\alpha}(\bar{\kappa}))$. In particular, for all $n \in \omega$, $H(\alpha, n) = j_{\alpha+1,\infty}(j_{0,\alpha}(\iota))(n)$ is a measurable cardinal in HOD which establishes Definition 7.30 property (3). Definition 7.30 property (4) holds by definition of h. For any $\alpha < \omega_1$, $j_{0,\alpha}(\bar{\delta}) > \alpha$ and $j_{0,\alpha}(\bar{\delta}) = \mathrm{dom}(j_{0,\alpha}(\bar{\rho}))$. By (4) above, $N_\alpha \models j_{0,\alpha}(\bar{\rho})(\alpha)$ is a measurable cardinal. Since $j_{\alpha,\alpha+1} : N_\alpha \to \mathrm{Ult}(N_\alpha, j_{0,\alpha}(\bar{\mu})) = N_{\alpha+1}$ is just the ultrapower map of $j_{0,\alpha}(\bar{\mu})$ and this ultrapower map fixes all strong limit cardinals of cofinality not equal to $j_{0,\alpha}(\bar{\delta})$, one has that $j_{0,\alpha+1}(\bar{\rho})(\alpha) = j_{\alpha,\alpha+1}(j_{0,\alpha}(\bar{\rho})(\alpha)) = j_{0,\alpha}(\bar{\rho})(\alpha)$. Clearly, $j_{0,\alpha}(\bar{\rho})(\alpha) < \sup(j_{0,\alpha}(\bar{\rho})) = j_{0,\alpha}(\bar{\kappa}) < j_{0,\alpha}(\lambda)$. Thus one has

$$\rho(\alpha) = j_{0,\infty}(\bar{\rho})(\alpha) = j_{\alpha+1,\infty}(j_{0,\alpha+1}(\bar{\rho})(\alpha)) = j_{\alpha+1,\infty}(j_{0,\alpha}(\bar{\rho})(\alpha)) < j_{\alpha+1,\infty}(j_{0,\alpha}(\lambda)) = h(\alpha)$$

Since $\alpha < \omega_1^{L(\mathbb{R})}$ was arbitrary and $\sup(\rho) = \kappa$, one has that $\sup(h) = \kappa$. Thus h is cofinal in κ which establishes Definition 7.30 (5). Note that since $\mathrm{Ult}(N_0, \bar{\mu}) \models \lambda^* = \lambda^+$ by (7), $\mathrm{Ult}(N_0, \bar{\mu}) \models \lambda^*$ is regular. Since N_0 and $\mathrm{Ult}(N_0, \bar{\mu})$ have the same $\bar{\delta}$-sequences, $N_0 \models \mathrm{cof}(\lambda^*) > \bar{\delta}$. Since $j_{0,\alpha}$ is a linear iteration by $\bar{\mu}$ which is a measure on $\bar{\delta}$ and its image, $j_{0,\alpha}$ is continuous at any ordinal whose cofinality is greater than $\bar{\delta}$. Hence $j_{0,\alpha}$ is continuous at λ^*. Since $\sup(\ell_1) = \lambda^*$, $\sup[j_{0,\alpha} \circ \ell_1] = \sup\{j_{0,\alpha}(\ell_1(n)) : n \in \omega\} = j_{0,\alpha}(\lambda^*)$. By (6), $N_{\alpha+1} \models j_{0,\alpha}(\lambda^*) = j_{0,\alpha}(\lambda)^+$. Thus $N_{\alpha+1} \models$ "$j_{0,\alpha}(\lambda^*)$ is regular non-measurable cardinal". The direct limit map $j_{\alpha+1,\infty}$ is continuous at all regular non-measurable cardinals of $N_{\alpha+1}$ (see the remark in the proof of [60] Lemma 8.25). Since $\sup[j_{0,\alpha} \circ \ell_1] = j_{0,\alpha}(\lambda^*)$, $\sup[j_{\alpha+1,\infty} \circ j_{0,\alpha} \circ \ell_1] = j_{\alpha+1,\infty}(j_{0,\alpha}(\lambda^*))$. Thus $\sup\{S_1(\alpha, n) : n \in \omega\} = \sup\{j_{\alpha+1,\infty}(j_{0,\alpha}(\ell_1(n))) : n \in \omega\} = j_{\alpha+1,\infty}(j_{0,\alpha}(\lambda^*))$. Since $N_{\alpha+1} \models j_{0,\alpha}(\lambda^*) =$

$j_{0,\alpha}(\lambda)^+$, one has that $M_\infty \models j_{\alpha+1,\infty}(j_{0,\alpha}(\lambda^*)) = j_{0,\alpha+1}(j_{0,\alpha}(\lambda))^+ = h(\alpha)^+$. Since $M_\infty \cap V_{\boldsymbol{\delta}^2_1} = \mathrm{HOD} \cap V_{\boldsymbol{\delta}^2_1}$, $\mathrm{HOD} \models j_{0,\alpha+1}(j_{0,\alpha}(\lambda^*)) = h(\alpha)^+$. This establishes Definition 7.30 (6). Definition 7.30 (7) is shown in an analogous manner. This completes the proof. □

To remove the restriction that $\kappa < \boldsymbol{\delta}^2_1$ in Theorem 7.34, one can perhaps use the more complex internal directed system of [64] Section 6. To remove the use of the external assumption of $M^\sharp_\omega$, one can possibly use the idea of [64] Section 7. Linear iterations cannot be used once the cofinality of κ is greater than ω_1. It is anticipated that the uniform cofinality of HOD successor and double successor hypothesis in its full generality should be established through the direct system analysis of HOD. In conversation with Schlutzenburg, it seems that Schlutzenburg's result on normalization [51] which implies HOD is a normal iterate of M_ω may suggest a path forward.

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