

THE COUNTABLE ADMISSIBLE ORDINAL EQUIVALENCE RELATION

WILLIAM CHAN

ABSTRACT. Let F_{ω_1} be the countable admissible ordinal equivalence relation defined on ${}^\omega 2$ by $x F_{\omega_1} y$ if and only if $\omega_1^x = \omega_1^y$. Some invariant descriptive set theoretic properties of F_{ω_1} will be explored using infinitary logic in countable admissible fragments as the main tool. Marker showed F_{ω_1} is not the orbit equivalence relation of a continuous action of a Polish group on ${}^\omega 2$. Becker strengthened this to show F_{ω_1} is not even the orbit equivalence relation of a Δ_1^1 action of a Polish group. However, Montalbán has shown that F_{ω_1} is Δ_1^1 reducible to an orbit equivalence relation of a Polish group action; in fact, F_{ω_1} is classifiable by countable structures. It will be shown here that F_{ω_1} must be classified by structures of high Scott rank. Let E_{ω_1} denote the equivalence of order types of reals coding well-orderings. If E and F are two equivalence relations on Polish spaces X and Y , respectively, $E \leq_{\text{a}\Delta_1^1} F$ denotes the existence of a Δ_1^1 function $f : X \rightarrow Y$ which is a reduction of E to F , except possibly on countably many classes of E . Using a result of Zapletal, the existence of a measurable cardinal implies $E_{\omega_1} \leq_{\text{a}\Delta_1^1} F_{\omega_1}$. However, it will be shown that in Gödel's constructible universe L (and set generic extensions of L), $E_{\omega_1} \leq_{\text{a}\Delta_1^1} F_{\omega_1}$ is false. Lastly, the techniques of the previous result will be used to show that in L (and set generic extensions of L), the isomorphism relation induced by a counterexample to Vaught's conjecture cannot be Δ_1^1 reducible to F_{ω_1} . This shows the consistency of a negative answer to a question of Sy-David Friedman.

1. INTRODUCTION

If $x \in {}^\omega 2$, ω_1^x denotes the supremum of the order types of x -recursive well-orderings on ω . Moreover, ω_1^x is also the minimum ordinal height of admissible sets containing x as an element. The latter definition will be more relevant for this paper.

The eponymous countable admissible ordinal equivalence relation, denoted by F_{ω_1} , is defined on ${}^\omega 2$ by:

$$x F_{\omega_1} y \Leftrightarrow \omega_1^x = \omega_1^y$$

It is a Σ_1^1 equivalence relation with all classes Δ_1^1 . Moreover, F_{ω_1} is a thin equivalence relation, i.e., it has no perfect set of inequivalence elements. Some further properties of F_{ω_1} as an equivalence relation will be established in this paper.

Some basic results in admissibility theory and infinitary logic that will be useful throughout the paper will be reviewed in Section 2. This section will cover briefly topics such as KP, admissible sets, Scott ranks, and the Scott analysis. In this section, aspects of Barwise's theory of infinitary logic in countable admissible fragments, which will be the main tool in many arguments, will be reviewed. As an example of an application, a proof of a theorem of Sacks (Theorem 2.16), which establishes that every countable admissible ordinal is of the form ω_1^x for some $x \in {}^\omega 2$, will be given. This proof serves as a template for other arguments. Sacks' theorem also explains why it is appropriate to call F_{ω_1} the "countable admissible ordinal equivalence relation".

There has been some early work on whether F_{ω_1} satisfies certain properties of equivalence relations related to generalization of Vaught's conjecture. For example, Marker in [15] has shown that F_{ω_1} is not induced by a continuous action of a Polish group on the Polish space ${}^\omega 2$. Becker in [3], page 782, strengthened this to show that the equivalence relation F_{ω_1} is not an orbit equivalence relation of a Δ_1^1 group action of a Polish group. A natural question following these results would be whether F_{ω_1} is Δ_1^1 reducible to equivalence relations induced by continuous or Δ_1^1 actions of Polish groups. If such reductions do exist, another question could

September 11, 2017

Research partially supported by NSF grants DMS-1464475 and EMSW21-RTG DMS-1044448

be what properties must these reductions have.

In Section 3, F_{ω_1} will be shown to be Δ_1^1 reducible to a continuous action of S_∞ , i.e., it is classifiable by countable structures. An explicit Δ_1^1 classification of F_{ω_1} by countable structures in the language with a single binary relation symbol, due to Montalbán, will be provided. The classification of F_{ω_1} will use an effective construction of the Harrison linear ordering. This classification, denoted f , has the additional property that for all $x \in {}^\omega 2$, $\text{SR}(f(x)) = \omega_1^x + 1$. This example was provided by Montalbán through communication with Marks and the author.

The explicit classification, f , mentioned above has images that are structures of high Scott rank. In Section 4, it will be shown that this is a necessary feature of all classification of F_{ω_1} by countable structures. The lightface version of the main result of this section is the following:

Theorem 4.2 *Let \mathcal{L} be a recursive language. Let $S(\mathcal{L})$ denote the set of reals that code \mathcal{L} -structures on ω . If $f : {}^\omega 2 \rightarrow S(\mathcal{L})$ is a Δ_1^1 function such that $x F_{\omega_1} y$ if and only if $f(x) \cong_{\mathcal{L}} f(y)$, then for all x , $\text{SR}(f(x)) \geq \omega_1^x$.*

The more general form considers reductions that are $\Delta_1^1(z)$ and involves a condition on the admissible spectrum of z . Intuitively, Theorem 4.2 (in its lightface form as stated above) asserts that any potential classification of F_{ω_1} must have high Scott rank in the sense that the image of any real under the reduction is a structure of high Scott rank. High Scott rank means that $\text{SR}(f(x))$ is either ω_1^x or $\omega_1^x + 1$.

Section 5 is concerned with a weak form of reduction of equivalence relations, invented by Zapletal, called the almost Δ_1^1 reduction. If E and F are two Σ_1^1 equivalence relations on Polish space X and Y , respectively, then E is almost Δ_1^1 reducible to F (in symbols: $E \leq_{a\Delta_1^1} F$) if and only there is a Δ_1^1 function $f : X \rightarrow Y$ and a countable set A such that if x and y are not E -related to any elements of A , then $x E y$ if and only if $f(x) F f(y)$.

An almost Borel reduction is simply a reduction that may fail on countably many classes. Often Σ_1^1 equivalence relations may have a few unwieldy classes. The almost Borel reduction is especially useful since it can be used to ignore these classes. One example of such a Σ_1^1 equivalence relation is E_{ω_1} which is the isomorphism relation of well-orderings with a single class of non-well-orderings. It is defined on ${}^\omega 2$ by:

$$x E_{\omega_1} y \Leftrightarrow (x, y \notin WO) \vee (\text{ot}(x) = \text{ot}(y))$$

E_{ω_1} is a thin Σ_1^1 equivalence with one Σ_1^1 class and all the other classes are Δ_1^1 .

Zapletal isolated an invariant of equivalence relations called the pinned cardinal. This invariant involves pinned names on forcings: an idea that appears implicitly or explicitly in the works of Silver, Burgess, Hjorth, and Zapletal in the study of thin Σ_1^1 equivalence relations. Zapletal showed that there is a deep connection between E_{ω_1} , almost Δ_1^1 reducibilities, and pinned cardinals under large cardinal assumptions:

Theorem 5.7 ([21] Theorem 4.2.1, [22] Theorem 4.4.1) *If there exists a measurable cardinal and E is a Σ_1^1 equivalence relation with infinite pinned cardinal, then $E_{\omega_1} \leq_{a\Delta_1^1} E$.*

Given that this result involves large cardinals, a natural question would be to explore the consistency results surrounding Zapletal's theorem. For example, a natural question is whether ZFC can prove the above result of Zapletal. More specifically, is this result true in Gödel constructible universe L ? This investigation leads to F_{ω_1} in the following way: It will be shown that F_{ω_1} has infinite pinned cardinal. Hence, with a measurable cardinal, $E_{\omega_1} \leq_{a\Delta_1^1} F_{\omega_1}$ via the result of Zapletal. (This also holds if 0^\sharp exists.)

The main result of this section is

Theorem 5.11 *The statement $E_{\omega_1} \leq_{a\Delta_1^1} F_{\omega_1}$ is not true in L (and set generic extensions of L).*

This result is proved by using infinitary logic in admissible fragments to show that if f is a $\Delta_1^1(z)$ function which witnesses $E_{\omega_1} \leq_{a\Delta_1^1} F_{\omega_1}$, then z has an admissibility spectrum which is full of gaps relative to the set of all admissible ordinals. No constructible real (or even a real set generic over L) can have such a property.

The final section addresses a question of Sy-David Friedman using the techniques of the previous section. Essentially, the question is:

Question 6.3 Is it possible that the isomorphism relation of a counterexample to Vaught's conjecture is Δ_1^1 bireducible to F_{ω_1} ?

The main result of this final section is:

Theorem 6.9 *In L (and set generic extensions of L), no isomorphism relation of a counterexample to Vaught's conjecture can be Δ_1^1 reducible to F_{ω_1} .*

This yields a negative answer to Friedman's question in L and set generic extensions of L .

The author would like to acknowledge and thank Sy-David Friedman, Su Gao, Alexander Kechris, Andrew Marks, and Antonio Montalbán for very helpful discussions and comments about what appears in this paper.

2. ADMISSIBILITY AND INFINITARY LOGIC

The reader should refer to [2] for definitions and further details about admissibility.

Let $\dot{\in}$ denote a binary relation symbol. Let \mathcal{L} be a language such that $\dot{\in} \in \mathcal{L}$. $\text{KP}_{\mathcal{L}}$ denotes Kripke-Platek Set Theory in the language \mathcal{L} with $\dot{\in}$ serving as the distinguished membership symbol. The \mathcal{L} subscript will usually be concealed. $\text{KP} + \text{INF}$ is KP augmented with the axiom of infinity.

Definition 2.1. Let \mathcal{L} be a language containing $\dot{\in}$. A \mathcal{L} -structure $\mathcal{A} = (A, \dot{\in}^{\mathcal{A}}, \dots)$ is an *admissible set* if and only if $\mathcal{A} \models \text{KP}$, A is a transitive set, and $\dot{\in}^{\mathcal{A}} = \dot{\in} \upharpoonright A$.

If \mathcal{A} is an admissible set, then $o(\mathcal{A}) = A \cap \text{ON}$.

An ordinal α is an *admissible ordinal* if and only if there is an admissible set \mathcal{A} such that $\alpha = o(\mathcal{A})$. More generally, if $x \in {}^\omega 2$, an ordinal α is *x -admissible* if and only if there is an admissible set \mathcal{A} such that $x \in A$ and $\alpha = o(\mathcal{A})$.

The *admissibility spectrum* of x is $\Lambda(x) = \{\alpha : \alpha \text{ is an } x\text{-admissible ordinal}\}$.

Definition 2.2. For $x \in {}^\omega 2$, let $\mathcal{HYP}(x)$ denote the \subseteq -smallest admissible set containing x .

Definition 2.3. If $x \in {}^\omega 2$, let $\omega_1^x = \min(\Lambda(x))$.

Proposition 2.4. *The function $(\alpha, x) \rightarrow L_\alpha(x)$, where $\alpha \in \text{ON}$ and x is a set, is a Σ_1 function in KP . In fact, it is Δ_1 .*

Proof. See [2], Chapter II, Section 5 - 7. Also note that the function is defined on a Δ_1 set. □

Proposition 2.5. *If \mathcal{A} is an admissible set with $x \in A$ and $\alpha = o(\mathcal{A})$, then $L_\alpha(x)$ is an admissible set. In fact, $L_\alpha(x)$ is the \subseteq -smallest admissible set \mathcal{A} such that $x \in A$ and $o(\mathcal{A}) = \alpha$.*

In particular, if α is an x -admissible ordinal, then $L_\alpha(x)$ is an admissible set.

Proof. See [2], Theorem II.5.7. □

Proposition 2.6. *If $x \subseteq \omega$, then $\mathcal{HYP}(x) = L_{\omega_1^x}(x)$.*

Proof. See [2], Theorem II.5.9. □

Definition 2.7. Let $x \in {}^\omega 2$. Let $\text{HYP}^x = {}^\omega 2 \cap \mathcal{HYP}(x) = {}^\omega 2 \cap L_{\omega_1^x}(x)$. HYP^x is the set of all x -hyperarithmetical reals.

In particular, x -hyperarithmetical reals are exactly those reals that appear in all admissible sets containing x .

Next, the relevant aspects of first order infinitary logic and admissible fragments will be reviewed. The detailed formalization can be found in [2], Chapter III.

Definition 2.8. Let \mathcal{L} denote a first order language (a set of constant, relation, and function symbols). Fix a Δ_1 class $\{v_\alpha : \alpha \in \text{ON}\}$, which will represent variables. $\mathcal{L}_{\omega\omega}$ denotes the collection of finitary \mathcal{L} -formulas using variables from $\{v_i : i < \omega\}$. $\mathcal{L}_{\infty\omega}$ denotes the collection of all infinitary formulas with finitely many free variables.

Proposition 2.9. *In KP + INF, $\mathcal{L}_{\omega\omega}$ is a set. In KP, $\mathcal{L}_{\infty\omega}$ is a Δ_1 class.*

Proof. See [2], Proposition III.1.4 and page 81. □

Proposition 2.10. (KP) “ $\mathcal{M} \models_{\mathcal{L}} \varphi(\bar{x})$ ” as a relation on the language \mathcal{L} , \mathcal{L} -structure \mathcal{M} , infinitary \mathcal{L} -formula φ , and tuple \bar{x} of M is equivalent to a Δ_1 predicate.

Proof. See [2], pages 82-83. □

Definition 2.11. Let \mathcal{L} be a language. Let \mathcal{A} be an admissible set such that \mathcal{L} is Δ_1 definable in \mathcal{A} . The admissible fragment of $\mathcal{L}_{\infty\omega}$ given by \mathcal{A} , denoted $\mathcal{L}_{\mathcal{A}}$, is defined as

$$\mathcal{L}_{\mathcal{A}} = \{\varphi \in A : \varphi \in L_{\infty\omega}\} = \{\varphi \in A : \mathcal{A} \models \varphi \in L_{\infty\omega}\}$$

The last equivalence follows from Δ_1 absoluteness.

Definition 2.12. Let \mathcal{L} be a language consisting of a binary relation $\dot{\in}$. Let \mathcal{M} be a \mathcal{L} -structure such that $(M, \dot{\in}^{\mathcal{M}})$ satisfies extensionality. Define $\text{WF}(\mathcal{M})$ as the substructure consisting of the well-founded elements of M . $\text{WF}(\mathcal{M})$ is called the well-founded part of \mathcal{M} .

\mathcal{M} is called *solid* if and only if $\text{WF}(\mathcal{M})$ is transitive.

Remark 2.13. The notion of solid comes from Jensen’s [9]. Every structure has an isomorphic solid model that is obtained by Mostowski collapsing the well-founded part.

The notion of solidness is mostly a convenience: In our usage, $\omega \subseteq M$. Therefore, Mostowski collapsing will not change reals. Transitivity is desired due to the definition of admissibility and in order to apply familiar absoluteness results. Rather than having to repeatedly Mostowski collapse $\text{WF}(\mathcal{M})$ and mention reals are not moved, one will just assume the well-founded part is transitive by demanding \mathcal{M} is solid.

Lemma 2.14. (*Truncation Lemma*) *If $\mathcal{M} \models \text{KP}$, then $\text{WF}(\mathcal{M}) \models \text{KP}$. In particular, if \mathcal{M} is a solid model, then $\text{WF}(\mathcal{M})$ is an admissible set.*

Proof. See [2], II.8.4. □

The following is the central technique used in the paper:

Theorem 2.15. (*Solid Model Existence Theorem*) *Let \mathcal{A} be a countable admissible set. Let \mathcal{L} be a language which is Δ_1 definable over \mathcal{A} and contains a binary relation symbol $\dot{\in}$ and constant symbols \bar{a} for each $a \in A$. Let T be a consistent \mathcal{L} -theory in the countable admissible fragment $\mathcal{L}_{\mathcal{A}}$, be Σ_1 definable over \mathcal{A} , and contains the following:*

- (i) KP
- (ii) For each $a \in A$, the sentence $(\forall v)(v \dot{\in} \bar{a} \Leftrightarrow \bigvee_{z \in a} v = \bar{z})$.

Then there exists a solid \mathcal{L} -structure \mathcal{B} such that $\mathcal{B} \models T$ and $ON \cap B = ON \cap A$.

Proof. See [5] and also see [9], Section 4, Lemma 11. □

Theorem 2.16. (*Sacks’ Theorem*) *If $\alpha > \omega$ is an admissible ordinal, then there exists some $x \in \omega^2$ such that $\alpha = \omega_1^x$.*

Let $z \in \omega^2$. If $\alpha \in \Lambda(z) \cap \omega_1$, then there exists $y \in \omega^2$ with $\omega_1^y = \alpha$ and $z \leq_T y$.

Proof. See [18], Corollary 3.16. The following proof is similar to [9], Section 4, Lemma 10. The second statement will be proved below:

Since $\alpha \in \Lambda(z)$, let \mathcal{A} be an admissible set such that $z \in A$ and $o(\mathcal{A}) = \alpha$. (For example, $\mathcal{A} = L_{\alpha}(z)$ by Proposition 2.5.)

Let \mathcal{L} be a language consisting of the following:

- (I) A binary relation symbol $\dot{\in}$.
- (II) Constant symbols \bar{a} for each $a \in A$.
- (III) One other distinguished constant symbol \dot{c} .

The elements of \mathcal{L} can be appropriately coded as elements of A so that \mathcal{L} is Δ_1 definable over \mathcal{A} .

Let T be a theory in the countable admissible fragment $\mathcal{L}_{\mathcal{A}}$ consisting of the following:

- (i) KP
- (ii) For each $a \in A$, $(\forall v)(v \dot{\in} \bar{a} \Leftrightarrow \bigvee_{z \in a} v = \bar{z})$.

- (iii) $\dot{c} \subseteq \bar{\omega}$.
- (iv) For each ordinal $\sigma \in \alpha$, “ $\bar{\sigma}$ is not admissible relative to \dot{c} ”. More formally, “ $L_{\bar{\sigma}}(\dot{c}) \not\models \text{KP} + \text{INF}$ ”.
- (v) $\bar{z} \leq_T \dot{c}$.

T can be coded as a class in A in such a way that it is Σ_1 in \mathcal{A} . T is consistent: Find any $u \in {}^\omega 2$ which codes an ordinal greater than α . Let $c = u \oplus z$. Consider the following \mathcal{L} -structure \mathcal{M} : The universe M is H_{\aleph_1} . For each $a \in \mathcal{A}$, $\bar{a}^{\mathcal{M}} = a$. (Since A is countable and transitive, $A \in H_{\aleph_1}$.) $\dot{c}^{\mathcal{M}} = \dot{c}$. \mathcal{M} clearly satisfy (i), (ii), (iii), and (v). For (iv), suppose there is an $\sigma < \alpha$ such that $L_\sigma(c) \models \text{KP}$. Since $c \in L_\sigma(c)$ and $L_\sigma(c) \models \text{KP}$, $u \in L_\sigma(c)$ because $c = u \oplus z$. Since the Mostowski collapse map is a Σ_1 definable function in KP, if reals code binary relations in the usual way, then KP proves the existence of $\text{ot}(u)$. Thus $\text{ot}(u) \in L_\sigma(c)$. However, $\text{ot}(u) > \alpha > \sigma$. Contradiction. It has been shown that \mathcal{M} also satisfy (iv). T is consistent.

The Solid Model Existence Theorem (Theorem 2.15) implies there is a solid \mathcal{L} -structure $\mathcal{B} \models T$ such that $\text{ON} \cap B = \text{ON} \cap A = \alpha$. Let $y = \dot{c}^{\mathcal{B}}$. The claim is that $\omega_1^y = \alpha$. By Lemma 2.14, $\text{WF}(\mathcal{B})$ is an admissible set containing y and z . $o(\text{WF}(\mathcal{B})) = \text{ON} \cap \text{WF}(\mathcal{B}) = \text{ON} \cap B = \text{ON} \cap A = \alpha$. Thus $\omega_1^y \leq \alpha$. Now suppose that $\omega_1^y < \alpha$. In V , $L_{\omega_1^y}(y) \models \text{KP}$. Since the function $(\alpha, x) \mapsto L_\alpha(x)$ is Δ_1 (by Proposition 2.4) and the satisfaction relation is Δ_1 (by Proposition 2.10), by Δ_1 absoluteness between the transitive sets $\text{WF}(\mathcal{B})$ and V , one has $\text{WF}(\mathcal{B}) \models L_{\omega_1^y}(y) \models \text{KP}$. Again by absoluteness of Δ_1 formulas between the transitive (in the sense of \mathcal{B}) sets $\text{WF}(\mathcal{B})$ and \mathcal{B} , $\mathcal{B} \models L_{\omega_1^y}(y) \models \text{KP}$. Letting $\sigma = \omega_1^y < \alpha$, $\mathcal{B} \models L_{\bar{\sigma}}(\dot{c}) \models \text{KP}$. This contradicts $\mathcal{B} \models T$. A similar absoluteness argument shows that $z \leq_T y$. \square

Remark 2.17. This proof of Sacks theorem is the basic template for several other arguments throughout the paper. This proof will be frequently referred.

Next, various aspects of the Scott analysis will be reviewed. Since there are some minor variations among the definitions of Scott rank, Scott sentences, canonical Scott sentences, etc., these will be provided below. See [16], page 57-60 or [17] for more information.

Definition 2.18. Let \mathcal{L} be a language. Define the binary relation $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{N}, \underline{b})$ where $\alpha \in \text{ON}$, $\underline{a} \in {}^{<\omega} M$, and $\underline{b} \in {}^{<\omega} N$ as follows:

- (i) $(\mathcal{M}, \underline{a}) \sim_0 (\mathcal{N}, \underline{b})$ if and only if for all atomic \mathcal{L} -formulas φ , $\mathcal{M} \models \varphi(\underline{a})$ if and only if $\mathcal{N} \models \varphi(\underline{b})$.
- (ii) If α is a limit ordinal, then $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{N}, \underline{b})$ if and only if for all $\beta < \alpha$, $(\mathcal{M}, \underline{a}) \sim_\beta (\mathcal{N}, \underline{b})$.
- (iii) If $\alpha = \beta + 1$, then $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{N}, \underline{b})$ if and only if for all $c \in M$, there exists a $d \in N$ such that $(\mathcal{M}, \underline{a}, c) \sim_\beta (\mathcal{N}, \underline{b}, d)$ and for all $d \in N$, there exists a $c \in M$ such that $(\mathcal{M}, \underline{a}, c) \sim_\beta (\mathcal{N}, \underline{b}, d)$.

Let \mathcal{M} be a \mathcal{L} -structure and $\underline{a} \in {}^k M$ for some $k \in \omega$. For $\alpha \in \text{ON}$, the $\mathcal{L}_{\infty\omega}$ formula $\Phi_{\underline{a}, \alpha}^{\mathcal{M}}(\underline{v})$ (in variables \underline{v} such that $|\underline{v}| = k$) is defined as follows:

- (I) Let X be the set of all atomic and negation atomic \mathcal{L} -formulas with free variables \underline{v} such that $|\underline{v}| = k$ which holds of \underline{a} . Let $\Phi_{\underline{a}, 0}^{\mathcal{M}}(\underline{v}) = \bigwedge X$.
- (II) If α is a limit ordinal, let $X = \{\Phi_{\underline{a}, \beta}^{\mathcal{M}}(\underline{v}) : \beta < \alpha\}$. Let $\Phi_{\underline{a}, \alpha}^{\mathcal{M}}(\underline{v}) = \bigwedge X$.
- (III) If $\alpha = \beta + 1$, then let $X = \{(\exists w)\Phi_{\underline{a}, \beta}^{\mathcal{M}}(\underline{v}, w) : b \in M\}$ and $Y = \{\Phi_{\underline{a}, \beta}^{\mathcal{M}}(\underline{v}, w) : b \in M\}$. Then let $\Phi_{\underline{a}, \alpha}^{\mathcal{M}}(\underline{v}) = \bigwedge X \wedge (\forall w) \bigvee Y$.

For \mathcal{M} , a \mathcal{L} -structure, and $\underline{a} \in {}^k M$ (for some k), define $\rho(\mathcal{M}, \underline{a})$ to be the least $\alpha \in \text{ON}$ such that for all $\underline{b} \in {}^k M$, $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{M}, \underline{b})$ if and only if for all β , $(\mathcal{M}, \underline{a}) \sim_\beta (\mathcal{M}, \underline{b})$.

Define $\text{SR}(\mathcal{M}) = \sup\{\rho(\mathcal{M}, \underline{a}) + 1 : \underline{a} \in {}^{<\omega} M\}$. Define $\text{R}(\mathcal{M}) = \sup\{\rho(\mathcal{M}, \underline{a}) : \underline{a} \in {}^{<\omega} M\}$.

Let $\alpha = \text{R}(\mathcal{M})$. Let

$$X = \{(\forall \bar{v})(\Phi_{\underline{a}, \alpha}^{\mathcal{M}}(\underline{v}) \Rightarrow \Phi_{\underline{a}, \alpha+1}^{\mathcal{M}}(\underline{v})) : \underline{a} \in {}^{<\omega} M\}$$

$$\text{CSS}(\mathcal{M}) = \Phi_{\emptyset, \alpha}^{\mathcal{M}} \wedge \bigwedge X$$

$\text{CSS}(\mathcal{M})$ is the *canonical Scott sentence* of \mathcal{M} . $\text{SR}(\mathcal{M})$ is the *Scott rank* of \mathcal{M} .

The following are well-known results. Usually, a careful inspection of the proof indicates what can be done in $\text{KP} + \text{INF}$ or ZFC .

Proposition 2.19. *The relation \sim is equivalent to a Δ_1 formula over $\text{KP} + \text{INF}$.*

Proof. It can be defined by Σ -recursion. \square

Proposition 2.20. *Let \mathcal{A} be an admissible set such that $\mathcal{A} \models \text{INF}$. Let $\mathcal{L} \in A$ be a language. Let $\mathcal{M} \in A$ range over \mathcal{L} -structure, \underline{a} range over elements of ${}^{<\omega}M$, and α range over $ON \cap A$. Then the function $f(\mathcal{M}, \underline{a}, \alpha) = \Phi_{\underline{a}, \alpha}^{\mathcal{M}}(\underline{v})$ is Δ_1 definable in \mathcal{A} .*

In particular, if $R(\mathcal{M}) \in A$, then $\text{CSS}(\mathcal{M}) \in A$.

Proof. It can be defined by Σ -recursion. \square

Proposition 2.21. *(KP + INF) Let \mathcal{L} be a language. Let \mathcal{M} and \mathcal{N} be \mathcal{L} -structures, $\underline{a} \in {}^k M$, $\underline{b} \in {}^k N$ for some $k \in \omega$, and $\alpha \in ON$. Then $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{N}, \underline{b})$ if and only if $\mathcal{N} \models \Phi_{\underline{a}, \alpha}^{\mathcal{M}}(\underline{b})$.*

Proof. This is proved by induction. See [16], Lemma 2.4.13. \square

Proposition 2.22. *(KP + INF) If $(M, \underline{a}) \equiv_{\mathcal{L}_{\infty\omega}} (N, \underline{b})$, then for all α , $(M, \underline{a}) \sim_\alpha (N, \underline{b})$.*

Proof. $\mathcal{M} \models \varphi_{\underline{a}, \alpha}(\underline{a})$. So $\mathcal{N} \models \varphi_{\underline{a}, \alpha}(\underline{b})$. By Proposition 2.21, $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{N}, \underline{b})$. \square

Definition 2.23. Let \mathcal{L} be a language. Let φ be a formula of $L_{\infty\omega}$. The *quantifier rank* of φ , denoted $\text{qr}(\varphi)$, is defined as follows:

- (i) $\text{qr}(\varphi) = 0$ if φ is an atomic formula.
- (ii) $\text{qr}(\neg\varphi) = \text{qr}(\varphi)$.
- (iii) $\text{qr}(\bigwedge X) = \text{qr}(\bigvee X) = \sup\{\text{qr}(\psi) : \psi \in X\}$.
- (iv) $\text{qr}(\exists v\varphi) = \text{qr}(\forall v\varphi) = \text{qr}(\varphi) + 1$.

Proposition 2.24. *The relation “ $\text{qr}(\varphi) = \alpha$ ” is Δ_1 definable in KP + INF.*

Proof. It can be defined by Σ -recursion. \square

Proposition 2.25. *(KP + INF) Let \mathcal{L} be a language. \mathcal{M}, \mathcal{N} be \mathcal{L} -structures. $\underline{a} \in {}^k M$ and $\underline{b} \in {}^k N$ for some $k \in \omega$. Then for all $\alpha \in ON$, $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{N}, \underline{b})$ if and only if for all φ with $\text{qr}(\varphi) \leq \alpha$, $\mathcal{M} \models \varphi(\underline{a})$ if and only if $\mathcal{N} \models \varphi(\underline{b})$.*

Proof. This is proved by induction. \square

Proposition 2.26. *(ZF) Let \mathcal{L} be some language. Let \mathcal{M} and \mathcal{N} be \mathcal{L} -structures. Suppose \mathcal{A} is an admissible set with $\mathcal{L}, \mathcal{M}, \mathcal{N} \in A$. Then $\mathcal{A} \models \mathcal{M} \equiv_{\mathcal{L}_{\infty\omega}} \mathcal{N}$ if and only if $\mathcal{M} \equiv_{\mathcal{L}_{\infty\omega}} \mathcal{N}$.*

Proof. See [17], Theorem 1.3. \square

Remark 2.27. A common phenomenon is that certain properties are reflected between appropriate admissible sets and the true universe. A useful observation is that if such a property holds from the point of view of an admissible set then it is true in the universe. The above proposition asserts that infinitary elementary equivalence is such a property.

Another familiar example is the effective boundedness theorem. Suppose $\varphi : \text{WO} \rightarrow \omega_1$ is a Π_1^1 rank. Let $B \subseteq \text{WO}$ be Σ_1^1 . Let \mathcal{A} be a countable admissible set containing the parameters used to define B . Inside of \mathcal{A} , $\varphi(B)$ is bounded by $o(\mathcal{A})$. A priori, the true bound on $\varphi(B)$ may be higher as the true universe has more countable ordinals and more members of B . However, the effective boundedness theorem asserts that in fact, in the true universe, $\varphi(B)$ is bounded by $o(\mathcal{A})$.

The following proposition with an included proof shows countable admissible sets can also be used to produce true bounds on the Scott rank.

Proposition 2.28. *Let \mathcal{L} be a countable language and \mathcal{M} be a countable \mathcal{L} -structure. One may identify \mathcal{M} as a real by associating it with an isomorphic structure on ω . If \mathcal{A} is an admissible set with $\mathcal{L}, \mathcal{M} \in A$, then $R(\mathcal{M}) \leq ON \cap A$. In particular, $R(\mathcal{M}) \leq \omega_1^{\mathcal{M}}$ and $SR(\mathcal{M}) \leq \omega_1^{\mathcal{M}} + 1$.*

Proof. See [17], Corollary 1.

Fix an admissible set \mathcal{A} as above. Let $\gamma = ON \cap A$. It suffices to show that $R(\mathcal{M}) \leq \gamma$. Suppose not. Then there exists \underline{a} and \underline{b} such that for all $\alpha < \gamma$, $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{M}, \underline{b})$ but $(\mathcal{M}, \underline{a}) \not\sim_\beta (\mathcal{M}, \underline{b})$ for some $\beta > \gamma$. By Δ_1 -absoluteness and Proposition 2.25, $\mathcal{A} \models (\mathcal{M}, \underline{a}) \equiv_{\mathcal{L}_{\infty\omega}} (\mathcal{M}, \underline{b})$. Thus by Proposition 2.26, $(\mathcal{M}, \underline{a}) \equiv_{\mathcal{L}_{\infty\omega}} (\mathcal{M}, \underline{b})$. However, $(\mathcal{M}, \underline{a}) \not\sim_\beta (\mathcal{M}, \underline{b})$ implies $\mathcal{M} \models \Phi_{\underline{a}, \beta}^{\mathcal{M}}(\underline{a})$ and $\mathcal{M} \not\models \Phi_{\underline{a}, \beta}^{\mathcal{M}}(\underline{b})$ by Proposition 2.21. This shows $(\mathcal{M}, \underline{a}) \not\equiv_{\mathcal{L}_{\infty\omega}} (\mathcal{M}, \underline{b})$. Contradiction. \square

Definition 2.29. Let \mathcal{L} be a language. Let \mathcal{M} be a \mathcal{L} -structure. φ is a Scott sentence if and only if for all \mathcal{L} -structures \mathcal{N} and \mathcal{M} , $\mathcal{N} \models \varphi$ and $\mathcal{M} \models \varphi$ implies $\mathcal{M} \equiv_{\mathcal{L}_{\infty\omega}} \mathcal{N}$.

Theorem 2.30. (ZFC) Let \mathcal{L} be a language. Let \mathcal{M} be a countable \mathcal{L} -structure. Then there exists a $L_{\infty\omega}$ -sentence φ such that for all countable \mathcal{L} -structure \mathcal{N} , $\mathcal{N} \cong_{\mathcal{L}} \mathcal{M}$ if and only if $\mathcal{N} \models \varphi$. In fact, φ can be chosen to be $CSS(\mathcal{M})$.

(KP + INF) If φ is a Scott sentence for a countable structure \mathcal{M} , then for all countable \mathcal{N} , $\mathcal{N} \models \varphi$ if and only if $\mathcal{N} \cong_{\mathcal{L}} \mathcal{M}$.

Proof. Observe the first statement asserts that there exists a sentence such that whenever a countable structure satisfies this sentence, there exists an isomorphism between it and \mathcal{M} . The existence of this sentence requires working beyond KP + INF. The second statement asserts that KP + INF can prove that if a Scott sentence happens to exist, then for any countable structure satisfying this sentence, there is an isomorphism between it and \mathcal{M} .

This is the Scott's isomorphism theorem. See [16], Theorem 2.4.15 for a proof. The results in KP + INF follows essentially the same proof with the assistance of some of the above propositions proved in KP + INF. \square

Definition 2.31. Let \mathcal{L} be a countable language. Let $S(\mathcal{L})$ denote the set of all \mathcal{L} -structures on ω .

Definition 2.32. Let $\dot{\in}$ be a binary relation symbol. Let S^* denote the subset of $S(\{\dot{\in}\})$ consisting of ω -models of KP + INF.

Proposition 2.33. Let $\{\phi_e : e \in \omega\}$ be a recursive enumeration of $\{\dot{\in}\}_{\omega\omega}$ -formulas. The relation on $x \in S(\{\dot{\in}\})$ and $e \in \omega$ asserting " $x \models \phi_e$ " is Δ_1^1 .

Also S^* is Δ_1^1 .

Proof. See [13], pages 14-16 for relevant definitions and proofs. \square

Remark 2.34. One can check that there is a Δ_1^1 function such that, given $A \in S^*$ and $n \in \omega$, the function gives the element of A which A thinks is n . Using this, one can determine in a Δ_1^1 way whether $A \in S^*$ thinks some $x \in \omega^2$ exists. In the following, if $A \in S^*$ and $x \in \omega^2$, the sentence " $x \in A$ " should be understood as this informally described Δ_1^1 relation.

Proposition 2.35. Let \mathcal{L} be a recursive language. Let $\varphi \in \mathcal{HYP}(x) \cap \mathcal{L}_{\infty\omega}$. Then $Mod(\varphi) = \{s \in S(\mathcal{L}) : s \models_{\mathcal{L}} \varphi\}$ is $\Delta_1^1(x)$.

Proof. Note that $s \in Mod(\varphi)$ if and only if

$$(\exists A)(A \in S^* \wedge x \in A \wedge s \in A \wedge A \models s \models_{\mathcal{L}} \varphi)$$

if and only if

$$(\forall A)(A \in S^* \wedge x \in A \wedge s \in A) \Rightarrow A \models s \models_{\mathcal{L}} \varphi$$

These equivalences are established using the absoluteness of satisfaction. This shows that $Mod(\varphi)$ is $\Delta_1^1(x)$. \square

Remark 2.36. Later, the paper will be concerned with relating countable admissible sets and isomorphism of countable structures. The second statement of Theorem 2.30 captures the essence of these types of arguments: Isomorphism of countable structures is reflected between the true universe and admissible sets which witness the countability of the relevant structures and possesses a Scott sentence for these structures.

The original arguments for some results of this paper used more directly the second statement of Theorem 2.30. The argument presented below is simpler using the Scott isomorphism theorem and Proposition 2.35 but may conceal this essential idea.

Now to introduce the main equivalence relation of this paper:

Definition 2.37. Let F_{ω_1} be the equivalence relation defined on ω^2 by $x F_{\omega_1} y$ if and only if $\omega_1^x = \omega_1^y$. F_{ω_1} is a Σ_1^1 equivalence relation with all classes Δ_1^1 .

The first claim from the above definition is well known and follows easily from the characterization of ω_1^x as the supremum of the x -recursive ordinals. The next proposition implies each class is Δ_1^1 . There will be much to say later about the complexity of each F_{ω_1} -equivalence class.

Proposition 2.38. *Let α be a countable admissible ordinal and $z \in {}^\omega 2$ be such that $\alpha < \omega_1^z$. Then the set $\{y \in {}^\omega 2 : \omega_1^y = \alpha\}$ is $\Delta_1^1(z)$.*

Proof. If u and v are reals coding linear orderings on ω , then $u \preceq v$ means there exists an order preserving injective function f from the linear ordering coded by u to the linear ordering coded by v . \preceq is a Σ_1^1 relation in the variables u and v .

Since $\alpha < \omega_1^z$, there exists some $e \in \omega$ such that $\{e\}^z$ is the characteristic function of a well-ordering isomorphic to α . Let $B = \{y \in {}^\omega 2 : \alpha = \omega_1^y\}$. Then

$$y \in B \Leftrightarrow (\forall n) \left((\{n\}^y \in \text{WO} \Rightarrow \{n\}^y \preceq \{e\}^z) \right) \wedge \\ (\forall k) (\exists j) (\{j\}^y \preceq \{e\}^z \wedge \{e\}^z \upharpoonright k \preceq \{j\}^y)$$

B is $\Sigma_1^1(z)$. Also

$$y \notin B \Leftrightarrow (\exists j) (\forall n) (\{n\}^y \in \text{WO} \Rightarrow (\{n\}^y \preceq \{e\}^z \upharpoonright j) \vee (\exists n) (\{n\}^y \in \text{LO} \wedge \{n\}^y \preceq \{e\}^z \wedge \{e\}^z \preceq \{n\}^y))$$

B is $\Pi_1^1(z)$. Hence B is $\Delta_1^1(z)$. \square

3. CLASSIFIABLE BY COUNTABLE STRUCTURES

Definition 3.1. Let $x \in {}^\omega 2$. A linear ordering R on ω is an x -recursive x -pseudo-wellordering if and only if R is an x -recursive linear ordering on ω which is not a wellordering but $L_{\omega_1^x}(x) \models R$ is a wellordering, i.e. R has no x -hyperarithmetical descending sequences.

Proposition 3.2. (Harrison, Kleene) *For all $x \in {}^\omega 2$, there exists an x -recursive x -pseudo-wellordering.*

Proof. See [10] or [20], III.2.1. A generalized form of this construction will be used below.

This can also be proved using Theorem 2.15 and infinitary logic in admissible fragments. In the application of Theorem 2.15, Barwise compactness is used to show the consistency of the appropriate theory in the countable admissible fragment. See Nadel's proof given in [1] VIII, Section 5.7 for more details. \square

The following characterizes the order type of x -recursive x -pseudo-wellorderings:

Theorem 3.3. (Harrison) *Let R be a x -recursive x -pseudo-wellordering, then $ot(R) = \omega_1^x(1 + \eta) + \rho$ where $\eta = ot(\mathbb{Q})$ and $\rho < \omega_1^x$.*

Proof. See [8] or [20], Lemma III.2.2. \square

Proposition 3.4. *Recall if $y \in {}^\omega 2$, then $\text{HYP}^y = L_{\omega_1^y}(y) \cap {}^\omega 2$, the set of y -hyperarithmetical reals.*

The relation $x \in \text{HYP}^y$ is a Π_1^1 relation in the variable x and y .

Proof. This result is classical. A proof is given to illustrate some ideas from ω -models of KP.

The claim is that:

$$x \in \text{HYP}^y \Leftrightarrow (\forall A) ((A \in S^* \wedge y \in A) \Rightarrow (x \in A))$$

See Remark 2.34 about what “ $y \in A$ ” should precisely mean. The latter part of the equivalence is Π_1^1 . Hence the result follows from the claim.

To prove the claim:

(\Rightarrow) Suppose $A \in S^*$. Let $n \in \omega$ be the representative of y in A . Since $A \models \text{KP}$, by Lemma 2.14 (Truncation Lemma), $\text{WF}(A) \models \text{KP}$. Let π be the Mostowski collapse of $\text{WF}(A)$ onto an admissible set B . $y \in B$ since $y = \pi(n)$. Since $x \in \text{HYP}^y$, x is in every admissible set containing y . $x \in B$. Then $\pi^{-1}(x)$ represents x in A .

(\Leftarrow) Recall $\mathcal{HYP}(y)$ is the smallest admissible set containing x and ω . The domain of $\mathcal{HYP}(y)$ is $L_{\omega_1^y}(y)$. It is countable. Let $\pi : L_{\omega_1^y}(y) \rightarrow \omega$ be any bijection. The bijection gives an element $A \in S^*$ isomorphic to $\mathcal{HYP}(y)$. $\pi(y)$ represents y in A . There exists some $n \in \omega$ such that n represents x in A , by the hypothesis. Then $x \in L_{\omega_1^y}(y)$ since $x = \pi^{-1}(n)$. $x \in \text{HYP}^y$. \square

The following propositions use the ideas from [20] III.1 and III.2.

Proposition 3.5. *There exists a recursive tree U on $2 \times \omega$ such that for all $x \in {}^\omega 2$, U^x has a path but has no x -hyperarithmetical paths.*

Proof. By Proposition 3.4, there is a recursive tree V on $2 \times 2 \times \omega$ such that $x \notin \text{HYP}^y$ if and only if $V^{(x,y)}$ is ill-founded. Define the relation Φ on ${}^\omega 2 \times {}^\omega \omega$ by

$$\Phi(y, f) \Leftrightarrow (\forall n)((f_0(n) = 0 \vee f_0(n) = 1) \wedge V(f_0 \upharpoonright n, y \upharpoonright n, f_1 \upharpoonright n))$$

where $f_i(n) = f(\langle i, n \rangle)$, for $i = 0, 1$. Φ is Π_1^0 . Let U be a recursive tree on $2 \times \omega$ such that

$$\Phi(y, f) \Leftrightarrow (\forall n)((y \upharpoonright n, f \upharpoonright n) \in U)$$

For any y , if U^y has a path f , then $\Phi(y, f)$. Therefore, $f_1 \in [V^{(f_0, y)}]$. $f_0 \notin \text{HYP}^y$. So U^y can not have a y -hyperarithmic path f , since otherwise $f_0 \in \text{HYP}^y$, which yields a contradiction. U^y has a path: Let x be any real which is not in HYP^y . $[V^{(x, y)}]$ is non-empty. Let $g \in [V^{(x, y)}]$. Let f be such that $f_0 = x$ and $f_1 = g$. Then $\Phi(y, f)$. $f \in [U^y]$. \square

Definition 3.6. The Kleene-Brouwer ordering $<_{\text{KB}}$ is defined on ${}^{<\omega}\omega$ as follows: $s <_{\text{KB}} t$ if and only if

(i) $t \preceq s$ and $|t| < |s|$

or

(ii) If there exists an $n \in \omega$ such that for all $k < n$, $s(k) = t(k)$ and $s(n) < t(n)$.

Proposition 3.7. *Let T be a tree on ω . T is wellfounded if and only if $<_{\text{KB}} \upharpoonright T$ is wellfounded. Moreover, if there is an x -hyperarithmic infinite descending sequence in $<_{\text{KB}} \upharpoonright T$, then there is an x -hyperarithmic path through T .*

Proof. If $f \in [T]$, then $\{f \upharpoonright n : n \in \omega\}$ is an infinite descending sequence in $<_{\text{KB}} \upharpoonright T$.

Let $S = \{s_n \in {}^{<\omega}2 : n \in \omega\}$ be an x -hyperarithmic descending sequence in $<_{\text{KB}} \upharpoonright T$. Define $f \in {}^\omega \omega$ by

$$f(n) = i \Leftrightarrow (\exists p)(\forall q \geq p)(s_q(n) = i)$$

$f \in [T]$ and f is $\Sigma_2^0(S)$. f is also x -hyperarithmic. \square

Now to produce a classification of F_{ω_1} by countable structures. The idea will be to send x to an x -Harrison linear ordering. Using Proposition 3.5 and applying the Kleene-Brouwer ordering, one can obtain a function g such that $g(x)$ is an x -recursive x -pseudo-wellordering. Now suppose $\omega_1^x = \omega_1^y$. Let α denote this admissible ordinal. By Theorem 3.3, $\text{ot}(g(x)) = \alpha(1 + \eta) + \rho_x$ and $\text{ot}(g(y)) = \alpha(1 + \eta) + \rho_y$, where $\rho_x < \alpha$ and $\rho_y < \alpha$. However, it could happen that $\rho_x \neq \rho_y$. One way to modify g to get a classification of F_{ω_1} would be to “cut off” the recursive tail of $g(x)$. To do this, one uses a trick, suggested Montalbán, to cut off the recursive tail of the order type by taking a product of ω copies of $g(x)$. The details follow:

Proposition 3.8. *Fix $x \in {}^\omega 2$. Let $\rho < \omega_1^x$ and $\eta = \text{ot}(\mathbb{Q})$. Then $(\omega_1^x(1 + \eta) + \rho)\omega = \omega_1^x(1 + \eta)$.*

Proof. Let P be any x -recursive x -pseudo-wellorderings of order type $\omega_1^x(1 + \eta) + \rho$. Let $P \times \omega$ be the x -recursive structure isomorphic to ω copies of P following each other. $P \times \omega$ is still an x -recursive x -pseudo-wellordering. It has no x -recursive tail. By Theorem 3.3, $\text{ot}(P \times \omega) = (\omega_1^x(1 + \eta) + \rho)\omega = \omega_1^x(1 + \eta)$. \square

Proposition 3.9. *There exists an $e \in \omega$ such that for all $x \in {}^\omega 2$, $\{e\}^x$ is isomorphic to $(<_{\text{KB}} \upharpoonright U^x) \cdot \omega$, where U comes from Proposition 3.5.*

Proof. This is basic recursion theory using the previous results. \square

Theorem 3.10. (Montalbán) *The equivalence relation F_{ω_1} is classifiable by countable structures. In fact, there is an $e \in \omega$ such that $f(x) = \{e\}^x$ is the desired classification.*

Proof. Let $\mathcal{L} = \{\dot{R}\}$, where \dot{R} is a binary relation symbol. F_{ω_1} will be classified by countable \mathcal{L} -structures. U^x is an x -hyperarithmic tree with paths but no x -hyperarithmic path. Hence $<_{\text{KB}} \upharpoonright U^x$ is an x -recursive linear ordering with infinite descending sequences but no x -hyperarithmic infinite descending sequences. So $<_{\text{KB}} \upharpoonright U^x$ is an x -recursive x -pseudo-wellordering. It has order type $\omega_1^x(1 + \eta) + \rho$ for some $\rho < \omega_1^x$. Therefore, $(<_{\text{KB}} \upharpoonright U^x) \cdot \omega$ has order type $\omega_1^x(1 + \eta)$, i.e., it is an x -Harrison linear ordering. Hence $x F_{\omega_1} y$ if and only if $\omega_1^x = \omega_1^y$ if and only if $\omega_1^x(1 + \eta) = \omega_1^y(1 + \eta)$ if and only if $(\leq_{\text{KB}} \upharpoonright U^x) \cdot \omega \cong_{\mathcal{L}} (\leq_{\text{KB}} \upharpoonright U^y) \cdot \omega$ if and only if $\{e\}^x \cong_{\mathcal{L}} \{e\}^y$. This gives a classification of F_{ω_1} . \square

4. FINER ASPECTS OF CLASSIFICATION BY COUNTABLE STRUCTURES

The previous section provided an explicit classification $f : {}^\omega 2 \rightarrow S(\mathcal{L})$ which was Δ_1^1 and for all $x \in {}^\omega 2$, $\text{SR}(f(x)) = \omega_1^x + 1$. This section will show that any classification of F_{ω_1} by countable structures must have a similar property.

The next result will calculate the complexity of each F_{ω_1} class according to effective descriptive set theory.

Theorem 4.1. *For any $x \in {}^\omega 2$, $[x]_{F_{\omega_1}}$ is not $\Pi_1^1(x)$.*

Proof. Suppose $[x]_{F_{\omega_1}}$ is $\Pi_1^1(x)$. Let $B = {}^\omega 2 - [x]_{F_{\omega_1}}$. B is then $\Sigma_1^1(x)$. Let U be a tree on $2 \times \omega$ recursive in x such that

$$y \in B \Leftrightarrow [U^y] \neq \emptyset$$

Observe $U \in L_{\omega_1^x}(x)$.

Let \mathcal{L} be the language consisting of the following:

- (I) A binary relation symbol $\dot{\in}$.
- (II) Constant symbol \bar{a} for each $a \in L_{\omega_1^x}(x)$.
- (III) Two other distinguished constant symbols \dot{c} and \dot{d} .

\mathcal{L} can be considered a Δ_1 definable subset of $L_{\omega_1^x}(x)$.

Let T be a theory in the countable admissible fragment $\mathcal{L}_{L_{\omega_1^x}(x)}$ consisting of the following:

- (i) KP
- (ii) For each $a \in L_{\omega_1^x}(x)$, $(\forall v)(v \dot{\in} \bar{a} \Leftrightarrow \bigvee_{z \in a} v = \bar{z})$.
- (iii) $\dot{c} \subseteq \bar{\omega}$ and $\dot{d} : \bar{\omega} \rightarrow \bar{\omega}$.
- (iv) For each ordinal $\sigma \in \omega_1^x$, " $\bar{\sigma}$ is not admissible relative to \dot{c} ".
- (v) $\dot{d} \in [\bar{U}^{\dot{c}}]$.

T can be considered a Σ_1 definable theory in $L_{\omega_1^x}(x)$.

T is consistent: Find any $y \in {}^\omega 2$ such that $\omega_1^y > \omega_1^x$. Then $y \in B$. There exists some $z \in {}^\omega \omega$ such that $z \in [U^y]$. Consider the \mathcal{L} -structure \mathcal{M} defined as follows: $M = H_{\aleph_1}$. $\dot{\in}^{\mathcal{M}} = \dot{\in} \upharpoonright H_{\aleph_1}$. For each $a \in L_{\omega_1^x}(x)$, let $\bar{a}^{\mathcal{M}} = a$. Let $\dot{c}^{\mathcal{M}} = y$ and $\dot{d}^{\mathcal{M}} = z$. $\mathcal{M} \models T$.

By Theorem 2.15, T has a solid model \mathcal{N} such that $\text{ON} \cap N = \text{ON} \cap L_{\omega_1^x}(x) = \omega_1^x$. Let $u = \dot{c}^{\mathcal{N}}$ and $v = \dot{d}^{\mathcal{N}}$. As in Theorem 2.16, $\omega_1^u = \omega_1^x$.

$\mathcal{N} \models v \in [U^u]$. By Δ_1 absoluteness, $\text{WF}(\mathcal{N}) \models v \in [U^u]$. Since \mathcal{N} is solid, $\text{WF}(\mathcal{N})$ is transitive as viewed in V . So by Δ_1 absoluteness, $V \models v \in [U^u]$. $[U^u] \neq \emptyset$. $u \in B$. $\omega_1^u \neq \omega_1^x$. Contradiction. \square

Suppose f is a classification of F_{ω_1} by countable structures in some recursive language. The Scott rank of the image of f must be high:

Theorem 4.2. *Let \mathcal{L} be a recursive language. If $f : {}^\omega 2 \rightarrow S(\mathcal{L})$ is a $\Delta_1^1(z)$ function such that $x F_{\omega_1} y$ if and only if $f(x) \cong_{\mathcal{L}} f(y)$, then for all x such that $\omega_1^x \in \Lambda(z)$, $\text{SR}(f(x)) \geq \omega_1^x$.*

Proof. Suppose there exists an $x \in {}^\omega 2$ with $\omega_1^x \in \Lambda(z)$ and $\text{SR}(f(x)) < \omega_1^x$. Let $\alpha = \omega_1^x$. By Proposition 2.16, there exists a y with $z \leq_T y$ and $\omega_1^y = \alpha$. Since $\omega_1^y = \alpha = \omega_1^x$, $x F_{\omega_1} y$. This implies that $f(x) \cong_{\mathcal{L}} f(y)$. Hence $\text{SR}(f(y)) = \text{SR}(f(x)) < \omega_1^x = \alpha = \omega_1^y$. $z \leq_T y$ implies that $z \in L_{\omega_1^y}(y)$, and in particular, z is in every admissible set containing y . Since f is $\Delta_1^1(z)$, $f(y)$ is $\Delta_1^1(z, y) = \Delta_1^1(y)$ since $z \leq_T y$. $f(y)$ is hyperarithmetic in y . $f(y)$ is in every admissible set that has y as a member. Since $\text{SR}(f(y)) < \omega_1^y$ and $f(y)$ is in every admissible set containing y , $\text{CSS}(f(y))$ is in every admissible set containing y . In particular $\text{CSS}(f(y)) \in L_{\omega_1^y}(y)$.

By Proposition 2.35, $\text{Mod}(\text{CSS}(f(y)))$ is $\Delta_1^1(y)$. Therefore,

$$v \in [y]_{F_{\omega_1}} \Leftrightarrow f(v) \in \text{Mod}(\text{CSS}(f(y)))$$

which is $\Delta_1^1(y, z) = \Delta_1^1(y)$ since $z \leq_T y$. This contradicts Theorem 4.1. \square

Remark 4.3. Let f be $\Delta_1^1(z)$ as above. For all $y \in [x]_{F_{\omega_1}}$, there is an ordinal α such that $\text{SR}(f(y)) = \alpha$. The previous result states that if $\omega_1^x \in \Lambda(z)$, then the Scott rank of $f(x)$ is greater than or equal to ω_1^x . So $\alpha \geq \omega_1^x$. Since $\omega_1^x \in \Lambda(z)$, by Theorem 2.16, there is an $x' \in {}^\omega 2$ such that $\omega_1^{x'} = \omega_1^x$ and $z \leq_T x'$. Then $f(x')$

is $\Delta_1^1(x', z) = \Delta_1^1(x')$. By Lemma 2.28, $\text{SR}(f(x')) \leq \omega_1^{x'} + 1 = \omega_1^x + 1$. So one has that $\omega_1^x \leq \alpha \leq \omega_1^x + 1$. One may ask if α must take the largest possible value.

Using the methods of infinitary logic as above, there is one obvious idea to try in order to force the Scott rank to be as high as possible:

Let \mathcal{J} be a countable recursive language. Suppose $f : \omega^2 \rightarrow S(\mathcal{J})$ is a $\Delta_1^1(z)$ function such that $x F_{\omega_1} y$ if and only if $f(x) \cong_{\mathcal{J}} f(y)$.

Since f is $\Delta_1^1(z)$, it is $\Sigma_1^1(z)$. There is a tree U on $2 \times 2 \times \omega$ recursive in z such that $(a, b) \in f$ if and only if $[U^{(a,b)}] \neq \emptyset$. Again, one may assume $x \geq_T z$: since one can find an x' with $\omega_1^{x'} = \omega_1^x$ and $x' \geq_T z$. This implies $\text{SR}(f(x')) = \text{SR}(f(x))$.

Let \mathcal{L} be the language consisting of the following:

- (I) A binary relation symbol $\dot{\in}$.
- (II) Constant symbols \bar{a} for each $a \in L_{\omega_1^x}(x)$.
- (III) Four distinguished constant symbols \dot{c} , \dot{d} , \dot{e} , and \dot{s} .

Let T be a theory in the countable admissible fragment $\mathcal{L}_{L_{\omega_1^x}(x)}$ consisting of the following:

- (i) KP
- (ii) For each $a \in L_{\omega_1^x}(x)$, $(\forall v)(v \dot{\in} \bar{a} \Leftrightarrow \bigvee_{u \in a} v = \bar{u})$.
- (iii) $\dot{c} \subseteq \bar{\omega}$, $\dot{d} \subseteq \bar{\omega}$, $\dot{e} : \bar{\omega} \rightarrow \bar{\omega}$, and $\dot{s} \dot{\in} <^{\bar{\omega}} \bar{\omega}$.
- (iv) “ $\bar{\alpha}$ is not admissible in \dot{c} ” for each $\alpha < \omega_1^x$.
- (v) $\dot{e} \in [\bar{U}^{(\dot{c}, \dot{d})}]$.
- (vi) $\rho(\dot{d}, \dot{s}) > \bar{\alpha}$ for each $\alpha < \omega_1^x$.

T can be considered a Σ_1 on $L_{\omega_1^x}(x)$ theory.

Next to show T is consistent: Find w such that $\omega_1^w > \omega_1^x$ and $w \in \Lambda(z)$. $(w, f(w)) \in f$, therefore, there exists some u such that $u \in [U^{(w, f(w))}]$. By Theorem 4.2, $\text{SR}(f(w)) \geq \omega_1^w$. Let $k \in <^{\omega} \omega$ such that $\rho(w, k) > \omega_1^x$. Define \mathcal{M} by $M = H_{\aleph_1}$. Let $\dot{\in}^{\mathcal{M}}$ be the $\dot{\in}$ relation of H_{\aleph_1} . For each $a \in L_{\omega_1^x}(x)$, $\bar{a}^{\mathcal{M}} = a$. $\dot{c}^{\mathcal{M}} = w$, $\dot{d}^{\mathcal{M}} = f(w)$, and $\dot{s} = k$. Then $\mathcal{M} \models T$. T is consistent.

By Theorem 2.15, T has a solid model \mathcal{N} such that $\text{ON} \cap N = \text{ON} \cap L_{\omega_1^x}(x) = \omega_1^x$. Let $v = \dot{c}^{\mathcal{N}}$, $w = \dot{d}^{\mathcal{N}}$, $u = \dot{e}^{\mathcal{N}}$, and $t = \dot{s}^{\mathcal{N}}$. As before, $\omega_1^v = \omega_1^x$. $\mathcal{N} \models u \in [U^{(v, w)}]$. $u, v, w \in \text{WF}(N)$ since $\mathcal{N} \models T$ implies $v \subseteq \omega$, $w \subseteq \omega$, and u is a function from ω to ω . By Δ_1 -absoluteness between transitive models, $\text{WF}(\mathcal{N}) \models u \in [U^{(v, w)}]$. Since \mathcal{N} is solid, by Δ_1 -absoluteness between transitive models, $V \models u \in [U^{(v, w)}]$. Hence $w = f(v)$.

Now, one would like to show that $\rho((w, t)) = \omega_1^x$. The problem occurs in how \mathcal{N} can satisfy (vi). It seems possible that there is an $\alpha < \omega_1^x$ such that for all (v, q) and $\beta < \omega_1^x$, $(w, t) \sim_{\alpha} (v, q)$ implies $(w, t) \sim_{\beta} (v, q)$, but there exists some ill founded ordinal $\gamma \in N$ such that $(w, t) \not\sim_{\gamma} (v, q)$. That is, in V , $\rho((w, t)) < \omega_1^x$ but in N , $\rho((w, t)) > \alpha$ for all $\alpha < \omega_1^x$.

The natural question is whether this is actually possible: Is there a structure w on ω , a tuple $t \in <^{\omega} \omega$, and an ill-founded model N of KP such that $V \models \rho((w, t)) < \text{ON} \cap N$ but for all $\alpha < \text{ON} \cap N$, $N \models \rho((w, t)) > \alpha$.

Proposition 4.4. (Makkai) *There is a hyperarithmetical (or even computable) structure P such that $\text{SR}(P) = \omega_1^{\emptyset}$.*

Proof. See [12]. Also see [4], Theorem 3.6. □

Before this, there had not been much difficulty proving the consistency of the desired theory by exhibiting some model with domain H_{\aleph_1} . A model of the next theory is not as easily produced. The classical Barwise compactness theorem will be useful in showing consistency in this case.

Theorem 4.5. (Barwise Compactness) *Let \mathcal{A} be a countable admissible set and \mathcal{L} be a Δ_1 in \mathcal{A} language. Let $\mathcal{L}_{\mathcal{A}}$ be the induced countable admissible fragment of $\mathcal{L}_{\infty \omega}$. Let T be a set of sentences of $\mathcal{L}_{\mathcal{A}}$ such that T is Σ_1 in \mathcal{A} . If every $F \subseteq T$ such that $F \in \mathcal{A}$ has a model, then T has a model.*

Proof. See [2], Theorem III.5.6. Also see [9], Section 4, Corollary 8. □

Proposition 4.6. *Let P be a computable structure on ω such that $\text{SR}(P) = \omega_1^{\emptyset}$. Then there exists an ill-founded model N of KP and some $t \in <^{\omega} \omega$ such that*

- (i) $N \cap \text{ON} = \omega_1^{\emptyset}$

- (ii) For all $\alpha < \omega_1^\emptyset$, $N \models \rho((P, t)) > \alpha$.
- (iii) $V \models \rho((P, t)) < \omega_1^\emptyset$.

Proof. Let \mathcal{L} be a language consisting of the following:

- (I) A binary relation symbol $\dot{\in}$.
- (II) Constant symbols \bar{a} for each $a \in L_{\omega_1^\emptyset}$.
- (III) A distinguished constant symbol \dot{s} .

\mathcal{L} can be considered a Δ_1 definable subset of $L_{\omega_1^\emptyset}$.

Let T be a theory in the countable admissible fragment $\mathcal{L}_{L_{\omega_1^\emptyset}}$ consisting of the following:

- (i) KP
- (ii) For each $a \in L_{\omega_1^\emptyset}$, $(\forall v)(v \dot{\in} \bar{a} \Leftrightarrow \bigvee_{z \in a} v = \bar{z})$.
- (iii) \dot{s} is a finite tuple on $\bar{\omega}$.
- (iv) For each $\alpha < \omega_1^\emptyset$, $\rho((P, \dot{s})) > \bar{\alpha}$.

T can be considered a Σ_1 on $L_{\omega_1^\emptyset}$ theory.

T is consistent: Let $F \subseteq T$ such that $F \in L_{\omega_1^\emptyset}$. Then there exists $\alpha < \omega_1^\emptyset$ such that all ordinals mentioned in sentences of type (iv) are less than α . Since $\text{SR}(P) = \omega_1^\emptyset$, there exists some $t \in {}^{<\omega}\omega$ such that $\rho((P, t)) > \alpha$. Consider the \mathcal{L} -structure defined as follows: $M = H_{\aleph_1}$. $\dot{\in}^M = \dot{\in} \upharpoonright H_{\aleph_1}$. For each $a \in L_{\omega_1^\emptyset}$, $\bar{a}^M = a$. $\dot{s}^M = t$. Then $M \models F$. F is consistent. By Barwise compactness (Theorem 4.5), T is consistent.

By Theorem 2.15, there is a solid structure $\mathcal{N} \models T$. Let $t = \dot{s}^{\mathcal{N}}$. Since $\mathcal{N} \models T$, for all $\alpha < \omega_1^\emptyset$, $\mathcal{N} \models \rho((P, t)) > \alpha$. However, since $\text{SR}(P) = \omega_1^\emptyset$, one has $V \models \rho((P, t)) < \omega_1^\emptyset$. \mathcal{N} and t are as desired. \square

As mentioned before, the Δ_1^1 classification f of F_{ω_1} from Theorem 3.10 has the property that $\text{SR}(f(x)) = \omega_1^x + 1$ for all x . Given the above remarks, one can ask the following:

Question 4.7. Does there exist a Δ_1^1 function f classifying F_{ω_1} such that $\text{SR}(f(x)) = \omega_1^x$ for all $x \in \omega_2$?

[4] produced a simple computable tree of Scott rank ω_1^\emptyset . However, their proof in [4], Section 2 uses Barwise-Kreisel compactness and their proof in [4], Section 4 uses an overspill into the illfounded portion of the Harrison linear ordering. It is unclear if their proof method can be made uniform enough to produce in a Δ_1^1 manner a map taking x to some x -relative version of their tree.

The distinction between structure of rank ω_1^x and $\omega_1^x + 1$ has had some role in works on the Vaught's conjecture. For example, [19], Theorem 4.2 shows that if a scattered $\varphi \in \mathcal{L}_{\omega_1\omega}$ has the property that for all countable $M \models \varphi$, $\text{SR}(M) \leq \omega_1^M$, then φ has only countably many models up to isomorphism (i.e., is not a counterexample to Vaught's conjecture).

Theorem 4.2 is only able to provide information about $f(x)$ when $\omega_1^x \in \Lambda(z)$ with z such that f is $\Delta_1^1(z)$. Some type of condition involving $\Lambda(z)$ is required:

Lemma 4.8. *Suppose $x \in \omega_2$ is such that ω_1^x is not a recursively inaccessible ordinal. Then there exists a $z \in \omega_2$ such that z is $\Delta_1^1(x)$ and $\{ot(z^{[n]}) : n \in \omega\} = \Lambda(\emptyset) \cap \omega_1^x$, where $z^{[n]} = \{y : \langle n, y \rangle \in z\}$.*

Proof. Since ω_1^x is not recursively inaccessible let β be the largest admissible ordinal less than ω_1^x . Since $\beta + 1 < \omega_1^x$, it is an x -recursive ordinal. There is an e such that $\{e\}^x$ has order type $\beta + 1$. The set

$$B = \{n \in \omega : \{e\}^x \upharpoonright n \text{ is an admissible ordinal}\}$$

is a set in $L_{\omega_1^x}(x)$ by Δ_1 separation. Let $f : \omega \rightarrow B$ be a bijection in $L_{\omega_1^x}(x)$. Now define z by $z^{[n]} = \{e\}^x \upharpoonright f(n)$. \square

In the proof above, one needed a bijection in $L_\alpha(x)$ between ω and $\Lambda(\emptyset) \cap \alpha$. Note that by Σ_1 collection, there is no Σ_1 function $f : \gamma \rightarrow \alpha$ with $\gamma < \alpha$ and f unbounded. If α is recursively inaccessible, then $\Lambda(\emptyset) \cap \alpha$ is unbounded in α . Hence when α is recursively inaccessible, there can not exist such a bijection.

Proposition 4.9. *Suppose $\alpha < \omega_1$ is an admissible but not recursively inaccessible ordinal. Let $\mathcal{L} = \{\dot{\in}\}$. There exists z with $\omega_1^z = \alpha$ such that:*

- (i) There is an $f : \omega_2 \rightarrow S(\mathcal{L})$ which is $\Delta_1^1(z)$.
- (ii) For all $x, y \in \omega_2$, $x F_{\omega_1} y$ if and only if $f(x) \cong_{\mathcal{L}} f(y)$.

(iii) For all x with $\omega_1^x < \omega_1^z$, $SR(f(x)) < \omega_1^x$.

Proof. By Lemma 4.8, let z be such that $\{\text{ot}(z^{[n]}) : n \in \omega\} = \Lambda(\emptyset) \cap \alpha$ and $\alpha = \omega_1^z$. Let $f : \omega_2 \rightarrow S(\mathcal{L})$ be the Δ_1^1 classification given in Theorem 3.10. Let $g : \omega \rightarrow S(\mathcal{L})$ be Δ_1^1 such that for all $n \in \omega$, $g(n) \cong_{\mathcal{L}} \omega + n$.

Define the set $B \subseteq \omega \times \omega_2$ by:

$$(m, x) \in B \Leftrightarrow \omega_1^x = \text{ot}(z^{[m]})$$

The claim is that B is $\Delta_1^1(z)$:

It is $\Sigma_1^1(z)$.

$$(m, x) \in B \Leftrightarrow (\forall n) \left((\{n\}^x \in \text{WO} \Rightarrow \{n\}^x \preceq z^{[m]}) \right) \wedge \\ (\forall k) (\exists j) (\{j\}^x \preceq z^{[m]} \wedge z^{[m]} \upharpoonright k \preceq \{j\}^x)$$

It is $\Pi_1^1(z)$.

$$(m, x) \notin B \Leftrightarrow (\exists j) (\forall n) (\{n\}^x \in \text{WO} \Rightarrow (\{n\}^x \preceq z^{[m]} \upharpoonright j) \vee (\exists n) (\{n\}^x \in \text{LO} \wedge \{n\}^x \preceq z^{[m]} \wedge z^{[m]} \preceq \{n\}^x)$$

Now define the following function $h : \omega_2 \rightarrow S(\mathcal{L})$.

$$(x, y) \in h \Leftrightarrow (\exists n) \left((n, x) \in B \wedge y = g(n) \right) \vee (\forall n) \left((n, x) \notin B \wedge y = f(x) \right)$$

h is $\Delta_1^1(z)$. For all x, y , $x F_{\omega_1} y$ if and only if $h(x) \cong_{\mathcal{L}} h(y)$. If $\omega_1^x < \omega_1^z$, then $SR(h(x)) = SR(\omega + n) < \omega_1^x$, where n is such that $\text{ot}(z^{[n]}) = \omega_1^x$. \square

Proposition 4.9 asserts that for each $\alpha < \omega_1$ which is admissible but not recursively inaccessible, there exists some z with $\omega_1^z = \alpha$ and some $\Delta_1^1(z)$ classification of F_{ω_1} such that the Scott rank condition of Theorem 4.2 fails on all the F_{ω_1} -classes associated with admissible ordinals less than α . Can this also be achieved when α is recursively inaccessible?

The most interesting question of this kind is: Is there some classification f of F_{ω_1} which is $\Delta_1^1(z)$ and the Scott rank condition fails for some class associated with an admissible ordinal $\alpha > \omega_1^z$?

5. ALMOST BOREL REDUCTIONS

Definition 5.1. ([21] Definition 3.1.1) Let E be a Σ_1^1 equivalence relation on a Polish space X . Let \mathbb{P} be a forcing and τ be a \mathbb{P} -name for an element of X , i.e. $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \tau \in X$. Let τ_{left} and τ_{right} be \mathbb{P}^2 -names for the evaluation of τ according to the left and right \mathbb{P} -generic coming from the \mathbb{P}^2 -generic. τ is an E -pinned \mathbb{P} -name if and only if $1_{\mathbb{P}^2} \Vdash \tau_{\text{left}} E \tau_{\text{right}}$.

([21] Definition 4.1.1 and 4.1.2) Let \mathbb{P} and \mathbb{Q} be two forcings. Let σ be an E -pinned \mathbb{P} name and τ be an E -pinned \mathbb{Q} -name. Define the relation $\sigma \bar{E} \tau$ if and only if $1_{\mathbb{P} \times \mathbb{Q}} \Vdash \sigma E \tau$ (where σ and τ are considered $\mathbb{P} \times \mathbb{Q}$ -names in the natural way). The pinned cardinal of E , denoted $\kappa(E)$, is the smallest cardinal κ such that every E -pinned \mathbb{P} -name is \bar{E} -related to an E -pinned \mathbb{Q} -name with $|\mathbb{Q}| < \kappa$, if this cardinal exists. Otherwise, $\kappa(E) = \infty$.

Definition 5.2. E_{ω_1} is the Σ_1^1 equivalence relation on ω_2 defined by $x E_{\omega_1} y$ if and only if $(x \notin \text{WO} \wedge y \notin \text{WO}) \vee (\text{ot}(x) = \text{ot}(y))$.

Proposition 5.3. $\kappa(E_{\omega_1}) = \infty$

Proof. See [21], Example 4.1.8 or see [22], Example 4.3.1. \square

Definition 5.4. Let E and F be two equivalence relations on Polish spaces X and Y , respectively. $E \leq_{a\Delta_1^1} F$ if and only if there is a Δ_1^1 function $f : X \rightarrow Y$ and a countable set $A \subseteq X$ such that if c and d are not E -related to any elements of A , then $c E d$ if and only if $f(c) F f(d)$. In this situation, one says E is almost Δ_1^1 reducible to F . (It is called a weak Borel reduction in [21] Definition 2.1.2.)

Proposition 5.5. Let E and F be Σ_1^1 equivalence relations on Polish spaces X and Y , respectively. If $E \leq_{a\Delta_1^1} F$, then $\kappa(E) \leq \kappa(F)$.

Proof. See [21], Theorem 4.1.3 or see [22], Theorem 4.2.2. \square

Proposition 5.6. $\kappa(F_{\omega_1}) = \infty$.

Proof. For any cardinal κ , consider the forcing $\text{Coll}(\omega, \kappa)$. Let τ be a $\text{Coll}(\omega, \kappa)$ name for a real such that $1_{\text{Coll}(\omega, \kappa)} \Vdash_{\text{Coll}(\omega, \kappa)} \omega_1^\tau = \check{\kappa}$. τ is a F_{ω_1} -pinned $\text{Coll}(\omega, \kappa)$ -name, since

$$1_{\text{Coll}(\omega, \kappa) \times \text{Coll}(\omega, \kappa)} \Vdash_{\text{Coll}(\omega, \kappa) \times \text{Coll}(\omega, \kappa)} \omega_1^{\tau_{\text{left}}} = \check{\kappa} = \omega_1^{\tau_{\text{right}}}$$

Now suppose \mathbb{Q} is a forcing and σ is an F_{ω_1} -pinned \mathbb{Q} -name with $\tau \bar{F}_{\omega_1} \sigma$. This implies that $1_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \omega_1^\sigma = \check{\kappa}$. $1_{\mathbb{Q}} \Vdash_{\mathbb{Q}} |\check{\kappa}| = \aleph_0$. Since any forcing \mathbb{Q} is $|\mathbb{Q}|^+$ -cc, \mathbb{Q} preserves cardinals greater than or equal to $|\mathbb{Q}|^+$. Since \mathbb{Q} makes κ countable, $|\mathbb{Q}| \geq \kappa$. $\kappa(F_{\omega_1}) \geq \kappa$. Since κ was arbitrary, $\kappa(F_{\omega_1}) = \infty$. \square

Theorem 5.7. (*Zapletal*) *Suppose there exists a measurable cardinal. Let E be a Σ_1^1 equivalence relation. $\kappa(E) = \infty$ if and only if $E_{\omega_1} \leq_{a\Delta_1^1} E$.*

Proof. See [21], Theorem 4.2.1 or [22] Theorem 4.4.1. \square

Proposition 5.8. (ZFC + Measurable Cardinal) $E_{\omega_1} \leq_{a\Delta_1^1} F_{\omega_1}$.

Proof. This follows from Theorem 5.7 and Proposition 5.6. \square

Since Theorem 5.7 assumes a measurable cardinal, a natural task would be to investigate the consistency strength of the statement “For all Σ_1^1 equivalence relation, $\kappa(E) = \infty$ if and only if $E_{\omega_1} \leq_{a\Delta_1^1} E$ ”.

Therefore, an interesting question is whether L satisfies the above statement. The rest of this section will consider this question.

Theorem 5.9. *Suppose $x \in WO$ and $y \in \omega_2$ such that $\omega_1^y < \text{ot}(x)$, then $[x]_{E_{\omega_1}}$ is not $\Sigma_1^1(y)$.*

Proof. Suppose $[x]_{E_{\omega_1}}$ was $\Sigma_1^1(y)$. Let U be a tree on $2 \times \omega$ which is recursive in y and

$$(\forall u)(u \in [x]_{E_{\omega_1}} \Leftrightarrow (\exists f)(f \in [U^u]))$$

Let \mathcal{L} be a language consisting of the following:

- (i) A binary relation symbol $\dot{\in}$.
- (ii) For each $a \in L_{\omega_1^y}(y)$, a constant symbol \bar{a} .
- (iii) Two distinct constant symbols \dot{c} and \dot{d} .

\mathcal{L} may be considered a Δ_1 definable language over $L_{\omega_1^y}(y)$.

Let T be a theory in the countable admissible fragment $\mathcal{L}_{L_{\omega_1^y}(y)}$ consisting of the following sentences:

- (I) KP
- (II) For each $a \in L_{\omega_1^y}(y)$, $(\forall v)(v \dot{\in} \bar{a} \Leftrightarrow \bigvee_{z \in a} v = \bar{z})$.
- (III) $\dot{c} \subseteq \bar{\omega}$, $\dot{d} : \bar{\omega} \rightarrow \bar{\omega}$
- (IV) $\dot{d} \in [U^{\dot{c}}]$.
- (V) For all $\alpha < \omega_1^y$, “ $\bar{\alpha}$ is not admissible in \dot{c} ”.

T may be considered a Σ_1 theory in $L_{\omega_1^y}(y)$.

Next, the claim is that T is consistent. Since $x \in [x]_{E_{\omega_1}}$, there exists g such that $g \in [U^x]$. Consider the \mathcal{L} -structure \mathcal{N} defined as follows: Let the universe N be H_{\aleph_1} . Let $\dot{\in}^{\mathcal{N}} = \dot{\in} \upharpoonright H_{\aleph_1}$. Let $\dot{c}^{\mathcal{N}} = x$ and $\dot{d}^{\mathcal{N}} = g$. $\mathcal{N} \models T$. For (V), observe that if \mathcal{A} is an admissible set with $x \in \mathcal{A}$, then $\text{ot}(x) \in \mathcal{A}$. Hence $\text{ON} \cap \mathcal{A} > \text{ot}(x) > \omega_1^y$.

By Theorem 2.15, let \mathcal{M} be a solid model of T with $\text{ON} \cap \mathcal{M} = \omega_1^y$. Let $z = c^{\mathcal{M}}$. $z \in [x]_{E_{\omega_1}}$ since $\dot{d}^{\mathcal{M}} \in [U^z]$. As in the proof of Sacks theorem, $\omega_1^z = \omega_1^y$. $z \in L_{\omega_1^z}(z)$. So $\text{ot}(z) \in L_{\omega_1^z}(z)$. This is impossible since $\omega_1^z = \omega_1^y < \text{ot}(x) = \text{ot}(z)$. \square

Theorem 5.10. *If $f : \omega_2 \rightarrow \omega_2$ is $\Delta_1^1(y)$ and witnesses $E_{\omega_1} \leq_{a\Delta_1^1} F_{\omega_1}$, then there exists a $\beta < \omega_1$ such that for all $\alpha \in \Lambda(y)$ with $\alpha > \beta$, the next admissible ordinal after α is not in $\Lambda(y)$.*

Proof. Let $f : \omega_2 \rightarrow \omega_2$ witness $E_{\omega_1} \leq_{a\Delta_1^1} F_{\omega_1}$. There exists some countable set $A \subseteq \omega_2$ such that $x E_{\omega_1} y$ if and only if $f(x) F_{\omega_1} f(y)$ whenever $x, y \notin [A]_E$. Let $\beta = \sup\{\text{ot}(x) : x \in A\}$. The claim is that this β works. So suppose not. There exists $\alpha', \alpha \in \Lambda(y)$ such that $\alpha > \beta$, $\alpha' > \beta$, and α is the next admissible ordinal after α' .

Since f is $\Delta_1^1(y)$, let U be a tree on $2 \times 2 \times \omega$ such that for all $a, b \in \omega_2$, $(a, b) \in f \Leftrightarrow [U^{(a,b)}]$ is ill-founded.

Claim: There exists $a, b \in \omega_2$ such that $\alpha' < \text{ot}(a) < \text{ot}(b) < \alpha$, $\omega_1^{f(a)} \geq \alpha$, and $\omega_1^{f(b)} \geq \alpha$.

To prove this claim: If there exists a $c \in {}^\omega 2$ such that $\alpha' < \text{ot}(c) < \alpha$ and $\omega_1^{f(c)} = \alpha'$, then fix such a c . If not, pick any $c \in {}^\omega 2$ such that $\alpha' < \text{ot}(c) < \alpha$. In this latter case, c will essentially be ignored.

Then for any $d \in {}^\omega 2$ with $\text{ot}(d) > \beta$ and $d \notin [c]_{E_{\omega_1}}$, $\omega_1^{f(d)} \neq \alpha'$ since f is a reduction. Pick any $d \in {}^\omega 2$ with $d \notin [c]_{E_{\omega_1}}$ and $\alpha' < \text{ot}(d) < \alpha$.

Suppose that $\omega_1^{f(d)} < \alpha'$. By Proposition 2.16, let z be any real such that $\omega_1^z = \alpha'$ and $y \leq_T z$. $[f(d)]_{F_{\omega_1}}$ is $\Delta_1^1(z)$ by Proposition 2.38.

$$k \in [d]_{E_{\omega_1}} \Leftrightarrow f(k) \in [f(d)]_{F_{\omega_1}}$$

Hence $[d]_{E_{\omega_1}}$ is $\Sigma_1^1(y, z) = \Sigma_1^1(z)$. However, $\omega_1^z = \alpha' < \text{ot}(d)$. This contradicts Theorem 5.9.

This shows that $\omega_1^{f(d)} \geq \alpha'$. Since $d \notin [c]_{E_{\omega_1}}$, $\omega_1^{f(d)} > \alpha'$. However, the next admissible ordinal greater than α' is α . Therefore, $\omega_1^{f(d)} \geq \alpha$.

Now let a, b be any two reals such that $a, b \notin [c]_{E_{\omega_1}}$ and $\alpha' < \text{ot}(a) < \text{ot}(b) < \alpha$. Since d in the above was arbitrary with these two properties, these two reals satisfy the claim.

Now fix $a, b \in {}^\omega 2$ satisfying the claim. $\text{SR}(a) = \text{ot}(a)$ and $\text{SR}(b) = \text{ot}(b)$. Thus their canonical Scott sentences are both elements of L_α since $\text{ot}(a), \text{ot}(b) \in L_\alpha$, $\text{SR}(\text{ot}(a)) < \alpha$, $\text{SR}(\text{ot}(b)) < \alpha$, and Proposition 2.20.

Let \mathcal{L} be a language consisting of:

- (i) A binary relation symbol $\dot{\in}$.
- (ii) For each $e \in L_\alpha(y)$, a constant symbol \bar{e} .
- (iii) Six distinct symbols $\dot{a}, \dot{b}, \dot{c}, \dot{d}, \dot{u}, \dot{v}$.

\mathcal{L} may be considered as a Δ_1 definable language in $L_\alpha(y)$.

Let T be a theory in the countable admissible fragment $\mathcal{L}_{L_\alpha(y)}$ consisting of the following sentences:

- (I) KP in the symbol $\dot{\in}$.
- (II) For each $e \in L_\alpha(y)$, $(\forall v)(v \dot{\in} \bar{e} \Leftrightarrow \bigvee_{z \in e} v = \bar{z})$.
- (III) $\dot{a}, \dot{b}, \dot{c}, \dot{d} \subseteq \bar{\omega}$. \dot{u}, \dot{v} are functions from $\bar{\omega} \rightarrow \bar{\omega}$.
- (IV) $\dot{u} \in [U^{\langle \dot{a}, \dot{c} \rangle}]$ and $\dot{v} \in [U^{\langle \dot{b}, \dot{d} \rangle}]$.
- (V) $\dot{a} \models \text{CSS}(a)$ and $\dot{b} \models \text{CSS}(b)$.
- (VI) For all $\beta < \alpha$, $\bar{\beta}$ is not admissible in \dot{c} and $\bar{\beta}$ is not admissible in \dot{d} .

T may be considered a Σ_1 theory in $L_\alpha(y)$.

Since $(a, f(a)) \in f$ and $(b, f(b)) \in f$, let $u, v \in {}^\omega \omega$, be such that $u \in [U^{(a, f(a))}]$ and $v \in [U^{(b, f(b))}]$.

To show that T is consistent: consider the following model of \mathcal{N} : The universe N is H_{\aleph_1} . $\dot{\in}^{\mathcal{N}} = \in \upharpoonright H_{\aleph_1}$. For each $e \in L_\alpha(y)$, $\bar{e}^{\mathcal{N}} = e$. $\dot{a}^{\mathcal{N}} = a$, $\dot{b}^{\mathcal{N}} = b$. Let $\dot{c}^{\mathcal{N}} = f(a)$ and $\dot{d}^{\mathcal{N}} = f(b)$. Let $\dot{u}^{\mathcal{N}} = u$ and $\dot{v}^{\mathcal{N}} = v$. Then $\mathcal{N} \models T$.

By Theorem 2.15, there exists a solid model $\mathcal{M} \models T$ with $M \cap \text{ON} = \alpha$. Let $a' = \dot{a}^{\mathcal{M}}$, $b' = \dot{b}^{\mathcal{M}}$. $f(a') = \dot{c}^{\mathcal{M}}$ and $f(b') = \dot{d}^{\mathcal{M}}$ since $\dot{u}^{\mathcal{M}} \in [U^{\langle \dot{a}', \dot{c}^{\mathcal{M}} \rangle}]$ and $\dot{v}^{\mathcal{M}} \in [U^{\langle \dot{b}', \dot{d}^{\mathcal{M}} \rangle}]$. As in the proof of Sacks theorem, $\omega_1^{f(a')} = \omega_1^{f(b')} = \alpha$. By absoluteness of satisfaction from \mathcal{M} to $\text{WF}(\mathcal{M})$ to V , $a' \models \text{CSS}(a)$ and $b' \models \text{CSS}(b)$. Hence in V , $\text{ot}(a') = \text{ot}(a)$ and $\text{ot}(b') = \text{ot}(b)$. In particular, $\text{ot}(a') \neq \text{ot}(b')$. Hence $\neg(a' E_{\omega_1} b')$. However, $\omega_1^{f(a')} = \omega_1^{f(b')} = \alpha$ implies $f(x) F_{\omega_1} f(y)$. This contradicts f being a reduction.

This proves the theorem for those $\alpha \in \Lambda(y) \cap \omega_1$. Note the statement that f and countable $A \subseteq {}^\omega 2$ witnesses $E_{\omega_1} \leq_{\Delta_1^1} F_{\omega_1}$ can be written as

$$(\forall x)(\forall y)((x \notin [A]_{E_{\omega_1}} \wedge y \notin [A]_{E_{\omega_1}}) \Rightarrow (x E_{\omega_1} y \Leftrightarrow f(x) F_{\omega_1} f(y)))$$

This is $\Pi_2^1(y, A)$ and so holds in all generic extensions by Schoenfield's absoluteness. To show the theorem holds for all $\alpha \in \Lambda(y)$ and $\alpha \geq \omega_1$, let $G \subseteq \text{Coll}(\omega, \alpha)$ be $\text{Coll}(\omega, \alpha)$ -generic over V . In $V[G]$, let $\beta = \sup\{\text{ot}(x) : x \in A\}$ be the same ordinal as before. Since $\beta < \omega_1^V \leq \alpha < \omega_1^{V[G]}$, the result above, applied in $V[G]$ for $\Lambda(y) \cap \omega_1^{V[G]}$, will show the theorem holds for α . This concludes the proof. \square

Theorem 5.11. $L \models \neg(E_{\omega_1} \leq_{\Delta_1^1} F_{\omega_1})$. This also holds in set generic extensions of L .

Proof. In L , for all $x \in {}^\omega 2$, there exists some $\alpha < \omega_1$ such that $x \in L_\alpha$. Then $\Lambda(x) - \alpha = \Lambda(\emptyset) - \alpha$. Hence there are no reals with admissible spectrum as described in Theorem 5.10. \square

6. COUNTEREXAMPLES TO VAUGHT'S CONJECTURE AND F_{ω_1}

Definition 6.1. Let \mathcal{L} be a recursive language. Let $\varphi \in \mathcal{L}_{\omega_1\omega}$. Define $E_{\mathcal{L}}^\varphi$ to be the Σ_1^1 equivalence relation on $S(\mathcal{L})$ defined by

$$x E_{\mathcal{L}}^\varphi y \Leftrightarrow (x \not\equiv \varphi \wedge y \not\equiv \varphi) \vee (x \cong_{\mathcal{L}} y)$$

See Proposition 2.35 for the Δ_1^1 definability of $x \not\equiv \varphi$.

Definition 6.2. A *counterexample to Vaught's conjecture* is a $\varphi \in \mathcal{L}_{\omega_1\omega}$ (for some recursive language \mathcal{L}) such that $E_{\mathcal{L}}^\varphi$ is a thin equivalence relation with uncountably many classes.

From a list of questions from the Vaught's Conjecture Workshop 2015 at the University of California at Berkeley, Sy-David Friedman asked the following question:

Question 6.3. (Sy-David Friedman) Is there some recursive language \mathcal{L} such that F_{ω_1} is Δ_1^1 bireducible to the \mathcal{L} -isomorphism relation restricted to some Δ_1^1 invariant set?

Every invariant Δ_1^1 set for the \mathcal{L} -isomorphism relation is of the form $\text{Mod}(\varphi)$ for some $\varphi \in \mathcal{L}_{\omega_1\omega}$ (see [6], Theorem 11.3.6). Therefore, the above question is equivalent to whether there exists some φ such that $F_{\omega_1} \equiv_{\Delta_1^1} E_{\mathcal{L}}^\varphi$.

$E_{\mathcal{L}}^\varphi \leq_{\Delta_1^1} F_{\omega_1}$ implies that $E_{\mathcal{L}}^\varphi$ is thin. $F_{\omega_1} \leq_{\Delta_1^1} E_{\mathcal{L}}^\varphi$ implies that $E_{\mathcal{L}}^\varphi$ has uncountably many classes. Hence any such φ is a counterexample to Vaught's conjecture.

Using the ideas from the previous section, it will be shown that in L , no counterexample φ of Vaught's conjecture has the property that $E_{\mathcal{L}}^\varphi \leq_{\Delta_1^1} F_{\omega_1}$. Hence, Friedman's question has a negative answer in L .

Theorem 6.4. *Let \mathcal{L} be a recursive language. Let $M \in S(\mathcal{L})$ and $y \in \omega^2$ be such that $\omega_1^y < R(M)$. Then $[M]_{\cong_{\mathcal{L}}}$ is not $\Sigma_1^1(y)$. ($\cong_{\mathcal{L}}$ denotes the equivalence relation of \mathcal{L} -isomorphism. Recall R is defined in Definition 2.18.)*

Proof. Suppose $[M]_{\cong_{\mathcal{L}}} \in \Sigma_1^1(y)$. Let U be a tree on $2 \times \omega$ which is recursive in y and

$$(\forall N)(N \in [M]_{\cong_{\mathcal{L}}} \Leftrightarrow (\exists f)(f \in [U^N]))$$

Let \mathcal{U} be the language consisting of the following:

- (i) A binary relation symbol $\dot{\in}$.
- (ii) For each $a \in L_{\omega_1^y}(y)$, a constant symbol \bar{a} .
- (iii) Two distinct constant symbols \dot{c} and \dot{d} .

\mathcal{U} may be considered a Δ_1 definable class in $L_{\omega_1^y}(y)$.

Let T be a theory in the countable admissible fragment $\mathcal{U}_{L_{\omega_1^y}(y)}$ consisting of the following sentences:

- (I) KP
- (II) For each $a \in L_{\omega_1^y}(y)$, $(\forall v)(v \dot{\in} \bar{a} \Leftrightarrow \bigvee_{z \in a} v = \bar{z})$.
- (III) $\dot{c} \subseteq \bar{\omega}$, $\dot{d} : \bar{\omega} \rightarrow \bar{\omega}$.
- (IV) $\dot{d} \in [U^{\dot{c}}]$.
- (V) For all $\alpha < \omega_1^y$, $\bar{\alpha}$ is not admissible in \dot{c} .

T may be considered a Σ_1 definable theory in $L_{\omega_1^y}(y)$.

T is consistent: Since $M \in [M]_{\cong_{\mathcal{L}}}$, there is some g such that $g \in [U^M]$. Define a \mathcal{U} -structure \mathcal{N} as follows: Let the universe N be H_{\aleph_1} . Let $\dot{\in}^{\mathcal{N}} = \dot{\in} \upharpoonright H_{\aleph_1}$. Let $\dot{c}^{\mathcal{N}} = M$ and $\dot{d}^{\mathcal{N}} = g$. \mathcal{N} is a model of T . To see (V), note that $\omega_1^M > \omega_1^y$. This is because, if otherwise, M would be an element of some admissible set \mathcal{A} such that $o(\mathcal{A}) \leq \omega_1^y$. By Proposition 2.28, $R(M) \leq \text{ON} \cap \mathcal{A} \leq \omega_1^y$, which is a contradiction.

By Theorem 2.15, let \mathcal{M} be a solid model of T . Let $P = \dot{c}^{\mathcal{M}}$. $P \in [M]_{\cong_{\mathcal{L}}}$ since $\dot{d} \in [U^P]$. Like in the proof of Sacks theorem, $\omega_1^P = \omega_1^y$. Therefore, $P \in L_{\omega_1^y}(P)$. By Proposition 2.28, $R(P) \leq \omega_1^y$. However $P \in [M]_{\cong_{\mathcal{L}}}$ implies that $P \cong_{\mathcal{L}} M$. $R(P) \leq \omega_1^y < R(M)$. Contradiction. \square

Fact 6.5. *Let \mathcal{L} be a recursive language. If φ is a counterexample to the Vaught conjecture, then for all limit ordinals $\beta > \text{qr}(\varphi)$, φ has a model of Scott rank β .*

Proof. See [11], Theorem 10.8. A similar result is also shown in [14], page 19. \square

Fact 6.6. *Let \mathcal{A} be a countable admissible set. Let $\alpha < o(\mathcal{A})$. Then there exists a countable admissible set \mathcal{B} extending \mathcal{A} with $o(\mathcal{B}) = o(\mathcal{A})$ and such that there exists a $c \in {}^\omega 2 \cap \mathcal{B}$ with $ot(c) = \alpha$.*

Proof. This can be proved using the techniques of infinitary logic in the countable admissible fragment \mathcal{A} using a Scott sentence for α as a linear ordering. Since this is similar to several previous arguments, the details are omitted. \square

Fact 6.7. *Let E be a $\Pi_1^1(z)$ equivalence relation on a Polish space ${}^\omega 2$ with countably many classes. Then for all $x \in {}^\omega 2$, there is a $\Delta_1^1(z)$ set U such that $x \in U \subseteq [x]_E$.*

Proof. See [7].

In the effective proof of Silver's dichotomy for Π_1^1 equivalence using the Gandy-Harrington topology, the two outcomes depend on whether the set

$$V = \{x \in {}^\omega 2 : \text{There exists } \Delta_1^1(z) \text{ set } U \text{ with } x \in U \subseteq [x]_E\}$$

is equal to ${}^\omega 2$. If $V = {}^\omega 2$, then E has only countable many classes. This gives the desired result above. See [6], Theorem 5.3.5 for a presentation of the effective proof of Silver's theorem. \square

Fact 6.8. *Let \mathcal{L} be a recursive language. Let $\varphi \in \mathcal{L}_{\omega_1\omega}$ be a counterexample to Vaught's conjecture. Let $z \in {}^\omega 2$ be such that $\varphi \in L_{\omega_1^z}(z)$. Let β be a z -admissible ordinal. Let $M, N \in S(\mathcal{L})$ be such that $M \models \varphi$, $N \models \varphi$, $R(M) < \beta$ and $R(N) < \beta$. Then there exists a countable admissible set \mathcal{A} extending $L_\beta(z)$ such that $o(\mathcal{A}) = \beta$ and $CSS(M), CSS(N) \in \mathcal{A}$.*

Proof. Let $\alpha < \beta$ be such that $R(M) < \alpha$ and $R(N) < \alpha$. $L_\beta(z)$ is an admissible set. By Fact 6.6, there exists some countable admissible set \mathcal{B} extending $L_\beta(z)$ containing some real c which codes α and $o(\mathcal{B}) = \beta$.

Let $\equiv_\alpha^\mathcal{L}$ be the relation of \mathcal{L} -elementary equivalence with respect to just the formulas of quantifier rank less than α . $\equiv_\alpha^\mathcal{L} \upharpoonright \text{Mod}(\varphi)$ is a $\Delta_1^1(c, z)$ equivalence relation. (Proposition 2.35 is used here.) Since φ is a counterexample to Vaught's conjecture, $\equiv_\alpha^\mathcal{L} \upharpoonright \text{Mod}(\varphi)$ has only countably many classes. By Fact 6.7, there exists $\Delta_1^1(c, z)$ sets U_M and U_N such that $M \in U_M \subseteq [M]_{\equiv_\alpha^\mathcal{L} \upharpoonright \text{Mod}(\varphi)}$ and $N \in U_N \subseteq [N]_{\equiv_\alpha^\mathcal{L} \upharpoonright \text{Mod}(\varphi)}$. Let T_M and T_N be the $c \oplus z$ recursive trees such that for all P

$$P \in U_M \Leftrightarrow T_M^P \text{ is illfounded}$$

$$P \in U_N \Leftrightarrow T_N^P \text{ is illfounded}$$

Let \mathcal{U} be the language consisting of the following:

- (i) A binary relation symbol $\dot{\in}$.
- (ii) For each $a \in B$, a constant symbol \bar{a} .
- (iii) Four new constant symbols, \dot{R} , \dot{S} , \dot{e} , and \dot{f} .

\mathcal{U} may be considered a Δ_1 definable language in \mathcal{B} .

Let T be a theory in the countable admissible fragment $\mathcal{U}_\mathcal{B}$ consisting of the following sentences:

- (I) KP
- (II) For each $a \in B$, $(\forall v)(v \dot{\in} \bar{a} \Leftrightarrow \bigvee_{z \in a} v = \bar{z})$.
- (III) $\dot{R} \subseteq \bar{\omega}$, $\dot{S} \subseteq \bar{\omega}$, $\dot{e} \in [T_M^{\dot{R}}]$, and $\dot{f} \in [T_N^{\dot{R}}]$.

T may be considered as a Σ_1 definable set in \mathcal{A} .

T is consistent: Since $M \in U_M$ and $N \in U_N$, find some $v \in [T_M^M]$ and $w \in [T_N^N]$. Consider the \mathcal{U} structure \mathcal{I} defined as follows: Let the universe I be H_{\aleph_1} . Let $\dot{\in}^\mathcal{I} = \dot{\in} \upharpoonright H_{\aleph_1}$. Let $\dot{R}^\mathcal{I} = M$, $\dot{S}^\mathcal{I} = N$, $\dot{e}^\mathcal{I} = v$, and $\dot{f}^\mathcal{I} = w$. Then $\mathcal{I} \models T$.

By Theorem 2.15, let \mathcal{J} be a solid model of T with $o(\mathcal{J}) = o(\mathcal{B})$. Let $R = \dot{R}^\mathcal{J}$, $S = \dot{S}^\mathcal{J}$, $e = \dot{e}^\mathcal{J}$, and $f = \dot{f}^\mathcal{J}$. By Δ_1 absoluteness (first between \mathcal{J} and $\text{WF}(\mathcal{J})$ and then between $\text{WF}(\mathcal{J})$ and V), $e \in [T_M^R]$ and $f \in [T_N^S]$. Hence $R \in U_M$ and $S \in U_N$.

Let $\mathcal{A} = \text{WF}(\mathcal{J})$. By Lemma 2.14, \mathcal{A} is an admissible set. It has been shown that \mathcal{A} has two elements R and S such that $R \in [M]_{\equiv_\alpha^\mathcal{L} \upharpoonright \text{Mod}(\varphi)}$ and $S \in [N]_{\equiv_\alpha^\mathcal{L} \upharpoonright \text{Mod}(\varphi)}$.

Since $CSS(M)$ and $CSS(N)$ have quantifier rank less than α , $M \equiv_\alpha^\mathcal{L} R$, and $N \equiv_\alpha^\mathcal{L} S$, the following must hold: $R \models CSS(M)$ and $S \models CSS(N)$. Hence $CSS(R) = CSS(M)$ and $CSS(S) = CSS(N)$.

Since $R(M), R(N) < \alpha$, Proposition 2.20 implies that $CSS(R) \in \mathcal{A}$ and $CSS(S) \in \mathcal{A}$. Therefore, $CSS(M) \in \mathcal{A}$ and $CSS(N) \in \mathcal{A}$. This completes the proof. \square

Theorem 6.9. *Let \mathcal{L} be a recursive language. Let $\varphi \in \mathcal{L}_{\omega_1}$ be a counterexample to Vaught's conjecture. Suppose f is a Δ_1^1 function witnessing $E_{\mathcal{L}}^\varphi \leq_{\Delta_1^1} F_{\omega_1}$, then there exists some ordinal γ and real z such that for all $\alpha \in \Lambda(z)$ with $\alpha > \gamma$, the next admissible ordinal greater than α is not in $\Lambda(z)$.*

Proof. First, the theorem will be shown for $\Lambda(z) \cap \omega_1$. At the end, this result will be used to obtain the theorem for the full $\Lambda(z)$.

Let $f : S(\mathcal{L}) \rightarrow {}^\omega 2$ be $\Delta_1^1(r)$ witnessing $E_{\mathcal{L}}^\varphi \leq_{\Delta_1^1} F_{\omega_1}$ where r is some real. Find any $s \in {}^\omega 2$ such that $\varphi \in L_{\omega_1^s}(s)$. Let $z = r \oplus s$. Note that f is $\Delta_1^1(z)$. Let $\gamma = \omega_1^z$. Certainly $\gamma > \text{qr}(\varphi)$.

Now suppose there exists some $\alpha, \beta \in \Lambda(z) \cap \omega_1$ with $\gamma < \alpha$ and β is the next admissible ordinal greater than α .

Between two consecutive admissible ordinals, there are infinitely many limit ordinals. Since φ is a counterexample to the Vaught's conjecture, Fact 6.5 implies that there are infinitely many models of φ with Scott ranks between α and β . Let P, M , and N be three models of φ with distinct Scott rank between α and β . Since f is a reduction of $E_{\mathcal{L}}^\varphi$ to F_{ω_1} , at most one $X \in \{P, M, N\}$ has the property that $\omega_1^{f(X)} = \alpha$. If such an X among these three exists, then without loss of generality, assume it was P . (If no X among these three has this property, then one can just ignore P for the rest of the proof.)

Now to show that $\omega_1^{f(M)} \geq \beta$ and $\omega_1^{f(N)} \geq \beta$: Suppose $\omega_1^{f(M)} < \beta$. Since P and M are not \mathcal{L} -isomorphic and f is a reduction to F_{ω_1} , $\omega_1^{f(M)} \neq \alpha$ (since one assumed that $\omega_1^{f(P)} = \alpha$, if this could occur among the three models). Thus, $\omega_1^{f(M)} < \alpha$ since β is the next admissible ordinal after α . Observe that

$$X \in [M]_{E_{\mathcal{L}}^\varphi} \Leftrightarrow f(X) \in [f(M)]_{F_{\omega_1}}$$

Let $y \in {}^\omega 2$ be such that $z \leq_T y$ and $\omega_1^y = \alpha$ (which exists due to Theorem 2.16). $[f(M)]_{F_{\omega_1}}$ is $\Delta_1^1(y)$ by Proposition 2.38. This shows that $[M]_{E_{\mathcal{L}}^\varphi} = \Sigma_1^1(y, z) = \Sigma_1^1(y)$. $\omega_1^y = \alpha < R(M)$. This contradicts Theorem 6.4.

So it has been shown that $\omega_1^{f(M)} > \alpha$. But since β is the smallest admissible ordinal greater than α , $\omega_1^{f(M)} \geq \beta$. The same exact argument shows $\omega_1^{f(N)} \geq \beta$.

By Fact 6.8, let \mathcal{A} be a countable admissible set extending $L_\beta(z)$ containing $\text{CSS}(M)$ and $\text{CSS}(N)$ with $o(\mathcal{A}) = \beta$.

Since f is $\Delta_1^1(z)$, let U be a z -recursive tree on $2 \times 2 \times \omega$ such that for all $X \in S(\mathcal{L})$ and $r \in {}^\omega 2$,

$$(X, r) \in f \Leftrightarrow [U^{(X, r)}] \neq \emptyset$$

Let \mathcal{U} be a language consisting of:

- (i) A binary relation symbol $\dot{\in}$.
- (ii) For each $e \in A$, a constant symbol \bar{e} .
- (iii) Six distinct symbols $\dot{R}, \dot{S}, \dot{c}, \dot{d}, \dot{u}$, and \dot{v} .

\mathcal{U} may be considered as a Δ_1 definable language in A .

Let T be the theory in the countable admissible fragment $\mathcal{U}_{\mathcal{A}}$ consisting of the following sentences:

- (I) KP
- (II) For each $e \in A$, $(\forall v)(v \dot{\in} \bar{e} \Leftrightarrow \bigvee_{z \in e} v = \bar{z})$.
- (III) $\dot{R}, \dot{S}, \dot{c}, \dot{d} \subseteq \bar{\omega}$. \dot{u} and \dot{v} are functions from $\bar{\omega} \rightarrow \bar{\omega}$.
- (IV) $\dot{u} \in [U^{(\dot{R}, \dot{c})}]$ and $\dot{v} \in [U^{(\dot{S}, \dot{d})}]$.
- (V) $\dot{R} \models \text{CSS}(M)$ and $\dot{S} \models \text{CSS}(N)$.
- (VI) For all $\xi < \beta$, $\bar{\xi}$ is not admissible in \dot{c} and $\bar{\xi}$ is not admissible in \dot{d} .

T may be considered a Σ_1 definable theory in \mathcal{A} . Note that \mathcal{A} was chosen so that (V) would be expressible.

Since $(M, f(M)) \in f$ and $(N, f(N)) \in f$, let $u, v \in {}^\omega \omega$ be such that $u \in [U^{(M, f(M))}]$ and $v \in [U^{(N, f(N))}]$.

Now to show T is consistent: Consider the following \mathcal{U} -structure \mathcal{G} . The domain of \mathcal{G} is $G = H_{\aleph_1}$. For each $e \in A$, $\bar{e}^{\mathcal{G}} = e$. $\dot{R}^{\mathcal{G}} = M$. $\dot{S}^{\mathcal{G}} = N$, $\dot{c}^{\mathcal{G}} = f(M)$, $\dot{d}^{\mathcal{G}} = f(N)$, $\dot{u}^{\mathcal{G}} = u$, and $\dot{v}^{\mathcal{G}} = v$. Then $\mathcal{G} \models T$.

By Theorem 2.15, there exists a solid model $\mathcal{H} \models T$ with $o(\mathcal{H}) = o(\mathcal{A})$. Let $R = \dot{R}^{\mathcal{H}}$ and $S = \dot{S}^{\mathcal{H}}$. Then $f(R) = \dot{c}^{\mathcal{H}}$ and $f(S) = \dot{d}^{\mathcal{H}}$ since $\dot{u}^{\mathcal{H}} \in [U^{(R, \dot{c}^{\mathcal{H}})}]$ and $\dot{v}^{\mathcal{H}} \in [U^{(S, \dot{d}^{\mathcal{H}})}]$. As in the proof of Sacks' theorem, $\omega_1^{f(R)} = \omega_1^{f(S)} = \beta$. By the absoluteness of satisfaction (from \mathcal{H} to $\text{WF}(\mathcal{H})$ to V), $R \models \text{CSS}(M)$ and $S \models \text{CSS}(N)$. Hence in V , R and S are not \mathcal{L} -isomorphic. However, $\omega_1^{f(R)} = \omega_1^{f(S)} = \beta$ implies that $f(R) F_{\omega_1} f(S)$. This contradicts f being a reduction.

This proves the theorem for $\alpha \in \Lambda(y) \cap \omega_1$. The statement f witnesses $E_{\mathcal{L}}^\varphi \leq_{\Delta_1^1} F_{\omega_1}$ is $\mathbf{\Pi}_2^1$. So the same argument as at the end of the proof of Theorem 5.10 shows the result holds for all $\alpha \in \Lambda(y)$. \square

Corollary 6.10. *In L (and any set generic extension of L), there are no recursive language \mathcal{L} and countereexample to Vaught's conjecture $\varphi \in \mathcal{L}_{\omega_1\omega}$ such that $E_{\mathcal{L}}^\varphi \leq_{\Delta_1^1} F_{\omega_1}$.*

Proof. There is no $z \in \omega_2$ having the property of Theorem 6.9 in L or set generic extensions of L . \square

Corollary 6.11. *In L (and set generic extensions of L), Question 6.3 has a negative answer.*

Proof. This follows from Corollary 6.11 and the remarks following Question 6.3. \square

This leaves open whether there is an answer to Question 6.3 in ZFC.

REFERENCES

1. J. Barwise and S. Feferman (eds.), *Model-theoretic logics*, Perspectives in Mathematical Logic, Springer-Verlag, New York, 1985. MR 819531 (87g:03033)
2. Jon Barwise, *Admissible sets and structures*, Springer-Verlag, Berlin-New York, 1975, An approach to definability theory, Perspectives in Mathematical Logic. MR 0424560 (54 #12519)
3. Howard Becker, *The topological Vaught's conjecture and minimal countereexamples*, J. Symbolic Logic **59** (1994), no. 3, 757–784. MR 1295968 (95k:03077)
4. Wesley Calvert, Julia F. Knight, and Jessica Millar, *Computable trees of Scott rank ω_1^{CK} , and computable approximation*, J. Symbolic Logic **71** (2006), no. 1, 283–298. MR 2210068 (2006j:03057)
5. Harvey Friedman, *Countable models of set theories*, Cambridge Summer School in Mathematical Logic (Cambridge, 1971), Springer, Berlin, 1973, pp. 539–573. Lecture Notes in Math., Vol. 337. MR 0347599
6. Su Gao, *Invariant descriptive set theory*, Pure and Applied Mathematics (Boca Raton), vol. 293, CRC Press, Boca Raton, FL, 2009. MR 2455198 (2011b:03001)
7. Leo Harrington, *A powerless proof of a theorem of silver*, Unpublished.
8. Joseph Harrison, *Recursive pseudo-well-orderings*, Trans. Amer. Math. Soc. **131** (1968), 526–543. MR 0244049 (39 #5366)
9. R. Björn Jensen, *Admissible sets*, <http://www.mathematik.hu-berlin.de/~%20raesch/org/jensen.html>.
10. S. C. Kleene, *On the forms of the predicates in the theory of constructive ordinals. II*, Amer. J. Math. **77** (1955), 405–428. MR 0070595 (17,5a)
11. P. B. Larson, *Scott processes*, ArXiv e-prints (2014).
12. M. Makkai, *An example concerning Scott heights*, J. Symbolic Logic **46** (1981), no. 2, 301–318. MR 613284 (82m:03049)
13. Richard Mansfield and Galen Weitkamp, *Recursive aspects of descriptive set theory*, Oxford Logic Guides, vol. 11, The Clarendon Press, Oxford University Press, New York, 1985, With a chapter by Stephen Simpson. MR 786122 (86g:03003)
14. David Marker, *Scott ranks of countereexamples to vaught's conjecture*, <http://homepages.math.uic.edu/~marker/harrington-vaught.pdf>.
15. David Marker, *An analytic equivalence relation not arising from a Polish group action*, Fund. Math. **130** (1988), no. 3, 225–228. MR 970906 (89k:03056)
16. ———, *Model theory*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002, An introduction. MR 1924282 (2003e:03060)
17. Mark Nadel, *Scott sentences and admissible sets*, Ann. Math. Logic **7** (1974), 267–294. MR 0384471 (52 #5348)
18. Gerald E. Sacks, *Countable admissible ordinals and hyperdegrees*, Advances in Math. **20** (1976), no. 2, 213–262. MR 0429523 (55 #2536)
19. ———, *On the number of countable models*, Southeast Asian conference on logic (Singapore, 1981), Stud. Logic Found. Math., vol. 111, North-Holland, Amsterdam, 1983, pp. 185–195. MR 723338 (85i:03095)
20. ———, *Higher recursion theory*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1990. MR 1080970 (92a:03062)
21. Jindřich Zapletal, *Forcing borel reducibility invariants*, <http://people.clas.ufl.edu/zapletal/files/t3.pdf>.
22. ———, *Reducibility invariants in higher set theory*, <http://people.clas.ufl.edu/zapletal/files/turbulence7.pdf>.

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA 91125
E-mail address: wcchan@caltech.edu