THE SIZE OF A SET FROM THE SIZE OF ITS COUNTABLE POWER SET

WILLIAM CHAN

ABSTRACT. For any set X, let $\mathscr{P}(X)$ be the set of subsets of X and let $\mathscr{P}_{\omega_1}(X)$ be the set of countable subsets of X. AD^+ is Woodin's extension of the axiom of determinacy, AD. The Countable Power Set Conjecture states under AD^+ that for all sets X which are surjective images of $\mathbb{R}$, $\mathscr{P}(\omega_1)$ injects into $\mathscr{P}_{\omega_1}(X)$ if and only if $\mathscr{P}(\omega_1)$ already injects into X. Since $\mathbb{R} \times \omega_1$ and ${}^\omega\omega_1$ (the set of ω-sequences of countable ordinals) represent two distinguished cardinalities strictly below $|\mathscr{P}(\omega_1)|$, the following provides evidence for the conjecture:

- Assume AD^+. If X is a surjective image of $\mathbb{R}$ and $|\mathscr{P}(\omega_1)| \leq |\mathscr{P}_{\omega_1}(X)|$, then $|\mathbb{R} \times \omega_1| < |{}^\omega\omega_1| < |X|$.

Let ${}^{<\omega_1}\omega_1$ be the set of all countable sequences of countable ordinals. The above result is closely related to primeness properties of ${}^{<\omega_1}\omega_1$. The ω-Primeness Conjecture for ${}^{<\omega_1}\omega_1$ states that assuming the strong partition property $\omega_1 \rightarrow_* (\omega_1)^{\omega_1}_2$, ${}^{<\omega_1}\omega_1$ is ω-prime which means for any sequence $\langle Y_n : n \in \omega \rangle$ of sets, ${}^{<\omega_1}\omega_1$ injects into $\prod_{n \in \omega} Y_n$ if and only if there is an $\bar{n} \in \omega$ so that ${}^{<\omega_1}\omega_1$ already injects into $Y_{\bar{n}}$. The following evidence for this conjecture will be given:

- Assume $\omega_1 \rightarrow_* (\omega_1)^{\omega_1}_2$. For any sequence $\langle Y_n : n \in \omega \rangle$, if $|{}^{<\omega_1}\omega_1| \leq |\prod_{n \in \omega} Y_n|$, then there exists an $\bar{n} \in \omega$ so that $|{}^\omega\omega_1| \leq |Y_{\bar{n}}|$.

1. Introduction

A cardinality is an equivalence class of a set under the bijection equivalence relation. If X is a set, then the cardinality of X, denoted $|X|$, is the bijection equivalence class of X. If X and Y are two sets, then one writes $|X| \leq |Y|$ if and only if there is an injection of X into Y. One write $|X| < |Y|$ if and only if $|X| \leq |Y|$ but $\neg(|Y| \leq |X|)$. A cardinal is an ordinal which does not inject into any smaller ordinal. If X is a wellorderable set, then $|X|$ has a unique cardinal as a member.

The axiom of determinacy, AD, fully classifies the cardinalities below $\mathscr{P}(\omega)$ or $\mathbb{R}$. Regularity properties for $\mathbb{R}$ (including the perfect set property, property of Baire, and the Lebesgue measurability) answer many questions concerning combinatorial properties of the cardinality of $\mathbb{R}$ or $\mathscr{P}(\omega)$.

A major goal of recent research under determinacy has been to classify the cardinalities below $\mathscr{P}(\omega_1)$. Many cardinalities below $\mathscr{P}(\omega_1)$ have been identified and distinguished from each other according to the injection comparison relation. A deep understanding of many of the cardinalities strictly below $\mathscr{P}(\omega_1)$ have been obtained. However, the full classification below $\mathscr{P}(\omega_1)$ and many basic combinatorial properties of $\mathscr{P}(\omega_1)$ are very far from fully understood. Further development toward understanding the cardinality of $\mathscr{P}(\omega_1)$ has been motivated by many conjectures concerning inaccessibility of the cardinality of $\mathscr{P}(\omega_1)$.

One such inaccessibility feature concerns the regularity of $\mathscr{P}(\omega_1)$ and the search for the cofinality of $\mathscr{P}(\omega_1)$. Chan, Jackson, and Trang developed the basic theory of regularity and cofinality in the choiceless determinacy context in [9]. The conjecture is that $\mathscr{P}(\omega_1)$ has globally regular cardinality which means that for all sets X which are images of $\mathbb{R}$ such that $\mathscr{P}(\omega_1)$ does not inject into X and all functions $\Phi : \mathscr{P}(\omega_1) \to X$, there is an $x \in X$ so that $|\Phi^{-1}[\{x\}]| = |\mathscr{P}(\omega_1)|$. [9] provides substantial evidence for the global regularity of $\mathscr{P}(\omega_1)$.

This paper will focus on another inaccessibility conjecture for $\mathscr{P}(\omega_1)$ which was investigated in [5] involving the countable power set operation. If X is a set, then $\mathscr{P}(X)$ is the power set of X consisting of all subsets of X. If κ is a cardinal, then $\mathscr{P}_\kappa(X) = \{Y \subseteq X : |Y| < \kappa\}$. (That is, the collection of all subsets Y of X so that Y injects into κ but κ does not inject into Y.) A particularly interesting case is $\mathscr{P}_{\omega_1}(X)$ which is the collection of all countable subsets of X.

The Countable Power Set Conjecture states that for any set X, if $\mathscr{P}(\omega_1)$ injects into $\mathscr{P}_{\omega_1}(X)$, then $\mathscr{P}(\omega_1)$ already injected into X. The intuition is that the countable power set operation applied to any set X cannot create a copy of $\mathscr{P}(\omega_1)$ in $\mathscr{P}_{\omega_1}(X)$ unless a copy of $\mathscr{P}(\omega_1)$ was already present in X.

May 16, 2025. This research was funded in whole or in part by the Austrian Science Fund (FWF) 10.55776/Y1498.

$\mathscr{P}(\omega_1)$ contains many important smaller cardinalities which include $\mathbb{R}$, ω_1, $\mathbb{R} \sqcup \omega_1$, $\mathbb{R} \times \omega_1$, $[\omega_1]^\omega$, and $[\omega_1]^{<\omega_1}$. Evidence for the above conjecture can be given by showing that for any set X, if $|\mathscr{P}(\omega_1)| \leq |\mathscr{P}_{\omega_1}(X)|$, then X must have an injective copy of $\mathbb{R}$, ω_1, $\mathbb{R} \sqcup \omega_1$, $\mathbb{R} \times \omega_1$, $[\omega_1]^\omega$, or $[\omega_1]^{<\omega_1}$.

An ∞-Borel code is a pair (S, φ) so that S is a set of ordinals and φ is a formula in the language of set theory. A set $A \subseteq \mathbb{R}$ is ∞-Borel if and only if there is an ∞-Borel code (S, φ) so that $A = \{x \in \mathbb{R} : L[S, x] \models \varphi(S, x)\}$. An ∞-Borel code for A is a highly absolute definition for A in the sense that to query membership of $x \in A$, one only needs to determine the truth of $\varphi(S, x)$ in $L[S, x]$ which is the minimal inner model of ZFC containing x and the code set S. The statement that all subsets of $\mathbb{R}$ have ∞-Borel codes and $\mathsf{DC}_\mathbb{R}$ are part of Woodin's extension of determinacy known as AD^+. [5] Theorem 6.10 began this investigation by showing that under AD^+, for any set X which is an image of $\mathbb{R}$, if $|\mathscr{P}(\omega_1)| \leq |\mathscr{P}_{\omega_1}(X)|$, then $|\mathbb{R} \sqcup \omega_1| < |X|$. [5] asked if X is an image of $\mathbb{R}$ and $|\mathscr{P}(\omega_1)| \leq |\mathscr{P}_{\omega_1}(X)|$, then does $|\mathbb{R} \times \omega_1| < |X|$ hold? Moreover, since $|\mathbb{R} \times \omega_1| < |[\omega_1]^\omega| < |\mathscr{P}(\omega_1)|$, an even stronger question would be under the same setting, does $|[\omega_1]^\omega| < |X|$ holds? The goal of the paper is to answer the latter question positively which will provide evidence for the Countable Power Set Conjecture:

- (Theorem 3.6) Assume AD, $\mathsf{DC}_\mathbb{R}$, and all subsets of $\mathbb{R}$ are ∞-Borel. For any set Y which is a surjective image of $\mathbb{R}$, if $|\mathscr{P}(\omega_1)| \leq |\mathscr{P}_{\omega_1}(Y)|$, then $|[\omega_1]^\omega| < |Y|$.

The key combinatorial property for the main theorem requires having a sufficiently deep understanding of cardinalities of sets to answer a class of related "finite-dimensional combinatorial questions". Vaguely, an n-dimensional combinatorial question on a set X requires finding an $\tilde{X} \subseteq X$ with $|\tilde{X}| = |X|$ for which one has complete control over all n-tuples from $\tilde{X}$. An important one-dimensional combinatorial question concerns regularity and cofinality. For example, for a particular set X and cardinal δ, X is δ-regular if and only if for all functions $\Phi : X \to \delta$, there exists a $\gamma < \delta$ such that $|\Phi^{-1}[\{\gamma\}]| = |X|$. One can asks for which ordinal δ is X δ-regular? There are many important two-dimensional combinatorial questions including primeness and basis for linear orderings. Important finite-dimensional questions (of arbitrary finite dimension) include the Jónssonness property and optimal colorings for partitions on n-tuple through a set X. Primeness is most relevant here. Chan, Jackson, and Trang defines the notion of primeness in [8]. A set X is a said to be prime (or 2-prime) if and only if for all sets A and B, if $|X| \leq |A \times B|$, then $|X| \leq |A|$ or $|X| \leq |B|$. [8] shows that under $\mathsf{AC}_\omega^\mathbb{R}$ and all subsets of $\mathbb{R}$ have the Baire property, $\mathbb{R}$ and $\mathbb{R}/E_0$ are prime. [8] shows that if κ is an uncountable cardinal satisfying $\kappa \to_* (\kappa)_2^{\omega+\omega}$, then $[\kappa]^\omega$ is prime. Moreover, [8] shows that for all $n < \omega$, $[\omega_n]^\omega$ is prime and even $[\omega_\omega]^\omega$ is prime (even through ω_n does not possess partition property since ω_n is singular when $3 \leq n \leq \omega$). More recently, Chan [2] has used a generalized Namba forcing (a form of Cox-Krueger Namba forcing [11] or diagonal Prikry forcing) over HOD-type inner models under AD^+ to determine the exact extent of ordinal regularity for ${}^\omega\kappa$ for all cardinals $\kappa < \Theta$. These Namba forcing methods can be adapted to establish the primeness of ${}^\omega\kappa$ for other $\kappa < \Theta$. These recent developments have provided a deep combinatorial understanding of the cardinality of ω-sequences of ordinals (which corresponds to the smallest nonwellorderable cardinal exponentiations). In particular, note that the 2-primeness of $[\omega_1]^\omega$ under the partition relation $\omega_1 \to_* (\omega_1)_2^{\omega+\omega}$ can be regarded as an inaccessibility phenomenon for $[\omega_1]^\omega$ with respect to the two-fold cartesian product operator: For any set A and B, $A \times B$ has an injective copy of $[\omega_1]^\omega$ if and only if one of the factors A or B already possessed an injective copy of $[\omega_1]^\omega$.

However, one is still missing such understanding for the cardinality of $[\omega_1]^{<\omega_1}$. If κ is an ordinals and X is a set, then say that X is κ-prime if and only if for any sequence $\langle Y_\alpha : \alpha < \kappa \rangle$, if $|X| \leq |\prod_{\alpha<\kappa} Y_\alpha|$, then there exists an $\bar{\alpha} < \kappa$ so that $|X| \leq |Y_{\bar{\alpha}}|$. Cardinals of countable cofinality are not ω-prime. It is shown in [8] that if κ is a cardinal and $\lambda < \kappa$, then κ is λ-prime if and only if $\lambda < \mathrm{cof}(\kappa)$ and for all $\delta < \kappa$, $\neg(|\kappa| \leq |{}^\lambda\delta|)$. Steel ([19] Theorem 8.26) and Woodin ([20] Theorem 2.16) showed that under AD^+, boldface GCH holds below Θ which is that statement that for all $\kappa < \Theta$, κ^+ does not inject into $\mathscr{P}(\kappa)$. Thus under AD^+, a cardinal κ is λ-prime if and only if $\lambda < \mathrm{cof}(\kappa)$. In particular, any cardinal of uncountable cofinality is ω-prime under AD^+. Note that $[\omega_1]^\omega$ is clearly not ω-prime although it is 2-prime under the partition relation $\omega_1 \to_* (\omega_1)_2^{\omega+\omega}$. There are currently no known example of a nonwellorderable ω-prime set but the natural candidate is $[\omega_1]^{<\omega_1}$. However, it is not presently known if $[\omega_1]^{<\omega_1}$ is even 2-prime. The ω-Primeness Conjecture for $[\omega_1]^{<\omega_1}$ is the statement that the strong partition property $\omega_1 \to_* (\omega_1)_2^{\omega_1}$ (or the very strong partition property $\omega_1 \to_* (\omega_1)_{<\omega_1}^{\omega_1}$) implies $[\omega_1]^{<\omega_1}$ is ω-prime. Intuitively this conjecture states that $[\omega_1]^{<\omega_1}$ is inaccessible to the countable cartesian product operation in the sense that for any sequence

$\langle Y_n : n \in \omega \rangle$ of sets, $[\omega_1]^{<\omega_1}$ injects into $\prod_{n\in\omega} Y_n$ if and only if there is a factor $Y_{\bar{n}}$ which already has an injective copy of $[\omega_1]^{<\omega_1}$. The key combinatorial fact needed to prove Theorem 3.6 is the following evidence for the ω-Primeness Conjecture for $[\omega_1]^{<\omega_1}$:

- (Theorem 2.11) Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$. Let $\langle Y_n : n \in \omega \rangle$ be a sequence of sets. If $|[\omega_1]^{<\omega_1}| \leq |\prod_{n<\omega} Y_n|$, then there is an $\bar{n} \in \omega$ so that $|[\omega_1]^{\omega}| < |Y_{\bar{n}}|$.

The last ingredient to prove the main result of the paper will involve some consequences of Woodin's analysis of nice models of AD^+ by OD ∞-Borel code forcings (a variation of Vopěnka forcing). This forcing will not be used directly here but two consequences will be needed. The first states in nice AD^+ models, there is a uniform procedure to transform ordinal definable (with parameters) definition of subsets of $\mathbb{R}$ into ordinal definable (with same parameters) ∞-Borel code for this subset of $\mathbb{R}$. Woodin's perfect set dichotomy ([1],[4]) states that if X is a surjective image of $\mathbb{R}$, then either X is wellorderable or $\mathbb{R}$ injects into X. The second necessary fact is a uniform version of Woodin's perfect set dichotomy which states that if X is a quotient of an equivalence relation E on $\mathbb{R}$ with ∞ Borel (S, φ) and X is wellorderable, then a wellordering of X can be founded uniformly from the ∞-Borel code (S, φ). If Y is a set, let $\mathcal{I}_{<\omega_1}(Y)$ be the set of all injective functions $f : \epsilon \to Y$ where $\epsilon < \omega_1$. The following will be shown:

- (Theorem 3.5) Assume AD, $\mathsf{DC}_{\mathbb{R}}$, and all subsets of $\mathbb{R}$ are ∞-Borel. If Y is a surjective image of $\mathbb{R}$ and $\Phi : \mathscr{P}(\omega_1) \to \mathscr{P}_{\omega_1}(Y)$, then there exists a function $\tilde{\Phi} : \mathscr{P}(\omega_1) \to \mathcal{I}_{<\omega_1}(Y)$ so that $\tilde{\Phi}(A)$ enumerate $\Phi(A)$ for all $A \in \mathscr{P}(\omega_1)$.

What is remarkable about Theorem 3.5 is that there is generally no map $\Gamma : \mathscr{P}_{\omega_1}(Y) \to \mathcal{I}_{<\omega_1}(Y)$ so that $\Gamma(B)$ is a wellordered enumeration of B for all $B \in \mathscr{P}_{<\omega_1}(Y)$. In fact, there is generally no injection from $\mathscr{P}_{\omega_1}(Y)$ into $\mathcal{I}_{<\omega_1}(Y)$. For example for $Y = \mathbb{R}$, one can show that $|\mathcal{I}_{<\omega_1}(\mathbb{R})| < |\mathscr{P}_{\omega_1}(\mathbb{R})|$.

Given the main result of the paper that for any set X which is an image of $\mathbb{R}$, $|\mathscr{P}(\omega_1)| \leq |\mathscr{P}_{\omega_1}(X)|$ implies $|[\omega_1]^{\omega}| < |X|$, the next natural evidence one would like for the Countable Power Set Conjecture would be to show that $|\mathscr{P}(\omega_1)| \leq |\mathscr{P}_{\omega_1}(X)|$ implies $|[\omega_1]^{<\omega_1}| < |X|$. If the ω-Primeness Conjecture for $[\omega_1]^{<\omega_1}$ can be shown, then the arguments of Section 3 will immediately show that $|\mathscr{P}(\omega_1)| \leq |\mathscr{P}_{\omega_1}(X)|$ implies $|[\omega_1]^{<\omega_1}| < |X|$. One will need a deeper grasp of the cardinality of $[\omega_1]^{<\omega_1}$ to handle ω-primeness and other finite dimensional combinatorial questions involving $[\omega_1]^{<\omega_1}$.

2. Evidence for Primeness

Definition 2.1. If $\epsilon \in \mathrm{ON}$ and $f : \epsilon \to \mathrm{ON}$.

- f is discontinuous everywhere if and only if for all $\beta < \epsilon$, $\sup(f \restriction \beta) = \sup\{f(\alpha) : \alpha < \beta\} < f(\beta)$.
- f has uniform cofinality ω if and only if there is a function $F : \epsilon \times \omega \to \mathrm{ON}$ so that for all $\alpha < \epsilon$ and $n \in \omega$, $F(\alpha, n) < F(\alpha, n+1)$ and $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$. (A function $F : \epsilon \times \omega \to \mathrm{ON}$ with this property will be a called a witness for the uniform cofinality ω of f.)
- f has the correct type if and only if f is both discontinuous everywhere and has uniform cofinality ω.

If $X \subseteq \mathrm{ON}$ and $\epsilon \in \mathrm{ON}$, then $[X]^{\epsilon}_*$ will denote the collection of $f : \epsilon \to X$ of the correct type.

The following is the correct type club partition relation.

Definition 2.2. Let κ be a cardinal.

- Let $\epsilon \leq \kappa$ and $\gamma < \kappa$. $\kappa \to_* (\kappa)^{\epsilon}_{\gamma}$ asserts that for all $P : [\kappa]^{\epsilon}_* \to \gamma$, there is a $\beta < \gamma$ and a club $C \subseteq \kappa$ so that for all $f \in [C]^{\epsilon}_*$, $P(f) = \beta$.
- Let $\epsilon \leq \kappa$ and $\gamma \leq \kappa$. $\kappa \to_* (\kappa)^{<\epsilon}_{<\gamma}$ asserts that for all $\epsilon' < \epsilon$ and $\gamma' < \gamma$, $\kappa \to_* (\kappa)^{\epsilon'}_{\gamma'}$.
- If κ satisfies $\kappa \to_* (\kappa)^{\kappa}_2$, then κ is called a strong partition cardinal.
- If κ satisfies $\kappa \to_* (\kappa)^{\kappa}_{<\kappa}$, then κ is called a very strong partition cardinal.
- If κ satisfies $\kappa \to_* (\kappa)^{<\kappa}_2$, then κ is called a weak partition cardinal.

See [17], [4], and [3] for more information on the combinatorial aspects of partition properties. The ordinary partition relation $\kappa \to (\kappa)^{\epsilon}_{\gamma}$ is the assertion that for all $P : [\kappa]^{\epsilon} \to \kappa$, there is a $\beta < \gamma$ and an $A \subseteq \kappa$ with $|A| = \kappa$ so that for all $f \in [A]^{\epsilon}$, $P(f) = \beta$. The correct type club partition and the ordinal partition relation are closely related after a small shift in exponent: $\kappa \to_* (\kappa)^{\epsilon}_2$ implies $\kappa \to (\kappa)^{\epsilon}_2$ and $\kappa \to (\kappa)^{\omega \cdot \epsilon}_2$ implies $\kappa \to_* (\kappa)^{\epsilon}_2$. The correct type partition relation is much more practical for combinatorial arguments and essentially all applications of partition properties in determinacy context directly or indirectly use the notion of a correct type partition.

It can be shown that for any $\epsilon < \kappa$, $\kappa \rightarrow_* (\kappa)^{\epsilon+\epsilon}_2$ implies that $\kappa \rightarrow (\kappa)^{\epsilon}_{<\kappa}$. Thus if κ is a weak partition cardinal, then $\kappa \rightarrow_* (\kappa)^{<\kappa}_{<\kappa}$ holds. It is open if a strong partition cardinal must always be a very strong partition cardinal. The most natural universes to study partition properties are those satisfying AD. Moreover, the only known method to establish the existence of a strong partition cardinal comes from Martin's argument which uses pointclass boundedness principles under determinacy to create good coding systems for functions from certain ordinals to certain ordinals. (See [15], [14], [4], and [3] for more information concerning good coding systems.)

If Γ is a pointclass, then let $\check{\Gamma}$ refer to the dual pointclass. $\delta(\Gamma)$ is the supremum of the length of the prewellordering in $\Gamma \cap \check{\Gamma}$. For each $n \in \omega$, $\boldsymbol{\delta}^1_n$ is defined to be $\delta(\boldsymbol{\Sigma}^1_n)$. By using the Kunen-Martin Theorem, Kunen and Martin showed that $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ for all $n \in \omega$. Jackson ([14], [13], and [12]) computed the value of all the projective ordinals $\boldsymbol{\delta}^1_n$ for $1 \leq n < \omega$. $\boldsymbol{\delta}^1_1 = \omega_1$ and thus $\boldsymbol{\delta}^1_2 = \omega_2$. Martin showed that $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ and thus $\boldsymbol{\delta}^1_4 = \omega_{\omega+2}$. $\boldsymbol{\delta}^2_1$ is defined to be $\delta(\boldsymbol{\Sigma}^2_1)$. Let $A \in \mathscr{P}(\omega_1)$. $\boldsymbol{\delta}_A$ is the least δ such that $L_\delta(A, \mathbb{R}) \prec_1 L(A, \mathbb{R})$ (Σ_1-elementarity in the language of set theory augmented with constant symbols for each real and the collection of all reals and the models are given the intended interpretation). It can be shown that $\boldsymbol{\delta}_\emptyset = (\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$. The following are some important examples of partition cardinals under the axiom of determinacy for which the results of this paper apply.

Fact 2.3. *Assume* AD.

- *(Martin;* [15] *Theorem 12.2,* [4] *Corollary 4.27)* ω_1 *is a very strong partition cardinal.*
- *(Martin-Paris;* [15] *Corollary 13.5,* [4] *Theorem 5.19 and Corollary 6.17)* ω_2 *is a weak partition cardinal but not a strong partition cardinal.*
- *(Jackson;* [13],[12]*)* $\boldsymbol{\delta}^1_{2n+1}$ *is a very strong partition cardinal.*
- *(Jackson;* [14], [13],[12]*)* $\boldsymbol{\delta}^1_{2n+2}$ *is a weak partition cardinal and not a strong partition cardinal.*
- *(Kechris-Kleinberg-Moschovakis-Woodin;* [16], [18] *Theorem 5.8) For any* $A \in \mathscr{P}(\kappa)$*,* $\boldsymbol{\delta}_A$ *is a very strong partition cardinal.*
- *(Kechris-Kleinberg-Moschovakis-Woodin;* [16], [18] *Theorem 5.5) (*$\mathsf{DC}_{\mathbb{R}}$*)* $\boldsymbol{\delta}^2_1$ *is a very strong partition cardinal.*

First, one will define a combinatorially useful injection of $[\omega_1]^\omega$ into $[\omega_1]^{<\omega_1}$.

Definition 2.4. Let κ be a cardinal and $\Delta : \kappa \times \omega \rightarrow \kappa$ be a function. Define $\mathfrak{d}_\Delta : \kappa \rightarrow \kappa$ be defined $\mathfrak{d}_\Delta(\alpha) = \sup\{\Delta(\alpha, n) : n \in \omega\}$. Say that $\Delta : \kappa \times \omega \rightarrow \kappa$ is a witness for uniform cofinality ω for some discontinuous function (namely $\mathfrak{d}_\Delta$) if and only if the following hold.

(1) For all $\alpha < \kappa$ and $n \in \omega$, $0 < \Delta(\alpha, n) < \Delta(\alpha, n+1)$.
(2) Then $\mathfrak{d}_\Delta : \kappa \rightarrow \kappa$ is a discontinuous function (that is, for all $\alpha < \kappa$, $\sup(\mathfrak{d}_\Delta \restriction \alpha) < \mathfrak{d}_\Delta(\alpha)$ and thus $\mathfrak{d}_\Delta$ must be a strictly increasing function).

(Note that Δ witnesses that $\mathfrak{d}_\Delta$ is a function of uniform cofinality ω and thus $\mathfrak{d}_\Delta \in [\kappa]^\kappa_*$.)

Let $\gamma < \kappa$ and $\alpha < \mathfrak{d}_\Delta(\gamma)$. Let n^γ_α be the least n so that $\alpha < \Delta(\gamma, n)$. Let L_Δ consists of all tuples $(\gamma, \alpha, \beta_0, ..., \beta_{n^\gamma_\alpha - 1})$ with the following properties.

(1) $\gamma < \kappa$ and $\alpha < \mathfrak{d}_\Delta(\gamma)$.
(2) If $n^\gamma_\alpha = 0$, then the tuple takes the form (γ, α).
(3) If $n^\gamma_\alpha > 0$, then $\beta_0 < \beta_1 < ... < \beta_{n^\gamma_\alpha - 1} < \gamma$.

Let $\prec_\Delta$ be the lexicographic ordering on L_Δ. Explicitly, $(\gamma^0, \alpha^0, \beta^0_0, ..., \beta^0_{n^{\gamma^0}_{\alpha^0} - 1}) \prec_\Delta (\gamma^1, \alpha^1, \beta^1_0, ..., \beta^1_{n^{\gamma^1}_{\alpha^1} - 1})$ if and only if the disjunction of the following holds.

(1) $\gamma^0 < \gamma^1$.
(2) $\gamma^0 = \gamma^1$ and $\alpha^0 < \alpha^1$.
(3) $\gamma^0 = \gamma^1$, $\alpha^0 = \alpha^1$, and there exists an $i < n^{\gamma^0}_{\alpha^0} = n^{\gamma^1}_{\alpha^1}$ such that for all $j < i$, $\beta^0_j = \beta^1_j$ and $\beta^0_i < \beta^1_i$.

Denote $\mathcal{L}_\Delta$ to be the linear ordering $(L_\Delta, \prec_\Delta)$.

Fact 2.5. *Let* κ *be a cardinal. Let* $\Delta : \kappa \times \omega \rightarrow \kappa$ *be a witness for uniform cofinality* ω *for some increasing discontinuous function.* $\mathcal{L}_\Delta = (L_\Delta, \prec_\Delta)$ *is a wellordering of ordertype* κ.

Proof. Suppose $\langle x_k : k \in \omega\rangle$ is a sequence in L_Δ so that for all $k \in \omega$, $x_{k+1} \prec_\Delta x_k$. Say $x_k = (\gamma^k, \alpha^k, \beta_0^k, ..., \beta^k_{n^{\gamma^k}_{\alpha^k}-1})$. Thus $\langle \gamma^k : k \in \omega\rangle$ is a non-increasing sequence of ordinals since for all $k \in \omega$, $x_{k+1} \prec_\Delta x_k$. By the wellfoundedness of the ordinals, there is a $K_0 \in \omega$ and a $\bar{\gamma}$ so that for all $K_0 < k < \omega$, $\gamma^k = \bar{\gamma}$. Since $x_{k+1} \prec_\Delta x_k$ for all $k \in \omega$, $\langle \alpha^k : K_0 < k < \omega\rangle$ is a non-increasing sequence of ordinals. By wellfoundedness of the ordinals, there is a $K_1 > K_0$ and an $\bar{\alpha} < \mathfrak{d}_\Delta(\bar{\gamma})$ so that for all $K_1 < k < \omega$, $\alpha^k = \bar{\alpha}$. Thus for all $K_1 < k < \omega$, $n^{\gamma^k}_{\alpha^k} = n^{\bar{\gamma}}_{\bar{\alpha}}$. Let $\bar{n} = n^{\bar{\gamma}}_{\bar{\alpha}}$. (If $\bar{n} = 0$, then the next steps are not relevant.) Suppose $i \leq \bar{n} - 1$, $\bar{\beta}_j$ for all $j < i$ have been defined, and $K_{i+1} \in \omega$ has been defined so that for all all $K_{i+1} < k < \omega$, $\gamma^k = \bar{\gamma}$, $\alpha^k = \bar{\alpha}$, and for all $j < i$, $\beta^k_j = \bar{\beta}_j$. Then $\langle \beta^k_i : K_{i+1} < k < \omega\rangle$ is a non-increasing sequence of ordinals. By wellfoundedness, there is a $K_{i+2} > K_{i+1}$ and $\bar{\beta}_i$ so that for all $K_{i+2} < k < \omega$, $\beta^k_i = \bar{\beta}_i$. Thus for all k such that $K_{\bar{n}+1} < k < \omega$, $x_k = (\bar{\gamma}, \bar{\alpha}, \bar{\beta}_0, ..., \bar{\beta}_{\bar{n}-1})$. Thus the sequence $\langle x_k : k \in \omega\rangle$ is eventually constant. This violates the assumption that $x_{k+1} \prec_\Delta x_k$ for all $k \in \omega$. It has been shown that $\mathcal{L}_\kappa$ is wellfounded.

For each $\gamma < \kappa$, let A_γ consist of those elements of L_Δ whose first coordinate is less than γ. Since A_γ injects into $\bigcup_{n<\omega} {}^n\Big(\mathfrak{d}_\Delta(\gamma)\Big)$, $|A_\gamma| = |\mathfrak{d}_\Delta(\gamma)| < |\kappa|$. Hence the order type of each A_γ is less than κ. Since for each $\gamma < \kappa$, A_γ is an initial segment of $\mathcal{L}_\Delta$ under $\prec_\Delta$, the ordertype of $\mathcal{L}_\Delta$ is κ. ☐

Definition 2.6. Let κ, Δ, and $\mathfrak{d}_\Delta$ be as in Definition 2.4. A pair (Δ, Σ) is suitable if and only if the following holds.

(1) $\Sigma : L_\Delta \to \kappa$ is an order preserving function from $\mathcal{L}_\Delta = (L_\Delta, \prec_\Delta)$ into $(\kappa, <)$ (the usual ordinal ordering on κ).
(2) For all $\gamma_0 < \gamma_1 < \kappa$ and $(\gamma_0, \alpha, \beta_0, ..., \beta_{n^{\gamma_0}_\alpha - 1}) \in L_\Delta$, $\mathfrak{d}_\Delta(\gamma_0) < \Sigma(\gamma_0, \alpha, \beta_0, ..., \beta_{n^{\gamma_0}_\alpha-1}) < \mathfrak{d}_\Delta(\gamma_1)$.
(3) Σ has the correct type in the following natural sense:
  (a) Σ is discontinuous: For all $x \in L_\Delta$, $\sup(\Sigma \restriction x) = \sup\{\Sigma(y) : y \prec_\Delta x\} < \Sigma(x)$.
  (b) Σ has uniform cofinality ω: There is a $\Pi : L_\Delta \times \omega \to \kappa$ so that for all $x \in L_\Delta$ and $n \in \omega$, $\Pi(x, n) < \Pi(x, n + 1)$ and $\Sigma(x) = \sup\{\Pi(x, n) : n \in \omega\}$.

For each $f \in [\kappa]^\omega$, define a function $\Lambda_{(\Delta,\Sigma)}(f) \colon \mathfrak{d}_\Delta(\sup(f)) \to \kappa$ by

$$\Lambda_{(\Delta,\Sigma)}(f)(\alpha) = \Sigma(\sup(f), \alpha, f(0), ..., f(n^{\sup(f)}_\alpha - 1))$$

for each $\alpha < \mathfrak{d}_\Delta(\sup(f))$. Thus $\Lambda_{(\Delta,\Sigma)} : [\kappa]^\omega \to {}^{<\kappa}\kappa$.

Fact 2.7. *Let κ, Δ, $\mathfrak{d}_\Delta$, $\mathcal{L}_\Delta = (L_\Delta, \prec_\Delta)$, Σ, $\Lambda_{(\Delta,\Sigma)}$ be as in Definition 2.4 and Definition 2.6. The following properties hold.*

(1) For all $f \in [\kappa]^\omega$ and $\alpha_0 < \alpha_1 < \mathfrak{d}_\Delta(\sup(f))$, $\Lambda_{(\Delta,\Sigma)}(f)(\alpha_0) < \Lambda_{(\Delta,\Sigma)}(f)(\alpha_1)$.
(2) For all $f \in [\omega_1]^\omega$, $\Lambda_{(\Delta,\Sigma)}(f)(0) > \mathfrak{d}_\Delta(\sup(f))$ and $\Lambda_{(\Delta,\Sigma)}(f) \in [\kappa]^{\mathfrak{d}_\Delta(\sup(f))}_*$.
(3) For all $f, g \in [\kappa]^\omega$, if $\sup(f) < \sup(g)$, then $\sup(\Lambda_{(\Delta,\Sigma)}(f)) < \mathfrak{d}_\Delta(\sup(g)) < \Lambda_{(\Delta,\Sigma)}(g)(0)$.
(4) Suppose $f, g \in [\kappa]^\omega$, $\sup(f) = \sup(g)$, $m \in \omega$ is least so that $f(m) \neq g(m)$, and $f(m) < g(m)$. For all $\alpha < \Delta(\sup(f), m) = \Delta(\sup(g), m)$, $\Lambda_{(\Delta,\Sigma)}(f)(\alpha) = \Lambda_{(\Delta,\Sigma)}(g)(\alpha)$. For all $\Delta(\sup(f), m) = \Delta(\sup(g), m) \leq \alpha < \mathfrak{d}_\Delta(\sup(f)) = \mathfrak{d}_\Delta(\sup(g))$, $\Lambda_{(\Delta,\Sigma)}(f)(\alpha) < \Lambda_{(\Delta,\Sigma)}(g)(\alpha) < \Lambda_{(\Delta,\Sigma)}(f)(\alpha + 1)$.
(5) $\Lambda_\Sigma : [\kappa]^\omega \to [\kappa]^{<\kappa}_*$ is an injection. Let $K = \Sigma[L_\Delta]$ and $J = \mathfrak{d}_\Delta[\kappa]$. For all $\epsilon \in J$, $\mathrm{cof}(\epsilon) = \omega$. $\Lambda_{(\Delta,\Sigma)} : [\kappa]^\omega \to \bigcup_{\epsilon\in J}[K \setminus (\epsilon + 1)]^\epsilon_*$ is an injection.

Proof. (1) Let $f \in [\kappa]^\omega$ and $\alpha_0 < \alpha_1 < \mathfrak{d}_\Delta(\sup(f))$. Observe $(\sup(f), \alpha_0, f(0), ..., f(n^{\sup(f)}_{\alpha_0} - 1)) \prec_\Delta (\sup(f), \alpha_1, f(0), ..., f(n^{\sup(f)}_{\alpha_1} - 1))$ by comparing the second coordinate. Thus $\Lambda_{(\Delta,\Sigma)}(f)(\alpha_0) < \Lambda_{(\Delta,\Sigma)}(f)(\alpha_1)$.

(2) By the suitability of the pair (Δ, Σ), $\Lambda_{(\Delta,\Sigma)}(f)(0) = \Sigma(\sup(f), 0) > \mathfrak{d}(\sup(f))$. Note that (1) shows that $\Lambda_{(\Delta,\Sigma)}(f)$ is an increasing function. For all $\alpha < \mathfrak{d}_\Delta(\sup(f))$, $\sup(\Lambda_{(\Delta,\Sigma)}(f) \restriction \alpha) = \sup(\Sigma \restriction (\sup(f), \alpha, f(0), ..., f(n^{\sup(f)}_\alpha - 1))) < \Sigma(\sup(f), \alpha, f(0), ..., f(n^{\sup(f)}_\alpha - 1)) = \Lambda_{(\Delta,\Sigma)}(f)(\alpha)$ using the discontinuity of Σ. Thus $\Lambda_{(\Delta,\Sigma)}(f)$ is discontinuous everywhere. Fix a function $\Pi : L_\Delta \times \omega \to \kappa$ which witnesses that Σ has uniform cofinality ω as in Definition 2.6 (3b). Define $F : \mathfrak{d}_\Delta(\sup(f)) \times \omega \to \kappa$ by $F(\alpha, n) = \Pi((\sup(f), \alpha, f(0), ..., f(n^{\sup(f)}_\alpha - 1)), n)$. F witnesses that $\Lambda_{(\Delta,\Sigma)}(f)$ has uniform cofinality ω. Thus $\Lambda_{(\Delta,\Sigma)}(f)$ has the correct type and hence $\Lambda_{(\Delta,\Sigma)}(f) \in [\kappa]^{\mathfrak{d}_\Delta(\sup(f))}_*$.

(3) Suppose $f, g \in [\kappa]^\omega$ and $\sup(f) < \sup(g)$. For all $\alpha < \mathfrak{d}_\Delta(\sup(f))$, $(\sup(f), \alpha, f(0), ..., f(n_\alpha^{\sup(f)} - 1)) \prec_\Delta (\sup(f)+1, 0)$ by comparing the first coordinates. Thus $\Lambda_{(\Delta,\Sigma)}(f)(\alpha) < \Sigma(\sup(f)+1, 0) < \mathfrak{d}_\Delta(\sup(g)) < \Lambda(g)(0)$ for all $\alpha < \mathfrak{d}_\Delta(\sup(f))$ by (2) established above and condition (2) in Definition 2.6. Hence $\sup(\Lambda_{(\Delta,\Sigma)}(f)) < \mathfrak{d}_\Delta(\sup(g)) < \Lambda_{(\Delta,\Sigma)}(g)(0)$.

(4) Suppose $f, g \in [\kappa]^\omega$, $\sup(f) = \sup(g)$, m is the least so that $f(m) \neq g(m)$, and $f(m) < g(m)$. Suppose $\alpha < \Delta(\sup(f), m) = \Sigma(\sup(g), m)$. Then $n_\alpha^{\sup(f)} = n_\alpha^{\sup(g)} \leq m$. Thus

$$\Lambda_{(\Delta,\Sigma)}(f)(\alpha) = \Sigma(\sup(f), \alpha, f(0), ..., f(n_\alpha^{\sup(f)} - 1)) = \Sigma(\sup(g), \alpha, g(0), ..., g(n_\alpha^{\sup(g)} - 1)) = \Lambda_{(\Delta,\Sigma)}(g)(\alpha).$$

Now suppose $\Delta(\sup(f), m) = \Delta(\sup(g), m) \leq \alpha$. Then $n_\alpha^{\sup(f)} = n_\alpha^{\sup(g)} > m$. Thus

$$(\sup(f), \alpha, f(0), ..., f(n_\alpha^{\sup(f)}-1)) \prec_\Delta (\sup(g), \alpha, g(0), ..., g(n_\alpha^{\sup(g)}-1)) \prec_\Delta (\sup(f), \alpha+1, f(0), ..., f(n_{\alpha+1}^{\sup(f)}))$$

by comparing the $(m+2)^{\text{th}}$ entries for the first inequality and comparing the second entry for the second inequality. Thus $\Lambda_{(\Delta,\Sigma}(f)(\alpha) < \Lambda_{(\Delta,\Sigma)}(g)(\alpha) < \Lambda_{(\Delta,\Sigma)}(f)(\alpha+1)$.

(5) Suppose $f, g \in [\kappa]^\omega$ and $f \neq g$. If $\sup(f) \neq \sup(g)$, then (3) implies $\Lambda_{(\Delta,\Sigma)}(f) \neq \Lambda_{(\Delta,\Sigma)}(g)$. Suppose $\sup(f) = \sup(g)$. There is a least $m \in \omega$ so that $f(m) \neq g(m)$. Then $\Lambda_{(\Delta,\Sigma)}(f) \neq \Lambda_{(\Delta,\Sigma)}(g)$ by (4). Thus $\Lambda_{(\Delta,\Sigma)}$ is an injection. Let $K = \Sigma[L_\Delta]$ and $J = \mathfrak{d}_\Delta[\kappa]$. By definition of Σ and $\Lambda_{(\Delta,\Sigma)}$ and (2), for all $f \in [\kappa]^\kappa$, $\Lambda_{(\Delta,\Sigma)}(f) \in [K]_*^{\mathfrak{d}_\Delta(\sup(f))}$. Since $\Lambda_{(\Delta,\Sigma)}(f)(0) > \mathfrak{d}_\Delta(\sup(f))$ by (2) and $\mathfrak{d}_\Delta(\sup(f)) \in J$, $\Lambda_{(\Delta,\Sigma)}(f) \in \bigcup_{\epsilon \in J} [K \setminus (\epsilon+1)]_*^\epsilon$. Note that for all $\gamma \in \kappa$, $\text{cof}(\mathfrak{d}_\Delta(\gamma)) = \omega$ as witnessed by the cofinal sequence $\langle \Delta(\gamma, n) : n \in \omega \rangle$. Thus for all $\epsilon \in J$, $\text{cof}(\epsilon) = \omega$. ☐

Let $X \subseteq \kappa$ with $|X| = \kappa$. Let $\text{enum}_X : \kappa \to X$ be the increasing enumeration of X. For $\alpha, \beta < \kappa$, let $\text{next}_X^\alpha(\beta)$ be the $(1+\alpha)^{\text{th}}$-element of X greater than β.

Fact 2.8. *Let κ be a cardinal and $C \subseteq \kappa$ be a club subset of κ. Then there is a suitable (Δ, Σ) such that $\Delta : \kappa \times \omega \to [C]_*^1$ and $\Sigma : L_\Delta \to C$. (Note that $\mathfrak{d}_\Delta : \kappa \to [C]_*^1$ and $\Sigma : L_\Delta \to [C]_*^1$ since Σ has the correct type by suitability.)*

Proof. The function $\Delta : \kappa \times \omega \to [C]_*^1$, $\Sigma : L_\Delta \to C$, and $\Pi : L_\Delta \times \omega \to \kappa$ will be defined by recursion. Let $\gamma < \kappa$. Suppose $\Delta \restriction \gamma \times \omega$ has been defined. This implies one can meaningfully define the objects $\mathfrak{d}_\Delta \restriction \gamma$, n_α^η for all $\eta < \alpha < \mathfrak{d}_\Delta(\eta)$, and a linear ordering A_γ using the definition of L_Δ from Definition 2.4 with the restriction that the first coordinate is less than γ. Suppose $\Sigma \restriction A_\gamma$ and $\Pi \restriction A_\gamma \times \omega$ have been defined. Suppose the following properties have been shown:

(1) $\Sigma \restriction A_\gamma$ is increasing.
(2) For all $x \in A_\gamma$ with first coordinate $\eta < \gamma$, $\mathfrak{d}_\Delta(\eta) < \Sigma(x)$.
(3) For all $\gamma_0 < \gamma_1 < \gamma$ and $x \in A_\gamma$ with first coordinate γ_0, $\Sigma(x) < \mathfrak{d}_\Delta(\gamma_1)$.
(4) $\Pi \restriction A_\gamma \times \omega : A_\gamma \times \omega \to \kappa$ witnesses that $\Sigma \restriction A_\gamma$ has uniform cofinality ω.

Let $\delta_\gamma = \sup(\Sigma \restriction A_\gamma)$. Let $\Delta(\gamma, n) = \text{next}_{[C]_*^1}^n(\delta_\gamma)$. Let $\mathfrak{d}_\Delta(\gamma) = \sup\{\Delta(\gamma, n) : n \in \omega\}$. Now one can meaningfully define $A_{\gamma+1}$ using the definition of L_Δ from Definition 2.4 restricted below $\gamma+1$. Let $B_\gamma = A_{\gamma+1} \setminus A_\gamma$ which are the elements of $A_{\gamma+1}$ with first coordinate γ. Note that $\text{ot}(A_{\gamma+1}) < \kappa$ and $\text{ot}(B_\gamma) < \kappa$ by the argument from Fact 2.5. Let $\zeta_\gamma = \text{ot}(B_\gamma)$ and $\pi_\gamma : B_\gamma \to \zeta_\gamma$ be the unique order preserving bijection. For $x \in B_\gamma$, let $\Sigma(x) = \text{next}_C^{\omega \cdot \pi_\gamma(x) + \omega}(\mathfrak{d}_\Delta(\gamma))$ and for $n \in \omega$, let $\Pi(x, n) = \text{next}_C^{\omega \cdot \pi_\gamma(x) + n}(\mathfrak{d}_\Delta(\gamma))$. One has defined $\Delta \restriction (\gamma+1) \times \omega$ and $\Sigma \restriction A_{\gamma+1}$. Using the induction hypothesis that $\Sigma \restriction A_\gamma$ is order preserving, $\Sigma \restriction A_{\gamma+1}$ is order preserving by construction. Note that $\mathfrak{d}_\Delta(\gamma) < \Sigma(x)$ for all $x \in B_\gamma$ by construction. Now suppose $\gamma_0 < \gamma$ and $x \in A_{\gamma+1}$ with first coordinate γ_0. $\Sigma(x) \leq \sup(\Sigma \restriction A_\gamma) = \delta_\gamma < \Delta(\gamma, 0) < \mathfrak{d}_\Delta(\gamma)$. $\Pi \restriction A_{\gamma+1} \times \omega$ witnesses that $\Sigma \restriction A_{\gamma+1}$ has uniform cofinality ω. Thus the four properties hold at $\gamma+1$ for $\Delta \restriction (\delta+1) \times \omega$, $\Sigma \restriction A_{\gamma+1}$, and $\Pi \restriction A_{\gamma+1} \times \omega$.

If γ is a limit ordinal and for all $\tilde{\gamma} < \gamma$, $\Delta \restriction \tilde{\gamma} \times \omega$, $\Sigma \restriction A_{\tilde{\gamma}}$, and $\Pi \restriction A_{\tilde{\gamma}} \times \omega$ have been defined with the above four properties. Note $A_\gamma = \bigcup_{\tilde{\gamma} < \gamma} A_{\tilde{\gamma}}$. $\Delta \restriction \gamma \times \omega$, $\Sigma \restriction A_\gamma$, and $\Pi \restriction A_\gamma \times \omega$ also have these four properties.

This completes the construction of $\Delta : \kappa \times \omega \to [C]_*^1$ and $\Sigma : L_\Delta \to C$. If $\alpha < \beta < \kappa$, then $\mathfrak{d}_\Delta(\alpha) < \sup(\Sigma \restriction A_{\alpha+1}) < \mathfrak{d}_\Delta(\beta)$ using properties (2) and (3) at $\beta+1$. Thus $\mathfrak{d}_\Delta$ is an increasing function. Suppose $\alpha < \kappa$, $\sup(\mathfrak{d}_\Delta \restriction \alpha) \leq \sup(\Sigma \restriction A_\alpha) = \delta_\alpha < \Delta(\alpha, 0) < \mathfrak{d}_\Delta(\alpha)$ by property (2) and (3) at $\alpha+1$. Thus $\mathfrak{d}_\Delta$ is discontinuous. $\Sigma : L_\Delta \to \kappa$ is increasing by (1). For all $\gamma_0 < \gamma_1 < \kappa$ and $x \in L_\Delta$ with first coordinate γ_0, $\mathfrak{d}_\Delta(\gamma_0) < \Sigma(x) < \mathfrak{d}_\Delta(\gamma_1)$ follows by property (2) and (3) at $\gamma_1 + 1$. This verifies property (2) of Definition 2.6. Suppose $x \in L_\Delta$ with first coordinate γ and hence $x \in B_\gamma$. If x is the least element of B_γ, then $\pi_\gamma(x) = 0$ and $\sup(\Sigma \restriction x) = \sup(\Sigma \restriction A_\gamma) = \delta_\gamma < \Delta(\gamma, 0) < \mathfrak{d}_\Delta(\gamma) < \mathrm{next}_C^\omega(\mathfrak{d}_\Delta(\gamma)) = \mathrm{next}_C^{\omega\cdot\pi_\gamma(x)+\omega}(\mathfrak{d}_\Delta(\gamma)) = \Sigma(x)$. Now suppose x is not the minimal element of B_γ. Then $\sup(\Sigma \restriction x) = \sup\{\Sigma(y) : y \prec_\Delta x\} = \sup\{\mathrm{next}_C^{\omega\cdot\pi_\gamma(y)+\omega}(\mathfrak{d}_\Delta(\gamma)) : y \prec_\Delta x\} = \sup\{\mathrm{next}_C^{\omega\cdot\nu+\omega}(\mathfrak{d}_\Delta(\gamma)) : \nu < \pi_\gamma(x)\} \leq \mathrm{next}_C^{\omega\cdot\pi_\gamma(x)}(\mathfrak{d}_\Delta(\gamma)) < \mathrm{next}_C^{\omega\cdot\pi_\gamma(x)+\omega}(\mathfrak{d}_\Delta(\gamma)) = \Sigma(x)$. Thus Σ is discontinuous. Π witnesses that Σ has uniform cofinality ω. Σ has the correct type. Thus (Δ, Σ) is suitable with the desired properties. ☐

If $\delta < \epsilon$ and $f : \epsilon \to \mathrm{ON}$, then let $\mathrm{drop}(f, \delta) : (\epsilon - \delta) \to \mathrm{ON}$ be defined by $\mathrm{drop}(f,\delta)(\alpha) = f(\delta + \alpha)$. If κ is a cardinals, then club_κ denote the set of all club subsets of κ. For any cardinal κ and $\epsilon \leq \kappa$, a relation $R \subseteq [\kappa]^\epsilon_* \times \mathsf{club}_\kappa$ is $\subseteq$-downward closed in the club_κ-coordinate if and only if for all $\ell \in [\kappa]^\epsilon_*$ and clubs $C \subseteq D$, if $R(\ell, D)$ holds, then $R(\ell, C)$ holds. Without of the axiom of choice, uniform choice of homogeneous sets for many partitions is a very subtle and challenging problem. "Club uniformization" results are often very important for combinatorial questions involving partition spaces. The club uniformization of [7] Theorem 3.10 is more than sufficient for the proof of Theorem 2.11 (for just the cardinal ω_1); however, the proof is not purely combinatorial. The following club uniformization results are proved from just the strong partition property and will allow for the selection of club homogeneous for various desired property in the main construction of Theorem 2.11

Fact 2.9. *Assume $\kappa \to_* (\kappa)^\kappa_2$ and $\epsilon < \kappa$. Let $R \subseteq [\kappa]^\epsilon_* \times \mathsf{club}_\kappa$ be $\subseteq$-downward closed in the club_κ-coordinate. Then there is a club $C \subseteq \kappa$ so that for all $\ell \in \mathrm{dom}(R) \cap [C]^\epsilon_*$, $R(\ell, C \setminus (\sup(\ell) + 1))$.*

Proof. Fix $\epsilon < \kappa$. If $g \in [\kappa]^\kappa_*$, then let $\mathcal{C}_g$ be the closure of $g[\kappa]$ which is a club. Define $P : [\kappa]^\kappa_* \to 2$ by $P(f) = 0$ if and only if $\mathrm{drop}(f, \epsilon) \in \mathrm{dom}(R)$ implies $R(f \restriction \epsilon, \mathcal{C}_{\mathrm{drop}(f,\epsilon)})$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C \subseteq \kappa$ and an $i \in 2$ so that for all $f \in [C]^\kappa_*$, $P(f) = i$. Pick any $\ell \in [C]^\epsilon_*$. First, suppose $\ell \notin \mathrm{dom}(R)$. Pick any $g \in [C \setminus (\sup(\ell) + 1)]^\kappa_*$ and let $f = \ell \hat{\ } g$. Then $f \in [C]^\kappa_*$ and $P(f) = 0$. Now suppose $\ell \in \mathrm{dom}(R)$. There is some club $D \subseteq \kappa$ so that $R(\ell, D)$. Let $g \in [D]^\kappa_*$ with $g(0) > \sup(\ell)$ and observe that $\mathcal{C}_g \subseteq D$. Since R is $\subseteq$-downward closed in the club_κ-coordinate, $R(\ell, \mathcal{C}_g)$. Let $f = \ell \hat{\ } g$. Note that $f \in [C]^\kappa_*$ and $g = \mathrm{drop}(f, \epsilon)$. Thus $P(f) = 0$. Since C was assumed to be homogeneous for P, this shows that $i = 0$. Fix an $h \in [C]^\kappa_*$. Let $E = \mathcal{C}_h$ which is a club subset of κ. Suppose $\ell \in \mathrm{dom}(R) \cap [E]^\epsilon_*$. Let $\alpha < \kappa$ be least so that $h(\alpha) > \sup(\ell)$. Let $f = \ell \hat{\ } \mathrm{drop}(h, \alpha)$ and note that $f \in [C]^\kappa_*$, $f \restriction \epsilon = \ell$, $\mathrm{drop}(f, \epsilon) = \mathrm{drop}(h, \alpha)$, $E \setminus (\sup(\ell) + 1) = \mathcal{C}_{\mathrm{drop}(h,\alpha)}$. $P(f) = 0$ implies that $R(f \restriction \epsilon, \mathcal{C}_{\mathrm{drop}(f,\epsilon)})$. Thus $R(\ell, E \setminus (\sup(\ell) + 1))$. E is the desired club. ☐

Fact 2.10. *Assume $\kappa \to_* (\kappa)^\kappa_2$. Let $R \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ be $\subseteq$-downward closed in the club_κ-coordinate. There is a club $C \subseteq \kappa$ so that for all $\ell \in \mathrm{dom}(R) \cap \bigcup_{\epsilon \in [C]^1_*} [C \setminus (\epsilon + 1)]^\epsilon_*$, $R(\ell, C \setminus (\sup(\ell) + 1))$.*

Proof. Fix $\epsilon < \kappa$. Define $R^\epsilon \subseteq [\kappa]^\epsilon_* \times \mathsf{club}_\kappa$ by $R^\epsilon(\ell, C)$ if and only if $R(\ell, C)$. R^ϵ is $\subseteq$-downward closed in the club_κ-coordinate. By Fact 2.9, there is a club D so that for all $\ell \in \mathrm{dom}(R^\epsilon) \cap [C]^\epsilon_*$, $R(\ell, D \setminus (\sup(\ell) + 1))$. Define $S \subseteq \kappa \times \mathsf{club}_\kappa$ by $S(\epsilon, C)$ if and only if for all $\ell \in [C]^\epsilon_*$, $R^\epsilon(\ell, C \setminus (\sup(\ell) + 1))$. By the above discussion, $\mathrm{dom}(S) = \kappa$. By Fact 2.9 applied with $\epsilon = 1$, there is a club $E \subseteq \kappa$ so that for all $\epsilon \in [E]^1_*$, $S(\epsilon, E \setminus (\epsilon + 1))$. Let $\epsilon \in [E]^1_*$ and $\ell \in [E \setminus (\epsilon + 1))$. $S(\epsilon, E \setminus (\epsilon + 1))$ holds. Since $\ell \in [E \setminus (\epsilon + 1)]^\epsilon_*$, $R^\epsilon(\ell, (E \setminus (\epsilon + 1)) \setminus (\sup(\ell) + 1))$. Since $(E \setminus (\epsilon + 1)) \setminus (\sup(\ell) + 1)) = E \setminus (\sup(\ell) + 1)$, $R^\epsilon(\ell, E \setminus (\sup(\ell) + 1))$. Thus $R(\ell, E \setminus (\sup(\ell) + 1))$. E is the desired club. ☐

The following will provide evidence for the conjecture that $[\omega_1]^{<\omega_1}$ is ω-prime. In the following argument, there will be a very large family of relevant partitions. Fact 2.10 will provide a club which is "simultaneously homogeneous" for all the relevant partition in the formal sense of Fact 2.10. Using this club, one will define an injection of the form presented in Definition 2.6. A number of fine technical details need to be verified to see that this injection interacts meaningfully with all the relevant partitions.

Theorem 2.11. *Assume κ is a cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$. Let $\lambda < \kappa$ and $\langle Y_\eta : \eta < \lambda \rangle$ be a sequence of sets. If $|[\kappa]^{<\kappa}| \leq |\prod_{\eta < \lambda} Y_\eta|$, then there is an $\bar{\eta} < \lambda$ so that $|[\kappa]^\omega| \leq |Y_{\bar{\eta}}|$.*

Proof. Since $|[\kappa]^{<\kappa}| \leq |\prod_{\eta \in \lambda} Y_\eta|$, let $\Phi : [\kappa]^{<\kappa} \to \prod_{\eta \in \lambda} Y_\eta$ be an injection. If $f, g \in [\kappa]^{<\kappa}$ and $f \neq g$, then let $\chi(f, g)$ be the least $\eta \in \lambda$ so that $\Phi(f)(\eta) \neq \Phi(g)(\eta)$. Let $Z \subseteq \kappa$ be the club of indecomposable ordinals below κ and work within Z. Note that $\kappa \rightarrow_* (\kappa)^\kappa_2$ implies $\kappa \rightarrow_* (\kappa)^{<\kappa}_2$ and hence $\kappa \rightarrow_* (\kappa)^{<\kappa}_{<\kappa}$ by the comments in the paragraphs following Definition 2.2.

For any $\epsilon \in Z$ and $\iota \in [Z]^\epsilon_*$, let $\iota^0, \iota^1 \in [Z]^\epsilon_*$ be defined by $\iota^i(\alpha) = \iota(2 \cdot \alpha + i)$. For any $(\delta, \epsilon) \in [Z]^2_*$ (hence $\delta < \epsilon$) and $\ell \in [Z]^\delta_*$, define $P_{\delta,\epsilon,\ell} : [Z]^\epsilon_* \to \lambda$ by $P_{\delta,\epsilon,\ell}(\iota) = \chi(\ell\hat{}\iota^0, \ell\hat{}\iota^1)$. By $\kappa \rightarrow_* (\kappa)^\epsilon_\lambda$, there is an $\eta_{\delta,\epsilon,\ell} \in \lambda$ and a club $C \subseteq Z$ so that for all $\iota \in [C]^\epsilon_*$, $P_{\delta,\epsilon,\ell}(\iota) = \eta_{\delta,\epsilon,\ell}$. For $(\delta, \epsilon) \in [Z]^2_*$, define $P_{\delta,\epsilon} : [Z]^\delta_* \to \lambda$ by $P_{\delta,\epsilon}(\ell) = \eta_{\delta,\epsilon,\ell}$. By $\kappa \rightarrow_* (\kappa)^\delta_\lambda$, there is a club $C \subseteq Z$ and an $\eta_{\delta,\epsilon} \in \lambda$ so that for all $\ell \in [C]^\delta_*$, $P_{\delta,\epsilon}(\ell) = \eta_{\delta,\epsilon,\ell} = \eta_{\delta,\epsilon}$. Define $P : [Z]^2_* \to \lambda$ by $P(\delta, \epsilon) = \eta_{\delta,\epsilon}$. By $\kappa \rightarrow_* (\kappa)^2_\lambda$, fix a club $C^0_0 \subseteq Z$ and an $\bar{\eta} \in \lambda$ so that for all $(\delta, \epsilon) \in [C^0_0]^2_*$, $P(\delta, \epsilon) = \eta_{\delta,\epsilon} = \bar{\eta}$. For $(\delta, \epsilon) \in [C^0_0]^2_*$, let $R_{\delta,\epsilon} \subseteq [C^0_0]^\delta_* \times \mathsf{club}_\kappa$ be defined by $R_{\delta,\epsilon}(\ell, C)$ if and only if for all $\iota \in [C]^\epsilon_*$, $P_{\delta,\epsilon,\ell}(\iota) = \bar{\eta}$. Fix $(\delta, \epsilon) \in [C^0_0]^2_*$. By definition of C^0_0, $\eta_{\delta,\epsilon} = \bar{\eta}$. By definition of $\eta_{\delta,\epsilon} = \bar{\eta}$, there is a club $D \subseteq Z$ so that for all $\ell \in [D]^\delta_*$, $P_{\delta,\epsilon}(\ell) = \eta_{\delta,\epsilon,\ell} = \eta_{\delta,\epsilon} = \bar{\eta}$. Fix $\ell \in [D]^\delta_*$. By definition of $\eta_{\delta,\epsilon,\ell} = \bar{\eta}$, there is a club $E \subseteq Z$ so that for all $\iota \in [E]^\epsilon_*$, $P_{\delta,\epsilon,\ell}(\iota) = \eta_{\delta,\epsilon,\ell} = \eta_{\delta,\epsilon} = \bar{\eta}$. Thus $R_{\delta,\epsilon}(\ell, E)$. So $\ell \in \mathrm{dom}(R_{\delta,\epsilon})$. Since $\ell \in [D]^\delta_*$ was arbitrary, $[D]^\delta_* \subseteq \mathrm{dom}(R_{\delta,\epsilon})$. $R_{\delta,\epsilon}$ is $\subseteq$-downward closed in the club_κ-coordinate. By Fact 2.9, there is a club $F \subseteq D$ so that for all $\ell \in \mathrm{dom}(R_{\delta,\epsilon}) \cap [F]^\delta_* = [F]^\delta_*$ (since $[D]^\delta_* \subseteq \mathrm{dom}(R_{\delta,\epsilon})$), $R_{\delta,\epsilon}(\ell, F \setminus (\sup(\ell) + 1))$. Since $(\delta, \epsilon) \in [C^0_0]^2_*$ was arbitrary, it has been shown that for all $(\delta, \epsilon) \in [C^0_0]^2_*$, there is a club C with the property that for all $\ell \in [C]^\delta_*$, $R_{\delta,\epsilon}(\ell, C \setminus (\sup(\ell) + 1))$. Define $R \subseteq [C^0_0]^2_* \times \mathsf{club}_\kappa$ by $R((\delta, \epsilon), C)$ if and only if for all $\ell \in [C]^\delta_*$, $R_{\delta,\epsilon}(\ell, C \setminus (\sup(\ell) + 1))$. Note that $\mathrm{dom}(R) = [C^0_0]^2_*$ by the above discussion. R is $\subseteq$-downward closed in the club_κ-coordinate. By Fact 2.9, there is a club $C^0_1 \subseteq C^0_0$ so that for all $(\delta, \epsilon) \in \mathrm{dom}(R) \cap [C^0_1]^2_* = [C^0_1]^2_*$, $R((\delta, \epsilon), C^0_1 \setminus (\epsilon + 1))$. In summary, for all $(\delta, \epsilon) \in [C^0_1]^2_*$, all $\ell \in [C^0_1 \setminus (\epsilon + 1)]^\delta_*$, and all $\iota \in [C^0_1 \setminus (\sup(\ell) + 1)]^\epsilon_*$, $P_{\delta,\epsilon,\ell}(\iota) = \chi(\ell\hat{}\iota^0, \ell\hat{}\iota^1) = \bar{\eta}$.

For $(\delta, \epsilon) \in [Z]^2_*$ and $\ell \in [Z]^\delta_*$, let $Q_{\delta,\ell,\epsilon} : [Z]^\epsilon_* \to 2$ be defined by $Q_{\delta,\ell,\epsilon}(\iota) = 0$ if and only if $\Phi(\ell)(\bar{\eta}) \neq \Phi(\iota)(\bar{\eta})$. By $\kappa \rightarrow_* (\kappa)^\epsilon_2$, there is a $j_{\delta,\ell,\epsilon} \in 2$ and a club $C \subseteq Z$ so that for all $\iota \in [C]^\epsilon_*$, $Q_{\delta,\ell,\epsilon}(\iota) = j_{\delta,\ell,\epsilon}$. For all $\delta \in [Z]^1_*$ and $\ell \in [Z]^\delta_*$, define $Q_{\delta,\ell} : [Z]^1_* \to 2$ by $Q_{\delta,\ell}(\epsilon) = j_{\delta,\ell,\epsilon}$. By $\kappa \rightarrow_* (\kappa)^1_2$, there is a $j_{\delta,\ell} \in 2$ and a club $C \subseteq Z$ so that for all $\epsilon \in [C]^1_*$, $Q_{\delta,\ell}(\epsilon) = j_{\delta,\ell}$. For all $\delta \in [Z]^1_*$, let $Q_\delta : [Z]^\delta_* \to 2$ be defined by $Q_\delta(\ell) = j_{\delta,\ell}$. By $\kappa \rightarrow_* (\kappa)^\delta_2$, there is $j_\delta \in 2$ and a club $C \subseteq Z$ so that for all $\ell \in [C]^\delta_*$, $Q_\delta(\ell) = j_{\delta,\ell} = j_\delta$. Define $Q : [Z]^1_* \to 2$ by $Q(\delta) = j_\delta$. By $\kappa \rightarrow_* (\kappa)^1_2$, there is a $\bar{j} \in 2$ and a fixed club $C^1_0 \subseteq Z$ so that for all $\delta \in [C^1_0]^1_*$, $Q(\delta) = j_\delta = \bar{j}$. Let $S \subseteq [Z]^1_* \times \mathsf{club}_\kappa$ be defined by $S(\delta, C)$ if and only if for all $\ell \in [C]^\delta_*$, $j_\ell = \bar{j}$. S is $\subseteq$-downward closed in the club_κ-coordinate. By Fact 2.9, there is a club $C^1_1 \subseteq C^1_0$ so that for all $\delta \in \mathrm{dom}(S) \cap [C^1_1]^1_*$, $S(\delta, C^1_1 \setminus (\delta + 1))$. Suppose $\delta \in [C^1_0]^1_*$. By definition of C^1_0, $j_\delta = \bar{j}$. By definition of $j_\delta = \bar{j}$, there is some club $C \subseteq Z$ so that for all $\ell \in [C]^\delta_*$, $j_{\delta,\ell} = j_\delta = \bar{j}$ and hence $S(\delta, C)$. So $\delta \in \mathrm{dom}(S)$. Since $\delta \in [C^1_0]^1_*$ was arbitrary, this shows that $[C^1_0]^1_* \subseteq \mathrm{dom}(S)$ and hence $[C^1_1]^1_* \subseteq [C^1_0]^1_* \subseteq \mathrm{dom}(S)$. Therefore for all $\delta \in [C^1_1]^1_*$, $S(\delta, C^1_1 \setminus (\delta + 1))$. Define $T \subseteq [Z]^{<\kappa}_* \times \mathsf{club}_\kappa$ by $T(\ell, C)$ if and only if for all $\epsilon \in [C]^1_*$, for all $\iota \in [C \setminus (\epsilon + 1)]^\epsilon_*$, $Q_{|\ell|,\ell,\epsilon}(\iota) = \bar{j}$. T is $\subseteq$-downward closed in the club_κ-coordinate and therefore by Fact 2.10, there is a club $C^1_2 \subseteq C^1_1$ so that for all $\ell \in \mathrm{dom}(T) \cap \bigcup_{\delta \in [C^1_2]^1_*} [C^1_2 \setminus (\delta + 1)]^\delta_*$, $T(\ell, C^1_2 \setminus (\sup(\ell) + 1))$. Fix $\ell \in \bigcup_{\delta \in [C^1_1]^1_*} [C^1_1 \setminus (\delta + 1)]^\delta_*$. Thus $|\ell| \in [C^1_1]^1_*$. Define $U_\ell \subseteq Z \times \mathsf{club}_\kappa$ by $U_\ell(\epsilon, C)$ if and only if for all $\iota \in [C]^\epsilon_*$, $Q_{|\ell|,\ell,\epsilon}(\iota) = \bar{j}$. Since $|\ell| \in [C^1_1]^1_*$, $\ell \in [C^1_1 \setminus (|\ell| + 1)]^{|\ell|}_*$, and $S(|\ell|, C^1_1 \setminus (|\ell| + 1))$, one has that $j_{|\ell|,\ell} = \bar{j}$. By definition of $j_{|\ell|,\ell} = \bar{j}$, there is a club $D \subseteq Z$ so that for all $\epsilon \in [D]^1_*$, $j_{|\ell|,\ell,\epsilon} = j_{|\ell|,\ell} = \bar{j}$. Fix $\epsilon \in [D]^1_*$. By definition of $j_{|\ell|,\ell,\epsilon} = \bar{j}$, there is a club $E \subseteq Z$ so that for all $\iota \in [E]^\epsilon_*$, $Q_{|\ell|,\ell,\epsilon}(\iota) = j_{|\ell|,\ell,\epsilon} = \bar{j}$ and thus $U_\ell(\epsilon, E)$. So $\epsilon \in \mathrm{dom}(U_\ell)$. Since $\epsilon \in [D]^1_*$ was arbitrary, this shows that $[D]^1_* \subseteq \mathrm{dom}(U_\ell)$. U_ℓ is $\subseteq$-downward closed in the club_κ-coordinate. Thus by Fact 2.9, there is a club $F \subseteq D$ so that for all $\epsilon \in \mathrm{dom}(U_\ell) \cap [F]^1_* = [F]^1_*$, $U_\ell(\epsilon, F \setminus (\epsilon + 1))$. Thus for all $\epsilon \in [F]^1_*$, for all $\iota \in [F \setminus (\epsilon + 1)]^\epsilon_*$, $Q_{|\ell|,\ell,\epsilon}(\iota) = \bar{j}$. Thus $T(\ell, F)$. This shows that $\ell \in \mathrm{dom}(T)$. Since $\ell \in \bigcup_{\delta \in [C^1_1]^1_*} [C^1_1 \setminus (\delta + 1)]^\delta_*$ was arbitrary, it has been shown that $\bigcup_{\delta \in [C^1_1]^1_*} [C^1_1 \setminus (\delta + 1)]^\delta_* \subseteq \mathrm{dom}(T)$ and hence $\bigcup_{\delta \in [C^1_2]^1_*} [C^1_2 \setminus (\delta + 1)]^\delta_* \subseteq \bigcup_{\delta \in [C^1_1]^1_*} [C^1_1 \setminus (\delta + 1)]^\delta_* \subseteq \mathrm{dom}(T)$. Thus for all $\ell \in \bigcup_{\delta \in [C^1_2]^1_*} [C^1_2 \setminus (\delta + 1)]^\delta_*$, $T(\ell, C^1_2 \setminus (\sup(\ell) + 1))$. In summary, for all $\delta \in [C^1_2]^1_*$, for all $\ell \in [C^1_2 \setminus (\delta + 1)]^\delta_*$, for all $\epsilon \in [C^1_2 \setminus (\sup(\ell) + 1)]^1_*$, and all $\iota \in [C^1_2 \setminus (\epsilon + 1)]^\epsilon_*$, $Q_{\delta,\ell,\epsilon}(\iota) = \bar{j}$.

Let $\bar{C} = C^0_1 \cap C^1_2$. Let $\delta \in [\bar{C}]^1_*$, $\ell_0 \in [\bar{C} \setminus (\delta + 1)]^\delta_*$, $\epsilon \in [\bar{C} \setminus (\sup(\ell_0) + 1)]^1_*$, $\ell_1 \in [\bar{C} \setminus (\epsilon + 1)]^\delta_*$, and $\iota \in [\bar{C} \setminus (\sup(\ell_1) + 1)]^\epsilon_*$. For $i \in 2$, let $\iota^i \in [\bar{C} \setminus (\sup(\ell_1) + 1)]^\epsilon_*$ be defined by $\iota^i(\alpha) = \iota(2 \cdot \alpha + i)$ and let $\sigma^i \in [\bar{C} \setminus (\epsilon + 1)]^\epsilon_*$ be defined by $\sigma^i = \ell_1\hat{}\iota^i$. Since $(\delta, \epsilon) \in [\bar{C}]^2_*$, $\ell_1 \in [\bar{C} \setminus (\epsilon + 1)]^\delta_*$, and $\iota \in [\bar{C} \setminus (\sup(\ell_1) + 1)]^\epsilon_*$, one has that $\chi(\sigma^0, \sigma^1) = \chi(\ell_1\hat{}\iota^0, \ell_1\hat{}\iota^1) = P_{\delta,\epsilon,\ell_1}(\iota) = \bar{\eta}$ by the definition of C^0_1. Thus $\Phi(\sigma^0)(\bar{\eta}) \neq \Phi(\sigma^1)(\bar{\eta})$. Hence there must be some $\hat{i} \in 2$ so that $\Phi(\sigma^{\hat{i}})(\bar{\eta}) \neq \Phi(\ell_0)(\bar{\eta})$. Thus $Q_{\delta,\ell_0,\epsilon}(\sigma^{\hat{i}}) = 0$. Since $\delta \in [\bar{C}]^1_*$, $\ell_0 \in [\bar{C} \setminus (\delta + 1)]^\delta_*$, $\epsilon \in [\bar{C} \setminus (\sup(\ell_0) + 1)]^1_*$, and $\sigma^{\hat{i}} \in [\bar{C} \setminus (\epsilon + 1)]^\epsilon_*$, $Q_{\delta,\ell_0,\epsilon}(\sigma^{\hat{i}}) = \bar{j}$. It has been shown that

$\bar{j} = 0$. In summary, for all $\delta \in [\bar{C}]^1_*$, for all $\ell \in [\bar{C} \setminus (\delta+1)]^\delta_*$, for all $\epsilon \in [\bar{C} \setminus (\sup(\ell)+1)]^1_*$, and for all $\iota \in [\bar{C} \setminus (\epsilon+1)]^\epsilon_*$, $\Phi(\ell)(\bar{\eta}) \neq \Phi(\iota)(\bar{\eta})$.

By Fact 2.8, let (Δ, Σ) be a suitable pair so that $\Delta : \kappa \times \omega \to [\bar{C}]^1_*$ and $\Sigma : L_\Delta \to [\bar{C}]^1_*$. Note that $\mathfrak{d}_\Delta : \kappa \to [\bar{C}]^1_*$. Let $\Lambda_{(\Delta,\Sigma)} : [\kappa]^\omega \to \bigcup_{\epsilon \in [\bar{C}]^1_*} [\bar{C} \setminus (\epsilon+1)]^\epsilon_*$ be the injection as defined in Definition 2.6 possessing the properties proved in Fact 2.7. Suppose $f, g \in [\kappa]^\omega$ and $f \neq g$. First, suppose $\sup(f) \neq \sup(g)$ and without loss of generality, $\sup(f) < \sup(g)$. $\mathfrak{d}_\Delta(\sup(f)) \in [\bar{C}]^1_*$. By Fact 2.7 (2) and (5), $\Lambda_{(\Delta,\Sigma)}(f) \in [\bar{C} \setminus (\mathfrak{d}_\Delta(\sup(f))+1)]^{\mathfrak{d}_\Delta(\sup(f))}_*$ and $\Lambda_{(\Delta,\Sigma)}(g) \in [\bar{C} \setminus (\mathfrak{d}_\Delta(\sup(g))+1)]^{\mathfrak{d}_\Delta(\sup(g))}_*$. $\mathfrak{d}_\Delta(\sup(g)) \in [\bar{C} \setminus (\sup(\Lambda_{(\Delta,\Sigma)}(f))+1)]^1_*$ by Fact 2.7 (3). By the observation shown above, $\Phi(\Lambda_{(\Delta,\Sigma)}(f))(\bar{\eta}) \neq \Phi(\Lambda_{(\Delta,\Sigma)}(g))(\bar{\eta})$. Secondly, suppose $\sup(f) = \sup(g)$, $m \in \omega$ is least so that $f(m) \neq g(m)$, and without loss of generality, suppose $f(m) < g(m)$. Let $\delta = \Delta(\sup(f), m) = \Delta(\sup(g), m)$ and note that $\delta \in [\bar{C}]^1_*$ by definition of Δ. Let $\epsilon = \mathfrak{d}_\Delta(\sup(f)) = \mathfrak{d}_\Delta(\sup(g))$ and note that $\epsilon \in [\bar{C}]^1_*$, $\delta < \epsilon$, and hence $(\delta, \epsilon) \in [\bar{C}]^2_*$. By Fact 2.7 (4), $\Lambda_{(\Delta,\Sigma)}(f) \restriction \delta = \Lambda_{(\Delta,\Sigma)}(g) \restriction \delta$. Let $\ell = \Lambda_{(\Delta,\Sigma)}(f) \restriction \delta = \Lambda_{(\Delta,\Sigma)}(g) \restriction \delta$. Note that $\epsilon < \ell(0)$ by Fact 2.7 (2). The set $A = \{\Lambda_{(\Delta,\Sigma)}(f)(\alpha), \Lambda_{(\Delta,\Sigma)}(g)(\alpha) : \delta = \Delta(\sup(f), m) = \Delta(\sup(g), m) < \alpha < \epsilon\}$ has ordertype ϵ (since $\epsilon \in \bar{C} \subseteq Z$ is indecomposable). Let $\iota : \epsilon \to A$ be the increasing enumeration of A. Since $\Lambda_{(\Delta,\Sigma)}(f)$ and $\Lambda_{(\Delta,\Sigma)}(g)$ have the correct type, ι has the correct type and thus $\iota \in [\bar{C} \setminus (\sup(\ell)+1)]^\epsilon_*$. By Fact 2.7 (4), $\iota^0 = \mathrm{drop}(\Lambda_{(\Delta,\Sigma)}(f), \delta)$ and $\iota^1 = \mathrm{drop}(\Lambda_{(\Delta,\Sigma)}(g), \delta)$. Therefore, $\ell\hat{\ }\iota^0 = \Lambda_{(\Delta,\Sigma)}(f)$ and $\ell\hat{\ }\iota^1 = \Lambda_{(\Delta,\Sigma)}(g)$. Thus $(\delta, \epsilon) \in [\bar{C}]^1_*$, $\ell \in [\bar{C} \setminus (\epsilon+1)]^\delta_*$, and $\iota \in [\bar{C} \setminus (\sup(\ell)+1)]^\epsilon_*$. By the observation shown above (the property of $C^0_1 \supseteq \bar{C}$), $\bar{\eta} = P_{\delta,\epsilon,\ell}(\iota) = \chi(\ell\hat{\ }\iota^0, \ell\hat{\ }\iota^1) = \chi(\Lambda_{(\Delta,\Sigma)}(f), \Lambda_{(\Delta,\Sigma)}(g))$. By definition of χ, $\Phi(\Lambda_{(\Sigma,\Delta)}(f))(\bar{\eta}) \neq \Phi(\Lambda_{(\Delta,\Sigma)}(g))(\bar{\eta})$.

Let $\Gamma : [\kappa]^\omega \to Y_{\bar{\eta}}$ be defined by $\Gamma(f) = \Phi(\Lambda_{(\Delta,\Sigma)}(f))(\bar{\eta})$. It has been shown that Γ is an injection. Thus $|[\kappa]^\omega| \leq |Y_{\bar{\eta}}|$. ☐

Fact 2.12. *([10]) Suppose κ is a cardinal such that $\kappa \to_* (\kappa)^{<\kappa}_2$. For all $\chi < \kappa$, there is no injection of $[\kappa]^{<\kappa}$ into ${}^\chi\mathrm{ON}$, the class of χ length sequence of ordinals.*

Corollary 2.13. *Suppose κ is a cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$, $\lambda < \kappa$, and Y be a set. If $|[\kappa]^{<\kappa}| \leq |{}^\lambda Y|$, then $|[\kappa]^\omega| < |Y|$.*

Proof. Let $\Phi : [\kappa]^{<\kappa} \to {}^\lambda Y$ be an injection. For $\eta < \lambda$, let $Y_\eta = Y$. By applying Theorem 2.11 to $\langle Y_\eta : \eta < \lambda \rangle$, $|[\kappa]^\omega| \leq |Y|$. Suppose $|Y| = |[\kappa]^\omega|$. Let $\Sigma : Y \to [\kappa]^\omega$ be an injection. Define $\Pi : {}^\lambda Y \to {}^{\omega \cdot \lambda}\kappa$ by $\Pi(f)(\omega \cdot \delta + n) = \Sigma(f(\delta))(n)$ for all $\delta < \lambda$ and $n \in \omega$. Π is an injection. Then $\Pi \circ \Phi : [\kappa]^{<\kappa} \to {}^{\omega\cdot\lambda}\kappa$ is an injection which violates Fact 2.12. This shows that $|[\kappa]^\omega| < |Y|$. ☐

Remark 2.14. In light of the strict cardinality inequality of Corollary 2.13, one would expect a strict inequality in statement of Theorem 2.11: that is, there is an $\bar{\eta}$ so that $|[\kappa]^\omega| < |Y_{\bar{\eta}}|$, in the notation for Theorem 2.11. Unfortunately, the argument for Corollary 2.13 would require the existence of a sequence $\langle \Sigma_\eta : \eta < \lambda \rangle$ such that $\Sigma_\eta : Y_\eta \to [\kappa]^\omega$ is an injection for each $\eta < \lambda$. When $\lambda > \omega$, there is insufficient general choice principle even under determinacy to find such a sequence. Of course, the ideal strategy to produce the strict inequality in Theorem 2.11 would be to answer the question of whether $[\kappa]^{<\kappa}$ is λ-prime for all $\lambda < \kappa$ when κ satisfies suitable partition properties: If $|[\kappa]^{<\kappa}| \leq |\prod_{\eta<\lambda} Y_\eta|$, then there is an $\bar{\eta} < \lambda$ so that $|[\kappa]^{<\kappa}| \leq |Y_{\bar{\eta}}|$.

The following asserts that if κ is a strong partition cardinal, then $\mathscr{P}(\kappa)$ is ON-regular.

Fact 2.15. *([3]) Suppose κ is a cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$. For all functions $\Psi : \mathscr{P}(\kappa) \to \mathrm{ON}$, there is an $\alpha \in \mathrm{ON}$ so that $|\Psi^{-1}[\{\alpha\}]| = |\mathscr{P}(\kappa)|$.*

Proof. Since $|\mathscr{P}(\kappa)| = |{}^\kappa 2| = |[\kappa]^\kappa_*|$, one will interchangeably use the various presentations of this cardinality. Assume the result fails. By replacement, for every function $\Psi : [\kappa]^\kappa_* \to \mathrm{ON}$, there is a $\delta \in \mathrm{ON}$ so that $\Psi : [\kappa]^\kappa_* \to \delta$. Let δ be the least ordinal so that there is a function $\Psi : [\kappa]^\kappa_* \to \delta$ with the property that for all $\alpha < \delta$, $|\Psi^{-1}[\{\alpha\}]| < |[\kappa]^\kappa_*|$. By minimality, δ must be a cardinal. If $f \in [\kappa]^\kappa_*$, then let $f^0, f^1 \in [\kappa]^\kappa_*$ be defined by $f^i(\alpha) = f(2 \cdot \alpha + i)$ for both $i \in 2$. Define $P : [\kappa]^\kappa_* \to 2$ by $P(f) = 0$ if and only if $\Psi(f^0) \leq \Psi(f^1)$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_0 \subseteq \kappa$ which is homogeneous for P. Suppose C_0 is homogeneous P taking value 1. Fix $g \in [C_0]^\kappa_*$. For each $n \in \omega$, let $g_n \in [C_0]^\kappa_*$ be defined by $g_n(\alpha) = g(\omega \cdot \alpha + n)$. For each $n \in \omega$, there is an $f_n \in [C_0]^\kappa_*$ so that $f^0 = g_n$ and $g^1 = g_{n+1}$. For all $n \in \omega$, since $P(f_n) = 1$, one has $\Psi(g_{n+1}) = \Psi(f^1) < \Psi(f^0) = \Psi(g_n)$. Thus $\langle \Psi(g_n) : n \in \omega \rangle$ is an infinite descending sequence of ordinals which is a contradiction. Thus C_0 is homogeneous for P taking value 0. Let $h \in [C_0]^\kappa_*$. Let

$g \in [C_0]^\kappa_*$ be defined by $g(\alpha) = g(3 \cdot \alpha + 2)$. Let $\Phi : {}^\kappa 2 \to [C_0]^\kappa_*$ be defined by $\Phi(p)(\alpha) = h(3 \cdot \alpha + p(\alpha))$ where $p \in {}^\kappa 2$ implies p takes the form of a function from κ into 2. $\Phi : {}^\kappa 2 \to [C_0]^\kappa_*$ is an injection. For each $p \in {}^\kappa 2$, there is a unique $f_p \in [C_0]^\kappa_*$ so that $f_p^0 = \Phi(p)$ and $f_p^1 = g$. $P(f_p) = 0$ implies that $\Psi(\Phi(p)) = \Psi(f_p^0) \leq \Psi(f_p^1) = \Psi(g)$. Let $\eta = \Psi(g)$ and note that $\eta + 1 < \delta$ since δ is a cardinal. Define $\Sigma : {}^\kappa 2 \to \eta + 1$ by $\Sigma(p) = \Psi(\Phi(p))$. For all $\alpha < \eta + 1$, $|\Sigma^{-1}[\{\alpha\}]| \leq |\Psi^{-1}[\{\alpha\}]| < |\mathscr{P}(\kappa)|$ by the assumption on Ψ and the fact that $\Phi : \Sigma^{-1}[\{\alpha\}] \to \Psi^{-1}[\{\alpha\}]$ is an injection. Thus Σ witnesses that $\eta + 1$ violates the minimality of δ. This contradiction implies that P has no homogeneous club. This contradicts $\kappa \to_* (\kappa)^\kappa_2$. This completes the proof. ☐

Fact 2.16. *If κ is a cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$, then $|[\kappa]^{<\kappa}| < |\mathscr{P}(\kappa)|$.*

Proof. Suppose $\Phi : \mathscr{P}(\kappa) \to [\kappa]^{<\kappa}$. Let $\Psi : \mathscr{P}(\kappa) \to \kappa$ be defined by $\Psi(X) = \text{dom}(\Phi(f))$. By Fact 2.15, there is an $\epsilon < \kappa$ so that $|\Psi^{-1}[\{\epsilon\}]| = |\mathscr{P}(\kappa)|$. Let $\Sigma : \mathscr{P}(\kappa) \to \Psi^{-1}[\{\epsilon\}]$ be an injection. Then $\Phi \circ \Sigma : \mathscr{P}(\kappa) \to [\kappa]^\epsilon$ is an injection. Since $|[\kappa]^{<\kappa}| \leq |\mathscr{P}(\kappa)|$, this implies there is an injection $\Lambda : [\kappa]^{<\kappa} \to [\kappa]^\epsilon$ which contradicts Fact 2.12. ☐

Theorem 2.17. *Suppose κ is a cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$. Let $\langle Y_\alpha : \alpha < \kappa \rangle$ be a sequence of sets. If $|\mathscr{P}(\kappa)| \leq |\bigcup_{\lambda < \kappa} \prod_{\alpha < \lambda} Y_\alpha|$, then there is $\bar{\lambda} < \kappa$ so that $|[\kappa]^\omega| \leq |Y_{\bar{\lambda}}|$.*

Proof. Let $\Gamma : \mathscr{P}(\kappa) \to \bigcup_{\lambda < \kappa} \prod_{\alpha < \lambda} Y_\alpha$ be an injection. Let $\Psi : \mathscr{P}(\kappa) \to \kappa$ be defined by $\Psi(A) = \text{dom}(\Gamma(A))$. (Note that $\Gamma(A) \in \bigcup_{\lambda < \kappa} \prod_{\alpha < \lambda} Y_\alpha$ implies that $\Gamma(A)$ is a function on some $\lambda < \kappa$ and thus $\Psi(A)$ is this λ.) By Fact 2.15, there is a $\lambda < \kappa$ so that $|\Psi^{-1}[\{\lambda\}]| = |\mathscr{P}(\kappa)|$. Let $\Sigma : \mathscr{P}(\kappa) \to \Psi^{-1}[\{\lambda\}]$ be an injection. Since $|[\kappa]^{<\kappa}| < |\mathscr{P}(\kappa)|$, let $\Pi : [\kappa]^{<\kappa} \to \mathscr{P}(\kappa)$ be an injection. Let $\Phi : [\kappa]^{<\kappa} \to \prod_{\alpha < \lambda} Y_\alpha$ be defined by $\Phi = \Gamma \circ \Sigma \circ \Pi$. Since Φ is an injection, there is some $\bar{\lambda} < \kappa$ so that $|[\kappa]^\omega| \leq |Y_{\bar{\lambda}}|$ by Theorem 2.11. ☐

Corollary 2.18. *Let κ be a cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$ and Y be a set. If $|\mathscr{P}(\kappa)| \leq |{}^{<\kappa}Y|$, then $|[\kappa]^\omega| < |Y|$.*

Proof. Let $\Phi : \mathscr{P}(\kappa) \to {}^{<\kappa}Y$ be an injection. For each $\alpha < \kappa$, let $Y_\alpha = Y$. By applying Theorem 2.17 to $\langle Y_\alpha : \alpha < \kappa \rangle$, $|[\kappa]^\omega| \leq |Y|$. Now suppose $|[\kappa]^\omega| = |Y|$. Let $\Sigma : Y \to [\kappa]^\omega$ be an injection. Define $\Pi : {}^{<\kappa}Y \to {}^{<\kappa}\kappa$ as follows: If $f \in {}^\lambda Y$ for some $\lambda < \kappa$, then $\Pi(f) \in {}^{\omega \cdot \lambda}\kappa$ is defined by $\Pi(f)(\omega \cdot \delta + n) = \Sigma(f(\delta))(n)$ for all $\delta < \kappa$ and $n \in \omega$. Π is an injection. Then $\Pi \circ \Phi : \mathscr{P}(\kappa) \to {}^{<\kappa}\kappa$ is an injection which violates Fact 2.16. This shows that $|[\kappa]^\omega| < |Y|$. ☐

3. Wellordered Power Set Operation

To study the wellordered power set operation, some results concerning determinacy in the presence of ∞-Borel codes will be necessary. Below the two relevant facts will be summarized. See [4] or [18] for more information about ∞-Borel codes.

Definition 3.1. A pair (S, φ) is an ∞-Borel code if and only if S is a set of ordinal and φ is a formula of set theory. A set $A \subseteq \mathbb{R}$ is ∞-Borel if and only if there is an ∞-Borel code (S, φ) so that $A = \{x \in \mathbb{R} : L[S, x] \models \varphi(S, x)\}$.

The following result states that if AD and $\mathsf{DC}_\mathbb{R}$ holds and J is a set of ordinals, then there is a uniform procedure which translate an $\text{OD}_{\{J,z\}}^{L(J,\mathbb{R})}$ definition for a set of reals to an $\text{OD}_{\{J,z\}}^{L(J,\mathbb{R})}$ ∞-Borel code for that set of reals. This is done using Woodin's result that models of the form $L(J, \mathbb{R}) \models \mathsf{AD} + \mathsf{DC}$ is a "symmetric collapse extension" of $\text{HOD}_{\{J\}}^{L(J,\mathbb{R})}$ when J is a set of ordinals. In particular, truth in $L(J, \mathbb{R})$ can be captured by a suitable remainder forcing of the ω-direct limit of the $\text{OD}_{\{J\}}$ ∞-Borel code forcing on the various $\mathbb{R}^n$. See [2] and [4] for the details concerning this forcing and these results in the model $L(\mathbb{R})$.

Fact 3.2. *(Woodin) Assume AD and $\mathsf{DC}_\mathbb{R}$. There is a function Ξ so that for all formula ϕ of set theory, sets of ordinals J, $z \in \mathbb{R}$, and finite sequences of ordinals $\vec{\alpha}$, $\Xi(\phi, J, z, \vec{\alpha})$ is an ∞-Borel code for the set $A = \{x \in \mathbb{R} : L(J, \mathbb{R}) \models \phi(x, J, z, \vec{\alpha})\}$. (Moreover, the definition of $\Xi(\phi, J, z, \vec{\alpha})$ is absolute into $L(J, \mathbb{R})$ and thus $\Xi(\phi, J, z, \vec{\alpha}) \in L(J, \mathbb{R})$.)*

Proof. This is shown for $L(\mathbb{R})$ in [4] Theorem 7.19. The same argument works for $L(J, \mathbb{R})$ after suitable relativization. Moreover, the proof provides a uniform manner of obtaining such ∞-Borel code from ϕ, J, z, and $\vec{\alpha}$. Also [2] provides a proof in full generality. ☐

Fact 3.3. *(Woodin Perfect Set Dichotomy; [4] Theorem 8.5) Assume* AD *and* $\mathsf{DC}_{\mathbb{R}}$*. Let* E *be an equivalence relation with* ∞*-Borel code* (S,φ)*. Then exactly one of the following holds.*

(1) $\mathbb{R}/E \subseteq \mathrm{OD}_{\{S\}}^{L(S,\mathbb{R})}$ *and hence* $\mathbb{R}/E$ *has an* $\mathrm{OD}_{\{S\}}^{L(S,\mathbb{R})}$ *wellordering (which is obtained uniformly from* (S,φ)*).*

(2) $|\mathbb{R}| \leq |\mathbb{R}/E|$ *(there is an injection of* $\mathbb{R}$ *into* $\mathbb{R}/E$*).*

Proof. The definability of the wellordering in the first case can be seen by the inspecting the argument corresponding to the first case. See the proof of this result as presented in [6] Theorem 3.2 which states this more explicitly. $\square$

Definition 3.4. Let X be a set. Let $\mathscr{P}_{\mathrm{WO}}(X)$ be the collection of all sets $A \subseteq X$ so that A is well orderable. If κ is a cardinal, let $\mathscr{P}_\kappa(X)$ be the collection of all $A \subseteq X$ so that $|A| < \kappa$ (that is, A injects into κ but κ does not inject into A). Note that for all cardinals κ, $\mathscr{P}_\kappa(X) \subseteq \mathscr{P}_{\mathrm{WO}}(X)$.

If $\delta \in \mathrm{ON}$, then let $\mathcal{I}_{<\delta}(X)$ be the set of all injective functions $f : \epsilon \to X$ with $\epsilon < \delta$. Let $\mathcal{I}_{<\mathrm{ON}}(X)$ be the set of all injective functions $f : \epsilon \to X$ for some ordinal ϵ. (By the axiom of replacement, there is a cardinal κ so that $\mathcal{I}_{<\kappa}(X) = \mathcal{I}_{<\mathrm{ON}}(X)$.) Let $\mathfrak{r} : \mathcal{I}_{<\mathrm{ON}}(X) \to \mathscr{P}_{\mathrm{WO}}(X)$ be defined by $\mathfrak{r}(f) = f[\mathrm{dom}(f)]$. $\mathfrak{r} : I_{<\mathrm{ON}}(X) \to \mathscr{P}_{\mathrm{WO}}(X)$ is a surjection. For any cardinal κ, the restriction of $\mathfrak{r}$ to $\mathcal{I}_{<\kappa}(X)$ is a surjection of the form $\mathfrak{r} : \mathcal{I}_{<\kappa}(X) \to \mathscr{P}_\kappa(X)$.

Theorem 3.5. *Assume* AD*,* $\mathsf{DC}_{\mathbb{R}}$*, and all sets of reals have an* ∞*-Borel code. Let* $\kappa < \Theta$ *and* Y *be a surjective image of* $\mathbb{R}$*. Let* $\Phi : \mathscr{P}(\kappa) \to \mathscr{P}_{\mathrm{WO}}(Y)$*. Let* $\mathfrak{r} : \mathcal{I}_{<\mathrm{ON}}(Y) \to \mathscr{P}_{\mathrm{WO}}(X)$ *be defined by* $\mathfrak{r}(f) = f[\mathrm{dom}(f)]$*. Then there is a function* $\tilde{\Phi} : \mathscr{P}(\kappa) \to \mathcal{I}_{<\mathrm{ON}}(Y)$ *such that* $\Phi = \mathfrak{r} \circ \tilde{\Phi}$*.*

Proof. Since $\kappa < \Theta$, let $\phi : \mathbb{R} \to \kappa$ be a surjection. Define a prewellordering $\preceq$ on $\mathbb{R}$ by $x \preceq y$ if and only if $\phi(x) \leq \phi(y)$. Let $\mathrm{rk}_\preceq(x)$ be the rank of x in the prewellordering $\preceq$ and note that $\mathrm{rk}_\preceq = \phi$. Let Γ be any nonselfdual pointclass closed under $\wedge$, $\exists^{\mathbb{R}}$, and contains $\preceq$ (for example, the pointclass $\mathbf{\Sigma}_1^1(\preceq)$ frequently used for the Moschovakis coding lemma). Let $U \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ be a set in Γ and universal for subsets of $\mathbb{R}^2$ in Γ (which exists by an application of the Wadge lemma). Let $\bar{0} \in \mathbb{R}$ be the constant 0 function. For each $e \in \mathbb{R}$, let $C_e = \{\alpha \in \kappa : (\exists x)(\mathrm{rk}_\preceq(x) = \alpha \wedge U(e, x, \bar{0}))\}$. For each $A \subseteq \kappa$, let $T_A \subseteq \mathbb{R} \times \mathbb{R}$ be defined by $T_A(x, y)$ if and only if $\mathrm{rk}_\preceq(x) \in A \wedge y = \bar{0}$. By the Moschovakis coding lemma, there is an $V \subseteq \mathbb{R}^2$ with $V \in \Gamma$ such that $V \subseteq T_A$ and for all $\alpha < \kappa$, $T_A \cap (\mathrm{rk}_\preceq^{-1}[\{\alpha\}] \times \mathbb{R}) \neq \emptyset$ if and only if $V \cap (\mathrm{rk}_\preceq^{-1}[\{\alpha\}] \times \mathbb{R}) \neq \emptyset$. Since U is universal for subsets of $\mathbb{R}^2$ in Γ, there is an $e \in \mathbb{R}$ so that $V = U_e = \{(x, y) : U(e, x, y)\}$. Then $C_e = A$. Thus the map $e \mapsto C_e$ is a surjection of $\mathbb{R}$ onto $\mathscr{P}(\kappa)$ via the explicit coding procedure described above using $\preceq$ and the universal set U. Note that this coding is absolute to any inner model M containing all the reals, $\preceq$, and U. This implies that $\mathscr{P}(\kappa) = \mathscr{P}(\kappa) \cap M$.

Since Y is a surjective image of $\mathbb{R}$, let $\pi : \mathbb{R} \to Y$ be a surjection. Define an equivalence relation E on $\mathbb{R}$ by $x \; E \; y$ if and only if $\pi(x) = \pi(y)$. Thus there is a bijection $\Upsilon : \mathbb{R}/E \to Y$. Define $R \subseteq \mathscr{P}(\kappa) \times \mathbb{R}$ by $R(A, z)$ if and only if $\Upsilon([z]_E) \in \Phi(A)$. Note that for all $A \in \mathscr{P}(\kappa)$, $R_A = \{z \in \mathbb{R} : R(A, z)\} = \bigcup \Upsilon^{-1}[\Phi(A)]$ and $\Upsilon : R_A/E \to \Phi(A)$ is a bijection. Define $\hat{R} \subseteq \mathbb{R} \times \mathbb{R}$ by $\hat{R}(e, z)$ if and only if $R(C_e, z)$.

Since all sets of reals have ∞-Borel codes, let (S_0, φ_0'), (S_1, φ_1'), (S_2, φ_2'), and (S_3, φ_3') be ∞-Borel codes for $\preceq$, U, E, and $\hat{R}$, respectively. For notational simplicity, merge all four sets of ordinals S_0, S_1, S_2, and S_3 into a single set of ordinals T in some simple manner. Then one can find four formulas φ_0, φ_1, φ_2, and φ_3 so that (T, φ_0), (T, φ_1), (T, φ_2), and (T, φ_3) are ∞-Borel codes for $\preceq$, U, E, and $\hat{R}$, respectively.

Thus $\preceq, U, E, \hat{R} \in L(T, \mathbb{R})$. By the discussion above, $\mathscr{P}(\kappa) \subseteq L(T, \mathbb{R})$. Thus $R \in L(T, \mathbb{R})$ since within $L(T, \mathbb{R})$, one can define $R(A, z)$ if and only if there exists an $e \in \mathbb{R}$ so that $\hat{R}(e, z)$ and $C_e = A$. Moreover using the ∞-Borel code (T, φ_0), (T, φ_1), (T, φ_2), and (T, φ_3), the above shows that R is $\mathrm{OD}_{\{T\}}^{L(T,\mathbb{R})}$. Let $A \in \mathscr{P}(\kappa)$. Define $E_A \subseteq \mathbb{R} \times \mathbb{R}$ by $x \; E_A \; y$ if and only if $x \; E \; y \wedge R(A, x) \wedge R(A, y)$ which is an equivalence relation on $R_A = \{x : R(A, x)\}$. Note that E_A and R_A are $\mathrm{OD}_{\{T,A\}}^{L(T,\mathbb{R})}$. Since T and A can be merged into a single set of ordinals and E_A is $\mathrm{OD}_{\{T,A\}}^{L(T,\mathbb{R})}$, Fact 3.2 implies there is ∞-Borel code for E_A which is $\mathrm{OD}_{\{T,A\}}^{L(T,\mathbb{R})}$. Let (S_A, φ_A) be the least such ∞-Borel according to the canonical ordering of $\mathrm{OD}_{\{T,A\}}^{L(T,\mathbb{R})}$ (or let it be the ∞-Borel code obtained by the uniform procedure of Fact 3.2). Since $R_A/E_A = R_A/E$ is in bijection with $\Phi(A)$, R_A/E_A is wellorderable. Thus by Fact 3.3, $\mathbb{R}/E$ has an $\mathrm{OD}_{\{T,A\}}^{L(T,\mathbb{R})}$ wellordering. Let $\sqsubset_A$ be the $\mathrm{OD}^{L(T,\mathbb{R})}_{\{T,A\}}$-least such wellordering (or more explicitly, the wellordering uniformly defined from (S_A, φ_A) as in the proof of Fact 3.3 found in [4]). Let $\psi_A : \mathrm{ot}(\sqsubset_A) \to R_A/E_A$ be the canonical order preserving enumeration of R_A/E_A via $\sqsubset_A$. Let $\tilde{\Phi}(A) = \Upsilon \circ \psi_A$. Thus $\tilde{\Phi} : \mathscr{P}(\kappa) \to \mathcal{I}_{<\mathrm{ON}}(Y)$. Since for $A \in \mathscr{P}(\kappa)$, $\Upsilon : R_A/E_A \to \Phi(A)$ is a bijection, $\mathfrak{r}(\tilde{\Phi}(A)) = \Phi(A)$. Thus $\mathfrak{r} \circ \tilde{\Phi} = \Phi$. $\square$

In other words, Theorem 3.5 states that for all $\Phi : \mathscr{P}(\kappa) \to \mathscr{P}_{\mathrm{WO}}(Y)$, there is a function $\tilde{\Phi} : \mathscr{P}(\kappa) \to \mathcal{I}_{<\mathrm{ON}}(Y)$ so that $\tilde{\Phi}(A)$ is an wellordered enumeration of $\Phi(A)$ for any $A \in \mathscr{P}(\kappa)$. Note that there is no map $\Gamma : \mathscr{P}_{\mathrm{WO}}(Y) \to \mathcal{I}_{<\mathrm{ON}}(Y)$ which assigns a wellorderable $B \subseteq Y$ to a wellordered enumerate $\Gamma(B)$ of B. In fact, there is in general no injection of $\mathscr{P}_{\mathrm{WO}}(Y)$ into $\mathcal{I}_{<\mathrm{ON}}(Y)$. For example, let $Y = \mathbb{R}$. Let E_0 be the equivalence relation on ${}^{\omega}2$ defined by $x \ E_0 \ y$ if and only if there exists an $m \in \omega$ so that for all $n \geq m$, $x(n) = y(n)$. If $\mathsf{AC}^{\mathbb{R}}_{\omega}$ holds and all subsets of $\mathbb{R}$ have the property of Baire, then $\mathbb{R}/E_0$ is not linear orderable. $\mathbb{R}/E_0$ injects into $\mathscr{P}_{\omega_1}(\mathbb{R})$ by the inclusion map since E_0 is an equivalence relation with all countable classes. Thus $\mathscr{P}_{\omega_1}(\mathbb{R})$ is not linear orderable. However $\mathcal{I}_{<\omega_1}(\mathbb{R})$ is linearly orderable by the lexicographic ordering. (Also one can inject $\mathcal{I}_{<\omega_1}(\mathbb{R})$ into $\mathscr{P}(\omega_1)$ and use the lexicographic ordering on $\mathscr{P}(\omega_1)$.) This shows that $\mathscr{P}_{\omega_1}(\mathbb{R})$ does not inject into $\mathcal{I}_{<\omega_1}(\mathbb{R})$. One can show $|\mathcal{I}_{<\omega_1}(\mathbb{R})| < |\mathscr{P}_{\omega_1}(\mathbb{R})|$.

Theorem 3.6. *Assume* AD, $\mathsf{DC}_{\mathbb{R}}$, *and all sets of reals have an* ∞*-Borel code. Let* $\kappa < \Theta$ *satisfies* $\kappa \to_* (\kappa)^{\kappa}_2$. *Let* Y *be a surjective image of* $\mathbb{R}$. *If* $|\mathscr{P}(\kappa)| \leq |\mathscr{P}_{\kappa}(Y)|$, *then* $|[\kappa]^{\omega}| < |Y|$.

Proof. Let $\Phi : \mathscr{P}(\kappa) \to \mathscr{P}_{\kappa}(Y)$ be an injection. By Theorem 3.5, there is a $\tilde{\Phi} : \mathscr{P}(\kappa) \to \mathcal{I}_{<\mathrm{ON}}(Y)$ so that $\Phi = \mathfrak{r} \circ \tilde{\Phi}$. Since Φ is injective, this implies that $\tilde{\Phi}$ must be injective. For all $A \in \mathscr{P}(\kappa)$, $\Phi(A) = \mathfrak{r}(\tilde{\Phi}(A)) = \tilde{\Phi}[\mathrm{dom}(\tilde{\Phi}(A))]$ implies that $\tilde{\Phi}(A)$ is a wellordered injective enumeration of $\Phi(A)$. Since $\Phi(A) \in \mathscr{P}_{\kappa}(Y)$, $\tilde{\Phi}(A) \in \mathcal{I}_{<\kappa}(Y)$ and thus $\tilde{\Phi} : \mathscr{P}(\kappa) \to \mathcal{I}_{<\kappa}(Y)$. Since $\mathcal{I}_{<\kappa}(Y) \subseteq {}^{<\kappa}Y$, $\tilde{\Phi}$ may be regarded as an injective function $\tilde{\Phi} : \mathscr{P}(\kappa) \to {}^{<\kappa}Y$. Then $|[\kappa]^{\omega}| < |Y|$ by Corollary 2.18. $\square$

References

1. Andrés Eduardo Caicedo and Richard Ketchersid, *A trichotomy theorem in natural models of* AD^+, Set theory and its applications, Contemp. Math., vol. 533, Amer. Math. Soc., Providence, RI, 2011, pp. 227–258. MR 2777751
2. William Chan, *Cardinality of the set of bounded subsets of a cardinal*, In preparation.
3. ______, *Definable combinatorics of strong partition cardinals*, In preparation.
4. William Chan, *An introduction to combinatorics of determinacy*, Trends in Set Theory, Contemp. Math., vol. 752, Amer. Math. Soc., Providence, RI, 2020, pp. 21–75. MR 4132099
5. William Chan and Stephen Jackson, *Applications of infinity-borel codes to definability and definable cardinals*, Submitted.
6. William Chan and Stephen Jackson, *Cardinality of wellordered disjoint unions of quotients of smooth equivalence relations*, Ann. Pure Appl. Logic **172** (2021), no. 8, 102988. MR 4256846
7. ______, *Definable combinatorics at the first uncountable cardinal*, Trans. Amer. Math. Soc. **374** (2021), no. 3, 2035–2056. MR 4216731
8. William Chan, Stephen Jackson, and Nam Trang, *The primeness property for cardinalties*, In preparation.
9. ______, *A search for the cofinality of the power set of the first uncountable cardinal*, In preparation.
10. ______, *Almost Everywhere Behavior of Functions According to Partition Measures*, Forum Math. Sigma **12** (2024), Paper No. e16. MR 4696011
11. Sean Cox and John Krueger, *Namba forcing, weak approximation, and guessing*, J. Symb. Log. **83** (2018), no. 4, 1539–1565. MR 3893287
12. Steve Jackson, AD *and the projective ordinals*, Cabal Seminar 81–85, Lecture Notes in Math., vol. 1333, Springer, Berlin, 1988, pp. 117–220. MR 960900
13. ______, *A computation of* δ^1_5, Mem. Amer. Math. Soc. **140** (1999), no. 670, viii+94. MR 1606035
14. ______, *Structural consequences of* AD, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1753–1876. MR 2768700
15. Alexander S. Kechris, AD *and projective ordinals*, Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77), Lecture Notes in Math., vol. 689, Springer, Berlin, 1978, pp. 91–132. MR 526915
16. Alexander S. Kechris, Eugene M. Kleinberg, Yiannis N. Moschovakis, and W. Hugh Woodin, *The axiom of determinacy, strong partition properties and nonsingular measures*, Cabal Seminar 77–79 (Proc. Caltech-UCLA Logic Sem., 1977–79), Lecture Notes in Math., vol. 839, Springer, Berlin-New York, 1981, pp. 75–99. MR 611168
17. Eugene M. Kleinberg, *Infinitary combinatorics and the axiom of determinateness*, Lecture Notes in Mathematics, Vol. 612, Springer-Verlag, Berlin-New York, 1977. MR 0479903
18. Paul B. Larson, *Extensions of the axiom of determinacy*, University Lecture Series, vol. 78, American Mathematical Society, Providence, RI, [2023] ©2023. MR 4658673
19. John R. Steel, *An outline of inner model theory*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1595–1684. MR 2768698

20. ———, *Ordinal definability in models of determinacy. Introduction to Part V*, Ordinal definability and recursion theory: The Cabal Seminar. Vol. III, Lect. Notes Log., vol. 43, Assoc. Symbol. Logic, Ithaca, NY, 2016, pp. 3–48. MR 3469165

Institute for Discrete Mathematics and Geometry, Vienna University of Technology, Vienna, Austria
Email address: William.Chan@tuwien.ac.at