DEFINABLE COMBINATORICS OF SOME BOREL EQUIVALENCE RELATIONS

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Abstract. If X is a set, E is an equivalence relation on X, and $n \in \omega$, then define

$$[X]^n_E = \{(x_0, ..., x_{n-1}) \in {}^nX : (\forall i,j)(i \neq j \Rightarrow \neg(x_i \; E \; x_j))\}.$$

For $n \in \omega$, a set X has the n-Jónsson property if and only if for every function $f : [X]^n_= \to X$, there exists some $Y \subseteq X$ with X and Y in bijection so that $f[[Y]^n_=] \neq X$. A set X has the Jónsson property if and only for every function $f : (\bigcup_{n\in\omega} [X]^n_=) \to X$, there exists some $Y \subseteq X$ with X and Y in bijection so that $f[\bigcup_{n\in\omega} [Y]^n_=] \neq X$.

Let $n \in \omega$, X be a Polish space, and E be an equivalence relation on X. E has the n-Mycielski property if and only if for all comeager $C \subseteq {}^nX$, there is some $\mathbf{\Delta}^1_1$ $A \subseteq X$ so that $E \leq_{\mathbf{\Delta}^1_1} E \restriction A$ and $[A]^n_E \subseteq C$.

The following equivalence relations will be considered: E_0 is defined on ${}^\omega 2$ by $x \; E_0 \; y$ if and only if $(\exists n)(\forall k > n)(x(k) = y(k))$. E_1 is defined on ${}^\omega({}^\omega 2)$ by $x \; E_1 \; y$ if and only if $(\exists n)(\forall k > n)(x(k) = y(k))$. E_2 is defined on ${}^\omega 2$ by $x \; E_2 \; y$ if and only if $\sum\{\frac{1}{n+1} : n \in x \triangle y\} < \infty$, where $\triangle$ denotes the symmetric difference. E_3 is defined on ${}^\omega({}^\omega 2)$ by $x \; E_3 \; y$ if and only if $(\forall n)(x(n) \; E_0 \; y(n))$.

Holshouser and Jackson have shown that $\mathbb{R}$ is Jónsson under AD. It will be shown that E_0 does not have the 3-Mycielski property and that E_1, E_2, and E_3 do not have the 2-Mycielski property. Under $\mathsf{ZF} + \mathsf{AD}$, ${}^\omega 2/E_0$ does not have the 3-Jónsson property.

1. Introduction

The Jónsson property and other combinatorial partition properties of well-ordered sets have been studied by set theorists under the axiom of choice, large cardinal axioms, and the axiom of determinacy. Holshouser and Jackson began the study of the Jónsson property using definability techniques for sets which generally cannot be well-ordered in a definable manner.

Let X be a set and E an equivalence relation on X. For each $n \in \omega$, let $[X]^n_E$ be the collection of tuples $(x_0, ..., x_{n-1}) \in {}^nX$ so that for all $i \neq j$, $\neg(x_i \; E \; x_j)$. Let $[X]^{<\omega}_E = \bigcup_{n\in\omega} [X]^n_E$. For each $n \in \omega$, X has the *n-Jónsson property* if and only if for every function $f : [X]^n_= \to X$, there is some $Y \subseteq X$ with Y in bijection with X and $f[[Y]^n_=] \neq X$. X has the *Jónsson property* if and only if for every function $f : [X]^{<\omega}_= \to X$, there is some $Y \subseteq X$ with Y in bijection with X, and $f[[Y]^{<\omega}_=] \neq X$.

Holshouser and Jackson showed that ${}^\omega 2$ has the Jónsson property under the axiom of determinacy, AD. Let $f : [{}^\omega 2]^{<\omega}_= \to {}^\omega 2$. For each $n \in \omega$, let $f_n : [X]^n_= \to X$ be $f \restriction [X]^n_=$. Their proof has two notable tasks:
(1) Holshouser and Jackson first (assuming all sets have the Baire property) choose comeager sets $C_n \subseteq {}^n({}^\omega 2)$ so that $f_n \restriction C_n$ is continuous. Then a single perfect set $P \subseteq {}^\omega 2$ is found so that for each n, $f_n \restriction [P]^n_=$ is continuous. To obtain this perfect set P, they use a classical theorem of Mycielski which states: If C_n is a sequence of comeager subsets of ${}^n({}^\omega 2)$, then there is some perfect set $P \subseteq {}^\omega 2$ so that $[P]^n_= \subseteq C_n$ for all n.
(2) Since each f_n is continuous on $[P]^n_=$, they use a fusion argument to simultaneously prune P to a smaller perfect set $Q \subseteq P$ so that there exists some real that is missed by each f_n on $[Q]^n_=$.

Holshouser and Jackson ask whether other sets which may not be well-ordered in some choiceless setting like AD could also have the Jónsson property. They observed that under $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$, every set $X \in L_\Theta(\mathbb{R})$ has a surjective function $f : \mathbb{R} \to X$. Define an equivalence relation on $\mathbb{R}$ by $x \; E \; y$ if and only if $f(x) = f(y)$. Then X is in bijection with $\mathbb{R}/E$. The study of the Jónsson property for sets in $L_\Theta(\mathbb{R})$ is equivalent to studying the Jónsson property for quotients of $\mathbb{R}$ by equivalence relations on $\mathbb{R}$. Note that $\mathbb{R}$ is in bijection with $\mathbb{R}/=$.

Through dichotomy results of Harrington, Hjorth, Kechris, Louveau, and others, the equivalence relations $=$, E_0, E_1, E_2, and E_3 occupy special positions in the structure of $\mathbf{\Delta}^1_1$ equivalence relations under $\mathbf{\Delta}^1_1$

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reducibilities. $=$ is the identity equivalence relation on ${}^\omega 2$. E_0 is defined on ${}^\omega 2$ by $x \ E_0 \ y$ if and only if $(\exists n)(\forall k > n)(x(k) = y(k))$. E_1 is defined on ${}^\omega({}^\omega 2)$ by $x \ E_1 \ y$ if and only if $(\exists n)(\forall k > n)(x(k) = y(k))$. E_2 is defined on ${}^\omega 2$ by $x \ E_2 \ y$ if and only if $\sum\{\frac{1}{n+1} : n \in x \triangle y\} < \infty$, where $\triangle$ denotes the symmetric difference. E_3 is defined on ${}^\omega({}^\omega 2)$ by $x \ E_3 \ y$ if and only if $(\forall n)(x(n) \ E_0 \ y(n))$.

Holshouser and Jackson asked whether the methods applicable for showing $\mathbb{R}/=$ has the Jónsson property could be used to show the quotients of these other $\mathbf{\Delta}_1^1$ equivalence relations could be Jónsson. An important aspect of their proof for $\mathbb{R}$ was the theorem of Mycielski. They defined the Mycielski property for arbitrary equivalence relations as follows: Let E be an equivalence relation on a Polish space X. For each $n \in \omega$, E has the *n-Mycielski property* if and only if for every comeager $C \subseteq {}^n X$, there exists some $\mathbf{\Delta}_1^1$ $A \subseteq X$ so that $E \leq_{\mathbf{\Delta}_1^1} E \upharpoonright A$ and $[A]^n_E \subseteq C$.

They asked whether any of the $\mathbf{\Delta}_1^1$ equivalence relations mentioned above have the n-Mycielski property for various $n \in \omega$ and whether the Mycielski property could be used to prove the Jónsson property for the quotient of any of these equivalence relations. Holshouser and Jackson began this study by showing that E_0 has the 2-Mycielski property and this can be used to show ${}^\omega 2/E_0$ has the 2-Jónsson property. This paper will show that the Mycielski property fails in most cases:

Theorem 8.1. The equivalence relation E_0 does not have the 3-Mycielski property.

Theorem 13.4. The equivalence relation E_1 does not have the 2-Mycielski property.

Theorem 15.1. The equivalence relation E_2 does not have the 2-Mycielski property.

Theorem 18.2. The equivalence relation E_3 does not have the 2-Mycielski property.

These results require understanding the structure of $\mathbf{\Delta}_1^1$ subsets of ${}^\omega 2$ or ${}^\omega({}^\omega 2)$ so that $E_0 \leq_{\mathbf{\Delta}_1^1} E_0 \upharpoonright A$ (or $E_1 \leq_{\mathbf{\Delta}_1^1} E_1 \upharpoonright A$, etc.) that come from the proofs of the dichotomy results. Kanovei, Sabok, and Zapletal in [12], [13], [20], and [21] have studied the forcing of such $\mathbf{\Delta}_1^1$ sets for each of these equivalence relations.

Given that the Mycielski property fails in general, a reflection on Holshouser and Jackson's proof of the Jónsson property for ${}^\omega 2$ shows that it is only used to find some perfect set P so that $f_n \upharpoonright [P]^n_=$ is nicely behaved (i.e., continuous). This paper will give a forcing style proof of Holshouser and Jackson results that ${}^\omega 2$ is Jónsson and ${}^\omega 2/E_0$ is 2-Jónsson assuming all functions satisfy a certain definability condition expressed in Lemma 3.3. This definability condition follows from the Mycielski property for the equivalence relation and the assumption that all sets have the Baire property. All $\mathbf{\Delta}_1^1$ functions have this definability condition and under the axiom of choice and large cardinal assumptions, projective and even more complex sets also satisfy this condition.

Following part (2) of Holshouser and Jackson's template for ${}^\omega 2$, suppose one could find some $\mathbf{\Delta}_1^1$ set $B \subseteq {}^\omega 2$ with $E_0 \leq_{\mathbf{\Delta}_1^1} E_0 \upharpoonright B$ and $f \upharpoonright [B]^3_{E_0}$ is continuous for some function f. Could one then somehow prune B to some $C \subseteq B$ so that $E_0 \leq_{\mathbf{\Delta}_1^1} E_0 \upharpoonright C$ and $f[[C]^3_{E_0}] \neq {}^\omega 2$, or even better, miss an E_0-class? This paper will have some discussion on how these continuity and surjectivity properties for E_0 and E_2 can fail.

This shows both part (1) and part (2) of the proof of Holshouser and Jackson establishing ${}^\omega 2$ is Jónsson fail for E_0 and several other $\mathbf{\Delta}_1^1$ equivalence relations. Moreover, for E_0, it will in fact be shown that ${}^\omega 2/E_0$ is not Jónsson under determinacy:

Theorem 10.4. $(\mathsf{ZF} + \mathsf{AD})$ ${}^\omega 2/E_0$ does not have the 3-Jónsson property and hence is not Jónsson.

Here are some historical remarks about the Jónsson property: Under the axiom of choice, the Jónsson property is usually studied on cardinals. Cardinals possessing the Jónsson property are called *Jónsson cardinals*. For $n \in \omega$, let $\mathscr{P}^n(X)$ denote the collection of all n-element subsets of X. Since there is a well-ordering, the Jónsson property is usually defined using $\mathscr{P}^n(X)$ rather than $[X]^n_=$. When this paper discusses the Jónsson property using $\mathscr{P}^n(X)$, it will be refered to as the classical Jónsson property.

Under the axiom of choice, Jónsson cardinals also have model-theoretic characterizations. The existence of Jónsson cardinals imply $V \neq L$. Moreover, it has large cardinal consistency strength: for instance, it implies $0^\sharp$ exists. Erdős and Hajnal ([4] and [3]) showed that if $2^\kappa = \kappa^+$, then κ^+ is not a Jónsson cardinal. Hence under CH, $2^{\aleph_0}$ is not a Jónsson cardinal. Every real valued measurable cardinal is Jónsson (see [3] Corollary 11.1). Solovay showed the consistency of a measurable cardinal implies the consistency of $2^{\aleph_0}$ being real valued measurable. Hence it is consistent relative to a measurable cardinal that $2^{\aleph_0}$ is a Jónsson cardinal. The sets ${}^\omega 2$, ${}^\omega 2/E_0$, ${}^\omega({}^\omega 2)/E_1$, ${}^\omega 2/E_2$, ${}^\omega({}^\omega 2)/E_3$ are all in bijection with each other using the axiom of choice. Hence if CH holds, these quotients do not have the Jónsson property and if $2^{\aleph_0}$ is real valued measurable, then all these quotients do have the Jónsson property.

Under AD, the Jónsson property and other combinatorial partition properties of cardinals were already studied during the 1960s and 1970s. Assuming AD, for each $n \in \omega$, $\aleph_n$ is a Jónsson cardinal ([17]). More recently Woodin had shown that under $\mathsf{ZF} + \mathsf{AD}^+$, every cardinal $\kappa < \Theta$ has the Jónsson property. Also [11] showed that in $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$, every cardinal $\kappa < \Theta$ is Jónsson. [11] asked whether ${}^\omega 2$, which cannot be well-ordered, has the Jónsson property. In analogy, they asked if every set in $L_\Theta(\mathbb{R})$ has the Jónsson property. Holshouser and Jackson's answer to this question for ${}^\omega 2$ begins the work that is carried out in this paper.

Throughout, results attributed to Holshouser and Jackson can be found in [10] and [9].

This paper is organized as follows:

Section 2 contains definitions of the main concepts and some basic facts about determinacy.

Section 3 will give a proof of the result of Holshouser and Jackson which shows ${}^\omega 2$ has the Jónsson property if all sets have the Baire property. The proof uses forcing arguments and fusion. This section will have some discussions about how absoluteness available under AD^+ can be used to prove this result without using the Mycielski property. However, throughout the paper, a flexible fusion argument is necessary for handling the combinatorics. It is unclear what the relation is between properness, fusion, and the Jónsson property for the five equivalence relations considered.

Upon considering the Jónsson property for ${}^\omega 2$, a natural question is whether there is a function $f : \mathscr{P}^\omega({}^\omega 2) \to {}^\omega 2$ so that for all $A \subseteq {}^\omega 2$ with $A \approx {}^\omega 2$, $f[\mathscr{P}^\omega(A)] = {}^\omega 2$. Such a function is called an *ω-Jónsson function* for ${}^\omega 2$. Under the axiom of choice, [4] showed that every set has an ω-Jónsson function. Section 4 gives an example under $\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_\omega$ (choice for countable sets of nonempty subsets of ${}^\omega 2$) of a $\mathbf{\Delta}^1_1$ ω-Jónsson function for ${}^\omega 2$.

From the effective proof of the E_0-dichotomy, every $\mathbf{\Sigma}^1_1$ set $A \subseteq {}^\omega 2$ so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \restriction A$ contains the body of a perfect tree with certain symmetry restrictions, known as an E_0-tree. Section 5 will modify the proof of the E_0-dichotomy using Gandy-Harrington methods to prove a structure theorem for $\mathbf{\Sigma}^1_1$ sets with the same E_0-saturation on which the restriction of E_0 is not smooth: For example, if A and B are two $\mathbf{\Sigma}^1_1$ sets with $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \restriction A$, $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \restriction B$, and $[A]_{E_0} = [B]_{E_0}$, then there are E_0 trees p and q with $[p] \subseteq A$, $[q] \subseteq B$, and p and q are the same except possibly at the stem. This is needed to show the failure of the weak 3-Mycielski property (see Definition 2.24) for E_0.

Section 6 will introduce the forcing $\widehat{\mathbb{P}}^2_{E_0}$. This forcing will be used to prove the result of Holshouser and Jackson stating that ${}^\omega 2/E_0$ has the 2-Jónsson property.

Let X and Y be sets. Let $n \in \omega$. Define $X \to (X)^n_Y$ to mean that for any function $f : \mathscr{P}^n(X) \to Y$, there is some $Z \subseteq X$ with $Z \approx X$ and $|f[\mathscr{P}^n(Z)]| = 1$. Define $X \mapsto (X)^n_Y$ to mean that for any function $f : [X]^n_= \to Y$, there is some $Z \subseteq X$ with $Z \approx X$ and $|f[[Z]^n_=]| = 1$. Section 7 will show that ${}^\omega 2/E_0 \mapsto ({}^\omega 2/E_0)^2_n$ holds for all $n \in \omega$.

Section 8 will show that E_0 does not have the 3-Mycielski property or weak 3-Mycielski property.

Section 9 will produce a continuous function $Q : [{}^\omega 2]^3_{E_0} \to {}^\omega 2$ so that for every $\mathbf{\Sigma}^1_1$ $A \subseteq {}^\omega 2$ with $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \restriction A$, $Q[[A]^3_{E_0}] = {}^\omega 2$. A modification of this function yields a $\mathbf{\Delta}^1_1$ function $K : {}^3({}^\omega 2) \to {}^\omega 2$ so that on any such set A, $K \restriction [A]^3_{E_0}$ is not continuous.

Section 10 will use the function produced in the previous section to show that ${}^\omega 2/E_0$ does not have the 3-Jónsson property or the classical 3-Jónsson property under $\mathsf{ZF} + \mathsf{AD}$. (In particular, ${}^\omega 2/E_0$ is not Jónsson under AD.)

Section 11 will use the classical 3-Jónsson map for ${}^\omega 2/E_0$ to show the failure of ${}^\omega 2/E_0 \to ({}^\omega 2/E_0)^3_2$.

The fusion argument related to the proper forcing $\widehat{\mathbb{P}}^2_{E_0}$ was used to establish many of the combinatorial properties of E_0 in dimension two. Given the failure of these properties in dimension three, a natural question would be whether the three dimensional analog $\widehat{\mathbb{P}}^3_{E_0}$ is proper and possesses a reasonable fusion. Section 12 will show that $\widehat{\mathbb{P}}^3_{E_0}$ is proper by having some type of fusion argument. However, there is far less control of this fusion.

Section 13 will show that E_1 does not have the 2-Mycielski property.

Section 14 will modify the proof of the E_2-dichotomy result using Gandy-Harrington methods to give structural result about E_2-big $\mathbf{\Sigma}_1^1$ sets with the same E_2-saturation: For example, if A and B are two $\mathbf{\Sigma}_1^1$ sets with $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \restriction A$, $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \restriction B$, and $[A]_{E_2} = [B]_{E_2}$, then there are two E_2-trees (perfect trees with certain properties) p and q so that $[p] \subseteq A$, $[q] \subseteq B$, $[[p]]_{E_2} = [[q]]_{E_2}$, and p and q resemble each other in specific ways.

Section 15 will use results of the previous section to show E_2 does not have the 2-Mycielski property and the weak 2-Mycielski property.

Section 16 will produce a continuous function $Q : [{}^\omega 2]^3_{E_2} \to {}^\omega 2$ so that on any $\mathbf{\Sigma}_1^1$ set A with $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \restriction A$, $Q[[A]^3_{E_2}] = {}^\omega 2$. There is also a $\mathbf{\Delta}_1^1$ function $P' : {}^3({}^\omega 2) \to {}^\omega 2$ so that for any such set A, $P' \restriction A$ is not continuous.

Section 17 contains no new results but just gives the rather lengthy characterization of $\mathbf{\Sigma}_1^1$ sets $A \subseteq {}^\omega({}^\omega 2)$ so that $E_3 \leq_{\mathbf{\Delta}_1^1} E_3 \restriction A$ which comes from the E_3-dichotomy result. This structure result is applied in Section 18 to show that E_3 does not have the 2-Mycielski property.

Section 19 will study the completeness of non-principal ultrafilters on quotients of Polish spaces by equivalence relations.

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2. Basic Information

Definition 2.1. Let $\sigma \in {}^{<\omega}2$. Suppose $|\sigma| = k$. Then $\tilde{\sigma} \in {}^\omega 2$ is defined by $\tilde{\sigma}(n) = \sigma(j)$ where $0 \leq j < k$ and $j \equiv n \mod k$.

For example, $\tilde{0}$, $\tilde{1}$, $\widetilde{01}$, etc. will appear frequently.

Definition 2.2. Let $\sigma \in {}^{<\omega}2$. Let $N_\sigma = \{x \in {}^\omega 2 : x \supseteq \sigma\}$.

$\{N_\sigma : \sigma \in {}^{<\omega}2\}$ is a basis for the topology on ${}^\omega 2$.

Let $\sigma, \tau \in {}^{<\omega}2$. Let $N_{\sigma,\tau} = \{(x, y) \in {}^2({}^\omega 2) : x \in N_\sigma \wedge y \in N_\tau\}$.

$\{N_{\sigma,\tau} : \sigma, \tau \in {}^{<\omega}2\}$ is a basis for the topology on ${}^2({}^\omega 2)$.

$N_{\sigma,\tau,\rho}$ is defined similarly for ${}^3({}^\omega 2)$.

Definition 2.3. Let $n \in \omega$ and $\sigma : n \to {}^{<\omega}2$. Let $N_\sigma = \{x \in {}^\omega({}^\omega 2) : (\forall k < n)(\sigma(k) \subseteq x(k))\}$.

$\{N_\sigma : \sigma \in {}^{<\omega}({}^{<\omega}2)\}$ is a basis for the topology on ${}^\omega({}^\omega 2)$.

Let $\sigma, \tau : n \to {}^{<\omega}2$. Let $N_{\sigma,\tau} = \{(x, y) \in {}^2({}^\omega({}^\omega 2)) : x \in N_\sigma \wedge y \in N_\tau\}$.

$\{N_{\sigma,\tau} : \sigma, \tau \in {}^{<\omega}({}^{<\omega}2) \wedge |\sigma| = |\tau|\}$ is a basis for the topology on ${}^2({}^\omega({}^\omega 2))$.

Definition 2.4. Let A and B be two sets. $A \approx B$ denotes that there is a bijection between A and B.

Often this paper will consider settings where the full axiom of choice may fail. In such contexts, not all sets have a cardinal, i.e. is in bijection with an ordinal. Similarity of size is more appropriately given by the existence of bijections. Recall the following method of producing bijections between sets which is provable in ZF:

Fact 2.5. *(Cantor-Schröder-Bernstein)* (ZF) *Let X and Y be two sets. Suppose there are injections $\Phi : X \to Y$ and $\Psi : Y \to X$. Then there is a bijection $\Lambda : X \to Y$.*

Definition 2.6. Let X and Y be sets. ${}^X Y$ is the set of functions from X to Y.

$\mathscr{P}(X)$ is the power set of X.

Let $n \in \omega$. Define

$$\mathscr{P}^n(X) = \{F \in \mathscr{P}(X) : F \approx n\}$$
$$\mathscr{P}^{<\omega}(X) = \bigcup_{n \in \omega} \mathscr{P}^n(X)$$

Let E be an equivalence relation on a set X. Let $n \in \omega$. Define

$$[X]^n_E = \{(x_0, ..., x_{n-1}) \in {}^n X : (\forall i, j < n)(i \neq j \Rightarrow \neg(x_i \; E \; x_j))\}$$

$$[X]^{<\omega}_E = \bigcup_{n \in \omega} [X]^n_E$$

Definition 2.7. Let X be a set and $n \in \omega$. A set X has the n-Jónsson property if and only for all functions $f : [X]^n_= \to X$, there is some $Y \subseteq X$ so that $Y \approx X$ and $f[[Y]^n_=] \neq X$. X has the Jónsson property if and only if for all $f : [X]^{<\omega}_= \to X$, there is some $Y \subseteq X$ so that $Y \approx X$ and $f[[Y]^{<\omega}_=] \neq X$.

A set X has the classical n-Jónsson property (or classical Jónsson property) if and only the above holds with $[X]^=_n$ (or $[X]^{<\omega}_=$) replaced with $\mathscr{P}^n(X)$ (or $\mathscr{P}^{<\omega}(X)$, respectively).

If X is a wellordered set, one can identify a finite set $F \subseteq X$ with the increasing enumeration of its elements. Such a presentation is helpful for defining useful functions on $\mathscr{P}^n(X)$. In the absence of choice, it is easier to define functions when one considers order tuples from $[X]^n_=$. For this reason, the paper will be mostly concerned about the Jónsson property as defined above rather than the classical Jónsson property, although the classical version will be discussed in Section 10.

Definition 2.8. Let X be a set. $[X]^\omega_=$ and $\mathscr{P}^\omega(X)$ are defined as above (with ω in place of $n \in \omega$).

Let $N \in \omega \cup \{\omega\}$. A N-Jónsson function for X is a function $\Phi : [X]^N_= \to X$ so that for any $Y \subseteq X$ with $Y \approx X$, $\Phi[[Y]^N_=] = X$.

A classical N-Jónsson function for X is defined in the same way as the above with $\mathscr{P}^N(X)$ instead of $[X]^N_=$.

With the axiom of choice, [4] showed that every set has an ω-Jónsson map. The existence of ω-Jónsson maps for certain cardinals is where Kunen's original proof of the Kunen inconsistency used the axiom of choice. Note that for $N \in \omega \cup \{\omega\}$, a counterexample to the N-Jónsson property for some set is equivalent to the existence of an n-Jónsson function for that set.

Definition 2.9. Let X and Y be Polish spaces. Let E and F be equivalence relations on X and Y, respectively. A $\mathbf{\Delta}^1_1$ reduction between X and Y is a $\mathbf{\Delta}^1_1$ function $\Phi : X \to Y$ such that for all $a, b \in X$, $a \; E \; b$ if and only if $\Phi(a) \; F \; \Phi(b)$.

This situtation is denoted by $E \leq_{\mathbf{\Delta}^1_1} F$. Define $E \equiv_{\mathbf{\Delta}^1_1} F$ if and only if $E \leq_{\mathbf{\Delta}^1_1} F$ and $F \leq_{\mathbf{\Delta}^1_1} E$.

Definition 2.10. Let E be an equivalence relation on a set X. If $x \in X$, then $[x]_E = \{y \in X : y \; E \; x\}$ is the E-class of x. Let $A \subseteq X$. $[A]_E = \{y \in X : (\exists x \in A)(x \; E \; y)\}$ is the E-saturation of A.

Definition 2.11. Let X be a Polish space and E be an equivalence relation on X. Let $n \in \omega$. X has the n-Mycielski property if and only if for every $C \subseteq {}^n X$ which is comeager in ${}^n X$, there is a $\mathbf{\Delta}^1_1$ set $A \subseteq X$ so that $E \equiv_{\mathbf{\Delta}^1_1} E \restriction A$ and $[A]^n_E \subseteq C$.

E has the Mycielski property if and only if for all sequences $(C_n : n \in \omega)$ such that for all $n \in \omega$, $C_n \subseteq {}^n X$ is comeager in ${}^n X$, there is a some set $A \subseteq X$ so that $E \equiv_{\mathbf{\Delta}^1_1} E \restriction A$ and for all $n \in \omega$, $[A]^n_E \subseteq C_n$.

The Mycielski property of equivalence relations comes from the following eponymous result:

Fact 2.12. *(Mycielski) Let $(C_n : n \in \omega)$ be a sequence such that for each $n \in \omega$, $C_n \subseteq {}^n({}^\omega 2)$ is a comeager subset of ${}^n({}^\omega 2)$. Then there is a perfect set $P \subseteq {}^\omega 2$ so that for all $n \in \omega$, $[P]^n_= \subseteq C_n$.*

Definition 2.13. Let E be an equivalence relation on a Polish space X. Let $n \in \omega$. E has the n-continuity property if and only if for every function $f : {}^n X \to X$, there is some $\mathbf{\Delta}^1_1$ $A \subseteq X$ so that $E \equiv_{\mathbf{\Delta}^1_1} E \restriction A$ and $f \restriction [A]^n_E$ is continuous.

Fact 2.14. *Let E be an equivalence relation on a Polish X which has the n-Mycielski property. Then for every function $f : {}^n X \to X$ with the property of Baire (i.e. $f^{-1}[U]$ has the Baire property for every open set U), there is some $\mathbf{\Delta}^1_1$ $A \subseteq X$ with $E \equiv_{\mathbf{\Delta}^1_1} E \restriction A$ so that $f \restriction [A]^n_E$ is continuous. Hence if every set has the Baire property, then E has the n-continuity property.*

Proof. Let $f : {}^n X \to X$. Since f is Baire measurable, there is some $C \subseteq {}^n X$ so that $f \restriction C$ is continuous. By the n-Mycielski property, there is some $A \subseteq X$ with $E \equiv_{\mathbf{\Delta}^1_1} E \restriction A$ so that $[A]^n_E \subseteq C$. $f \restriction [A]^n_E$ is continuous. ☐

In place of the axiom of choice, the paper will often use the axiom of determinacy. The following is a quick description of determinacy:

Definition 2.15. Let X be a set. Let $A \subseteq {}^\omega X$. The game G_A is defined as follows: Player 1 plays $a_i \in X$, and player 2 plays $b_i \in X$ for each $i \in \omega$. At turn $2i$, player 1 plays a_i, and at turn $2i+1$, player 2 plays b_i. Let $f \in {}^\omega X$ be defined by $f(2i) = a_i$ and $f(2i+1) = b_i$. Player 1 wins this play of G_A if and only if $f \in A$. Player 2 wins otherwise.

A winning strategy for player 1 is a function $\tau : {}^{<\omega}X \to X$ so that for any $(b_i : i \in \omega)$ if $(a_i : i \in \omega)$ is defined recursive by $a_0 = \tau(\emptyset)$ and $a_{n+1} = \tau(a_0...a_n b_n)$, then player 1 wins the resulting play of G_A. A winning strategy for player 2 is defined similarly.

The Axiom of Determinacy for X, denoted AD_X, is the statement that for all $A \subseteq {}^\omega X$, G_A has a winning strategy for some player.

AD refers to AD_2 or equivalently AD_ω. $\mathsf{AD}_\mathbb{R}$ will also be used. Note that $\mathsf{AD}_\mathbb{R}$ often will refer to $\mathsf{AD}_{{}^\omega 2}$ or $\mathsf{AD}_{{}^\omega \omega}$.

AD implies classical regularity properties for sets of reals: Every set of reals has the Baire property and is Lebesgue measurable. Every uncountable set of reals has a perfect subset. Every function on the reals is continuous on a comeager set.

Uniformization however is more subtle:

Definition 2.16. Let $R \subseteq {}^\omega 2 \times {}^\omega 2$. Let $R^x = \{y : (x,y) \in R\}$. Suppose for all $x \in {}^\omega 2$, $R^x \neq \emptyset$. R can be uniformized if and only if there is a some function $f : {}^\omega 2 \to {}^\omega 2$ so that for all $x \in {}^\omega 2$, $(x, f(x)) \in R$. Such a function f is called a uniformization of R.

Fact 2.17. $(\mathsf{ZF} + \mathsf{AD}_\mathbb{R})$ *Every relation can be uniformized.*

Proof. Suppose $R \subseteq {}^\omega 2 \times {}^\omega 2$ with the property that for all x, $R^x \neq \emptyset$. Consider the two step game where player 1 plays $a \in {}^\omega 2$ and player 2 responds with $b \in {}^\omega 2$. Player 2 wins if and only if $(a,b) \in R$. Clearly player 1 can not have a winning strategy. Any winning strategy for player 2 yields a uniformization of R. ☐

Woodin has shown that if there is a measurable cardinal with infinitely many Woodin cardinals below it, then $L(\mathbb{R}) \models \mathsf{AD}$. Solovay showed in [18] Lemma 2.2 and Corollary 2.4 that the relation $R(x,y)$ if and only y is not ordinal definable from x can not be uniformized in $L(\mathbb{R})$. Hence $\mathsf{AD}_\mathbb{R}$ is stronger than than AD. AD is not capable of proving full uniformization.

Definition 2.18. Let E be an equivalence relation on ${}^\omega 2$ and $n \in \omega$. Let $f : ({}^\omega 2/E)^n \to {}^\omega 2/E$. A lift of f is a function $F : {}^n({}^\omega 2) \to {}^\omega 2$ with the property that for all $(x_0, ..., x_n) \in {}^n({}^\omega 2)$, $[F(x_0, ..., x_{n-1})]_E = f([x_0]_E, ..., [x_{n-1}]_E)$.

Fact 2.19. *Let E be an equivalence relation on ${}^\omega 2$ and $n \in \omega$. Let $f : {}^n({}^\omega 2/E) \to {}^\omega 2/E$. Define $R_f(x_0, ..., x_{n-1}, y) \Leftrightarrow y \in f([x_0]_E, ..., [x_{n-1}]_E)$. If F is a uniformization of R_f (with respect to the last variable), then F is a lift of f.*

Under $\mathsf{AD}_\mathbb{R}$, every such function has a lift.

Many natural models of AD such as $L(\mathbb{R})$ are not models of $\mathsf{AD}_\mathbb{R}$. However, functions on quotients of equivalence relations with all classes countable can still be uniformized: AD^+ is a strengthening of AD which holds in all known models of AD (in particular $L(\mathbb{R})$). See [19] Definition 9.6 for the definition of AD^+. It is open whether AD and AD^+ are equivalent. Also $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$ implies AD^+, and it is open whether this holds without DC.

Fact 2.20. *(Countable Section Uniformization) (Woodin)* (AD^+) *Let $R \subseteq {}^\omega 2 \times {}^\omega 2$ have the property that for all $x \in {}^\omega 2$, R^x is countable. Then R can be uniformized.*

Proof. See [16] Theorem 3.2 for a proof. ☐

Fact 2.21. (AD^+) *Let E be an equivalence relation on ${}^\omega 2$ with all classes countable. Let $n \in \omega$ and $f : {}^n({}^\omega 2/E) \to {}^\omega 2/E$. Then f has a lift.*

For the results of this paper, all results that require lifts can be replace by lift on some comeager set. The benefit is that such lift follows from comeager uniformization which is provable in just $\mathsf{ZF} + \mathsf{AD}$.

Fact 2.22. *(Comeager Unformization)* $(\mathsf{ZF}+\mathsf{AD})$ *Let* $R \subseteq {}^\omega 2 \times {}^\omega 2$ *be a result such that* $(\forall x)(\exists y)R(x,y)$, *then there is a comeager* $C \subseteq {}^\omega 2$ *and some function* $f : C \to {}^\omega 2$ *so that* $(\forall x \in C)R(x, f(x))$.

By shrinking to an appropriate comeager set, one can assume that the uniformizing function is also continuous.

Often to use the techniques of forcing over countable elementary structures, the axiom of determinacy will need to be augmented by dependent choice (DC). Kechris [14] proved that AD and $\mathsf{AD}+\mathsf{DC}$ have the same consistency strength by showing if $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R}) \models \mathsf{DC}$. However, Solovay [18] showed that $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$ has strictly stronger consistency strength than $\mathsf{AD}_\mathbb{R}$.

If one is ultimately interested in functions $F : {}^n({}^\omega 2) \to {}^\omega 2$ which are lifts of some function $f : {}^n({}^\omega 2/E) \to {}^\omega 2/E$ only in order to infer information about f, then the demand in the Mycielski property that one considers tuples coming from a single set $A \subseteq {}^\omega 2$ such that $E \equiv_{\boldsymbol{\Delta}^1_1} E \restriction A$ seems restrictive. If one ultimately will collapse back to the quotient, two sets A and B with the same E-saturation should work equally well. This motivates the following concepts:

Definition 2.23. Let $n \in \omega$ and E be an equivalence relation on some Polish space X. Let $(A_i : i < n)$ be a sequence of subsets of X. Define

$$\prod_{i<n}^E A_i = \{(x_0, ..., x_{n-1}) : (\forall i)(x_i \in A_i) \wedge (\forall i \neq j)(\neg(x_i \; E \; x_j))\}.$$

This set will sometimes be denoted $A_0 \times_E ... \times_E A_{n-1}$.

Definition 2.24. Let E be an equivalence relation on a Polish space X. Let $n \in \omega$. E has the n-weak-Mycielski property if and only if for any $C \subseteq {}^n X$ which is comeager in ${}^n X$, there are $\boldsymbol{\Delta}^1_1$ sets $(A_i : i < n)$ with the property that for each $i < n$, $E \equiv_{\boldsymbol{\Delta}^1_1} E \restriction A_i$ and $\prod_{i<n}^E A_i \subseteq C$.

3. ${}^\omega 2$ Has the Jónsson Property

This section will give a forcing style proof of Holshouser and Jackson's result that ${}^\omega 2$ has the Jónsson property under some determinacy assumptions. The Jónsson property for ${}^\omega 2$ will follow from a flexible fusion argument for Sacks forcing and the fact that under determinacy assumptions, every function is definable (on some perfect set) with certain absoluteness properties between countable structures and the real universe. Continuous functions will satisfy this property, and so the Baire property and the Mycielski property for $=$ can be used to show every function has such a definition on some perfect set. This definability can also be achieved by absoluteness phenomena that occur under AD^+. Later, it will be shown that the Mycielski property fails for all the other simple equivalence relations considered; the hope is that such a definability and absoluteness approach could establish Jónsson type properties without the Mycielski property. In the following, the fusion argument is essential for the combinatorics of the forcing argument. It is unclear what the relation is between fusion (or properness), the Mycielski property, and the Jónsson property.

Definition 3.1. A tree p on 2 is a subset of ${}^{<\omega}2$ so that if $s \in p$ and $t \subseteq s$, then $t \in p$. p is a perfect tree if and only if for all $s \in p$, there is a $t \supseteq s$ so that $t\hat{}0, t\hat{}1 \in p$.

Let $\mathbb{S}$ denote the collection of all perfect trees on 2, $\leq_\mathbb{S} = \subseteq$, and $1_\mathbb{S} = {}^{<\omega}2$. $(\mathbb{S}, \leq_\mathbb{S}, 1_\mathbb{S})$ is Sacks forcing, denoted by just $\mathbb{S}$.

Let $p \in \mathbb{S}$. $s \in p$ is a split node if and only if $s\hat{}0, s\hat{}1 \in p$. $s \in p$ is a split of p if and only if $s \restriction (|s| - 1)$ is a split node of p. For $n \in \omega$, s is a n-split of p if and only if s is a $\subseteq$-minimal element of p with exactly n-many proper initial segments which are split nodes of p.

Let $\text{split}^n(p)$ denote the set of n-splits of p. Note that $|\text{split}^n(p)| = 2^n$ and $\text{split}^0(p) = \{\emptyset\}$.

If $p, q \in \mathbb{S}$, define $p \leq^n_\mathbb{S} q$ if and only if $p \leq_\mathbb{S} q$ and $\text{split}^n(p) = \text{split}^n(q)$.

If $p \in \mathbb{S}$ and $s \in p$, then define $p_s = \{t \in p : t \subseteq s \vee s \subseteq t\}$.

Let $p \in \mathbb{S}$. Let Λ be defined as follows:

(i) $\Lambda(p, \emptyset) = \emptyset$.

(ii) Suppose $\Lambda(p, s)$ has been defined for all $s \in {}^n 2$. Fix an $s \in {}^n 2$ and $i \in 2$. Let $t \supseteq \Lambda(p, s)$ be the minimal split node of p extending $\Lambda(p, s)$. Let $\Lambda(p, s\hat{}i) = t\hat{}i$.

Let $\Xi(p, s) = p_{\Lambda(p,s)}$.

For $n \in \omega$, let $\mathbb{S}^n$ denote the n-fold product of $\mathbb{S}$. If $p \in \mathbb{S}$, then let $p^n \in \mathbb{S}^n$ be defined so that for all $i < n$, $p^n(n) = p$.

Let $n \in \omega$ and $m < n$. There is an $\mathbb{S}^n$-name $x^{n,m}_{\text{gen}}$ which names the m^{th} Sacks-generic real coming from an $\mathbb{S}^n$-generic filter.

Fact 3.2. *A fusion sequence is a sequence $\langle p_n : n \in \omega\rangle$ in $\mathbb{S}$ so that for all $n \in \omega$, $p_{n+1} \leq^n_{\mathbb{S}} p_n$. The fusion of this sequence is $p_\omega = \bigcap_{n\in\omega} p_n$.*

p_ω is a condition in $\mathbb{S}$.

Lemma 3.3. *Let $f : [^\omega 2]^{\leq\omega}_= \to {}^\omega 2$. Let $f_n = f \restriction [^\omega 2]^n_=$. Suppose there is a countable model M of some sufficiently large fragment of ZF, $p \in \mathbb{S} \cap M$, and a $\mathbb{S}^n$-name $\tau_n \in M$ so that $p^n \Vdash \tau_n \in {}^\omega 2$ and whenever $G^n \subseteq \mathbb{S}^n$ is $\mathbb{S}^n$-generic over M with $p^n \in G^n$, $\tau_n[G^n] = f_n(x^{n,0}_{\text{gen}}[G^n], ..., x^{n,n-1}_{\text{gen}}[G^n])$. Then there exists a $q \in \mathbb{S}$ so that $f[[[q]]^{\leq\omega}_=] \neq {}^\omega 2$.*

Proof. For each $n \in \omega$, let $(D^n_m : m \in \omega)$ be a sequence of dense open subsets of $\mathbb{S}^n$ in M so that for all m, $D^n_{m+1} \subseteq D^n_m$ and if D is a dense open subset of $\mathbb{S}^n$ in M, then there is some m so that $D^n_m \subseteq D$.

Let $z \in {}^\omega 2 \setminus ({}^\omega 2)^M$.

A fusion sequence $\langle p_n : n \in \omega\rangle$ with $p_0 = p$ will be constructed with the following properties:

For all $n > 0$, $m \leq n$, and $(\sigma_0, ..., \sigma_{m-1}) \in {}^m({}^n 2)$ so that $\sigma_i \neq \sigma_j$ if $i \neq j$:

(i) $(\Xi(p_n, \sigma_0), ..., \Xi(p_n, \sigma_{m-1})) \in D^m_n$.

(ii) There are some $k \in \omega$ and $i \in 2$ so that $z(k) \neq i$ and $(\Xi(p_n, \sigma_0), ..., \Xi(p_n, \sigma_{m-1})) \Vdash^M_{\mathbb{S}^m} \tau_m(\check{k}) = \check{i}$.

Suppose this fusion sequence $\langle p_n : n \in \omega\rangle$ could be constructed. Let q be its fusion. Fix $m > 0$. Suppose $(x_0, ..., x_{m-1}) \in [[q]]^m_=$. Let $G^m_{(x_0,...,x_{m-1})} = \{(p_0, ..., p_{m-1}) \in \mathbb{S}^m \cap M : (\forall i < m)(x_i \in [p_i])\}$. Note that $G^m_{(x_0,...,x_{m-1})}$ is a $\mathbb{S}^m$ generic filter over M: There is some L so that for all $k \geq L$, there are $\sigma^k_i \in {}^k 2$ with the property that for all $i < m$, $x_i \in \Xi(p_k, \sigma^k_i)$ and for all $i \neq j$, $\sigma^k_i \neq \sigma^k_j$. Then for all $k \geq L$, $(\Xi(p_k, \sigma^k_0), ..., \Xi(p_k, \sigma^k_{m-1})) \in G^m_{(x_0,...,x_{m-1})}$. (i) asserts that this element belongs to D^m_k. Hence $G^m_{(x_0,...,x_{m-1})}$ is $\mathbb{S}^m$-generic over M.

By (ii), $\tau_m[G^m_{(x_0,...,x_{m-1})}] \neq z$. Also

$$\tau_m[G^m_{(x_0,...,x_{m-1})}] = f_m(x^{m,0}_{\text{gen}}[G^m_{(x_0,...,x_{m-1})}], ..., x^{m,m-1}_{\text{gen}}[G^m_{(x_0,...,x_{m-1})}]) = f_m(x_0, ..., x_{m-1}).$$

Hence $z \notin f_m[[[q]]^m_=]$. Thus $f[[[q]]^{\leq\omega}_=] \neq {}^\omega 2$.

The construction of the fusion sequence remains: Let $p_0 = p$.

Suppose p_n has been constructed with the above properties. For some $J \in \omega$, let $(\bar{\sigma}_k : k < J)$ enumerate all tuples of strings $(\sigma_0, ..., \sigma_{m-1})$ where $m \leq n+1$, $\sigma_i \in {}^{n+1}2$, and if $i \neq j$, $\sigma_i \neq \sigma_j$.

Next, one construct a sequence $r_{-1}, ..., r_{J-1}$ as follows: Let $r_{-1} = p_n$. Suppose r_k for $k < J-1$ has been constructed. Suppose $\bar{\sigma}_{k+1} = (\sigma_0, ..., \sigma_{m-1})$.

(Case I) There is some $(u_0, ..., u_{m-1}) \leq_{\mathbb{S}^m} (\Xi(r_k, \sigma_0), ..., \Xi(r_k, \sigma_{m-1}))$ and $c \in ({}^\omega 2)^M$ so that

$$(u_0, ..., u_{m-1}) \Vdash^M_{\mathbb{S}^m} \tau_m = \check{c}.$$

Also as D^m_{n+1} is dense open in $\mathbb{S}^m$, one may choose $(u_0, ..., u_{m-1})$ satisfying the above and $(u_0, ..., u_{m-1}) \in D^m_{n+1}$. Note that since $z \notin M$ and $c \in M$, there must be some $j \in \omega$ and $i \in 2$ so that $c(j) \neq i$ and $z(j) = i$. Now let $r_{k+1} \in \mathbb{S}$ be so that for all $\sigma \in {}^{n+1}2$

$$\Xi(r_{k+1}, \sigma) = \begin{cases} u_i & (\exists i)(0 \leq i \leq m-1)(\sigma = \sigma_i) \\ \Xi(r_k, \sigma) & \text{otherwise} \end{cases}.$$

(Case II) For all $(u_0, ..., u_{m-1}) \leq_{\mathbb{S}^m} (\Xi(r_k, \sigma_0), ..., \Xi(r_k, \sigma_{m-1}))$,

$$(u_0, ..., u_{m-1}) \Vdash^M_{\mathbb{S}^m} \tau_m \notin M.$$

Hence there are $(u_0, ..., u_{m-1}) \leq_{\mathbb{S}^m} (\Xi(r_k, \sigma_0), ..., \Xi(r_k, \sigma_{m-1}))$, $(v_0, ..., v_{m-1}) \leq_{\mathbb{S}^m} (\Xi(r_k, \sigma_0), ..., \Xi(r_k, \sigma_{m-1}))$, and $j \in \omega$ so that

$$(u_0, ..., u_{m-1}) \Vdash^M_{\mathbb{S}^m} \tau_m(\check{j}) = \check{0}$$
$$(v_0, ..., v_{m-1}) \Vdash^m_{\mathbb{S}^m} \tau_m(\check{j}) = \check{1}$$

Without loss of generality, suppose that $z(j) = 1$. Moreover since D^m_{n+1} is dense open, one may assume that $(u_0, ..., u_{m-1}) \in D^m_{n+1}$. Define r_{k+1} in the same way as in Case I.

Finally, let $p_{n+1} = r_{J-1}$. $p_{n+1} \leq^n_{\mathbb{S}} p_n$ and condition (i) and (ii) are satisfied. This completes the construction. $\square$

Fact 3.4. *($\mathsf{ZF} + \mathsf{DC}$) Let $p \in \mathbb{S}$ and $n \in \omega$. If $f_n : [[p]]^n_= \to {}^\omega 2$ is continuous, then there is a countable elementary $M \prec V_\Xi$ (for Ξ some sufficiently large cardinal) and a name τ_n so that M, τ_n, and p satisfy the conditions of Lemma 3.3.*

Proof. If f_n is continuous, then f_n has Σ^1_1 and Π^1_1 formulas with parameters from ${}^\omega 2$ defining it. Let $M \prec V_\Xi$ be a countable elementary substructure containing p and all the parameters used to define f_n. (This requires DC.) Using Mostowski's absoluteness, f_n (as defined by this formula) continues to define a function in the forcing extension $M[G]$, where $G \subseteq \mathbb{S}^n$ is $\mathbb{S}^n$-generic over M. So there is some $\mathbb{S}^n$-name $\tau_n \in M$ so that $p^n \Vdash^M_{\mathbb{S}^n} \tau_n = f_n(x^{n,0}_{\text{gen}}, ..., x^{n,n-1}_{\text{gen}})$. Suppose $G^n \subseteq \mathbb{S}^n$ is $\mathbb{S}^n$-generic over M and contains p^n. Then $M[G^n] \models \tau_n[G^n] = f_n(x^{n,0}_{\text{gen}}[G^n], ..., x^{n,n-1}_{\text{gen}}[G^n])$. Let $\pi : M[G^n] \to N$ be the Mostowski collapse of $M[G^n]$. Since reals are not moved by the Mostowki collapse map π, $\pi(f_n)$ is still defined by the same formula. So

$$N \models \pi(\tau_n[G^n]) = \tau_n[G^n] = \pi(f_n(x^{n,0}_{\text{gen}}[G^n], ..., x^{n,n-1}_{\text{gen}}[G^n])) = f_n(x^{n,0}_{\text{gen}}[G^n], ..., x^{n,n-1}_{\text{gen}}[G^n]).$$

Then applying Mostowski absoluteness, $\tau_n[G^n] = f_n(x^{n,0}_{\text{gen}}[G^n], ..., x^{n,n-1}_{\text{gen}}[G^n])$. $\square$

Theorem 3.5. *(Holshouser-Jackson) Assume $\mathsf{ZF} + \mathsf{DC}$ and all sets of reals have the Baire property. Then ${}^\omega 2$ has the Jónsson property.*

Proof. Let $f : [{}^\omega 2]^{\leq\omega}_= \to {}^\omega 2$. Let $f_n : [\mathbb{R}]^n_= \to \mathbb{R}$ be defined by $f_n = f \restriction [{}^\omega 2]^n_=$. Since all sets of reals have the Baire property, there are comeager subsets $C_n \subseteq {}^n({}^\omega 2)$ so that $f_n \restriction C_n$ is continuous. By the theorem of Mycielski (i.e. $=$ has the Mycielski property), there is a perfect tree p so that $[[p]]^n_= \subseteq C_n$ for all $n \in \omega$. Hence for all $n \in \omega$, $f_n \restriction [[p]]^n_=$ is a continuous function. Then Fact 3.4 and Lemma 3.3 imply that ${}^\omega 2$ has the Jónsson property. $\square$

Remark 3.6. As a consequence of phrasing this argument using forcing, one needed to introduce countable elementary substructures. DC is needed in general to obtain useful countable elementary substructures. A more direct topological argument can be used to avoid DC.

4. ω-Jónsson Function for ${}^\omega 2$

Let $\mathsf{AC}^{\mathbb{R}}_\omega$ be the axiom of countable choice for ${}^\omega 2$: If $\mathcal{E}$ is a countable set of nonempty subsets of ${}^\omega 2$, then $\mathcal{E}$ has a choice function.

Note that $\mathsf{ZF} + \mathsf{AD}$ implies $\mathsf{AC}^{\mathbb{R}}_\omega$.

Using the axiom of choice, every set has an ω-Jónsson function. However, just $\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_\omega$ implies there is a Δ^1_1 classical ω-Jónsson function for ${}^\omega 2$. In fact, a slightly stronger statement holds:

Theorem 4.1. ($\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_\omega$) *There is a Δ^1_1 function $\Phi : \mathscr{P}^\omega({}^\omega 2) \to {}^\omega 2$ so that if $B \subseteq {}^\omega 2$ is uncountable, then $\Phi[\mathscr{P}^\omega(B)] = {}^\omega 2$.*

There is a Δ^1_1 classical ω-Jónsson function for ${}^\omega 2$.

Proof. Let A be a countable subset of ${}^\omega 2$.

Let $a^A_\emptyset$ be the longest element of ${}^{<\omega}2$ which is an initial segment of every element of A.

If a^A_σ is not defined, then $a^A_{\sigma\hat{\ }i}$ is not defined for $i \in 2$. If a^A_σ is defined, let $a^A_{\sigma\hat{\ }i}$ be the longest element of ${}^{<\omega}2$ which is an initial segment of every element of $A \cap N_{a^A_\sigma\hat{\ }i}$, if it exists. Otherwise $a^A_{\sigma\hat{\ }i}$ is undefined. (Note this happens if and only if $A \cap N_{a_\sigma\hat{\ }i}$ is a singleton.)

For $A \in \mathscr{P}^\omega({}^\omega 2)$, let $\Psi(A)$ be the collection of $\sigma \in {}^{<\omega}2$ so that a^A_σ is defined. $\Psi(A)$ is an infinite tree on 2 with possibly dead nodes and is not perfect. Ψ is a Δ^1_1 function.

Using some recursive coding, let X be the collection of reals coding infinite binary trees (which may have dead branches). X is an uncountable Π^0_1 set.

Let $T \in X$. Let $\hat{T} = \{\sigma\hat{\ }\tilde{0}, \sigma\hat{\ }1\hat{\ }\tilde{0} : \sigma \in T\}$. $\hat{T} \in \mathscr{P}^\omega({}^\omega 2)$. One seeks to show that $\Psi(\hat{T}) = T$. To see this, the following claim is helpful: If $\sigma \in T$, then $a^{\hat{T}}_\sigma = \sigma$ and if $\sigma \notin T$, then $a^{\hat{T}}_\sigma$ is undefined.

This claim is proved by induction: $\emptyset \in T$ so $\tilde{0}, 1\hat{\ }\tilde{0} \in \hat{T}$. $a^{\hat{T}}_\emptyset = \emptyset$. Suppose this holds for σ. Suppose $\sigma\hat{\ }i \in T$. Then $\sigma\hat{\ }i\hat{\ }0\hat{\ }\tilde{0}$ and $\sigma\hat{\ }i\hat{\ }1\hat{\ }\tilde{0}$ are both in $\hat{T}$. By induction, $a^{\hat{T}}_\sigma = \sigma$. The longest string which is an initial segment of every element of $\hat{T} \cap N_{a_\sigma^{\hat{T}} \hat{} i} = \hat{T} \cap N_{\sigma \hat{} i}$ is $\sigma \hat{} i$. This shows $a_{\sigma \hat{} i}^{\hat{T}} = \sigma \hat{} i$. Suppose $\sigma \hat{} i \notin T$. Either $a_\sigma^{\hat{T}}$ is undefined or $a_\sigma^{\hat{T}}$ is defined. If $a_\sigma^{\hat{T}}$ is undefined, then $a_{\sigma \hat{} i}^{\hat{T}}$ is undefined. Suppose $a_\sigma^{\hat{T}}$ is defined. By induction, $a_\sigma^{\hat{T}} = \sigma$. $\sigma \in T$ implies that $\sigma \hat{} 0 \hat{} \tilde{0}$ and $\sigma \hat{} 1 \hat{} \tilde{0}$ are both in $\hat{T}$. Since $\sigma \hat{} i \notin T$, $N_{\sigma \hat{} i} \cap \hat{T} = N_{a_\sigma^{\hat{T}} \hat{} i} \cap \hat{T} = \{\sigma \hat{} i \hat{} \tilde{0}\}$. $a_{\sigma \hat{} i}^{\hat{T}}$ is undefined. This completes the proof of the claim.

This shows $\Phi(\hat{T}) = T$. Hence $\Psi[\mathscr{P}^\omega({}^\omega 2)] = X$. Let $\Gamma : X \to {}^\omega 2$ be a Δ_1^1 bijection. Let $\Phi = \Gamma \circ \Psi$.

Let B be an uncountable subset of ${}^\omega 2$. Then there is an uncountable $C \subseteq B$ which has no isolated points. One way to see this is to note that using a countable basis, the Cantor-Bendixson process must stop at a countable ordinal. The fixed point starting from B would be an uncountable set with no isolated points.

Fix such a set C. Let $\mathcal{E} = \{N_\sigma \cap C : \sigma \in {}^{<\omega}2 \wedge N_\sigma \cap C \neq \emptyset\}$. $\mathcal{E}$ is a countable set. Using $\mathsf{AC}_\omega^\mathbb{R}$, let Λ be a choice function for $\mathcal{E}$. Let T be any infinite binary tree on 2.

The following objects will be constructed:

(I) $c_s \in C$ for each $s \in {}^{<\omega}2$.

(II) A strictly increasing sequence $(k_i : i \in \{-1\} \cup \omega)$ of integers.

For each $n \in \omega$, let $A_n = \{c_s : s \in {}^n 2\}$. The objects above will satisfy the following properties:

(i) If $s \in T$, then $a_s^{A_{|s|+1}}$ is defined and has length less than $k_{|s|}$. If $s \notin T$, then $A_{|s|+1} \cap N_{c_s \restriction k_{|s|}} = \{c_s\}$.

(ii) If $s \in T$, then $c_{s \hat{} i} \supseteq a_s^{A_{|s|+1}} \hat{} i$ for each $i \in 2$.

(iii) For all m, if $n > m$, then $\{x \restriction k_m : x \in A_m\} = \{x \restriction k_m : x \in A_n\}$.

Let $c_\emptyset$ be any element of C. Let $k_{-1} = 0$. Let $A_0 = \{c_\emptyset\}$.

Suppose for $m \in \omega$, $c_s \in C$ for all $s \in {}^m 2$ and k_{m-1} have been defined. Suppose properties (i) to (iii) hold for $t \in {}^{<\omega}2$ with $|t| < m$. Let $s \in T \cap {}^m 2$. Since $c_s \in C$ and C has no isolated points, there is some $m_s > k_{m-1}$ so that $N_{(c_s \restriction m_s) \hat{} (1 - c_s(m_s))} \cap C \neq \emptyset$. Let $c_{s \hat{} c_s(m_s)} = c_s$ and let $c_{s \hat{} (1 - c_s(m_s))} = \Lambda(N_{(c_s \restriction m_s) \hat{} (1 - c_s(m_s))})$. If $s \notin T$, then let $c_{s \hat{} i} = c_s$ for each $i \in 2$. Let $k_m = \sup\{m_s + 1 : s \in {}^m 2 \cap T\}$.

Since for each $s \in T \cap {}^m 2$, $m_s > k_{m-1}$, (i) to (iii) still hold for t with $|t| < m$. Let $s \in T$. Using the induction hypothsis for (ii) on $s \restriction m-1$, one has that $c_s \supseteq a_{s \restriction m-1}^{A_m} \hat{} s(m-1)$. $c_{s \hat{} 0}$ and $c_{s \hat{} 1}$ extend $a_{s \restriction m-1}^{A_m} \hat{} s(m-1)$. This shows that $a_s^{A_{m+1}}$ is defined. In fact, $a_s^{A_{m+1}} = c_s \restriction m_s$. If $s \notin T$, (i) is clear from the construction. Properties (i) to (iii) hold for $s \in {}^m 2$.

Let $A = \bigcup_{n \in \omega} A_n$. Note that A is countably infinite and $A \subseteq C \subseteq B$. From the above properties, if $s \in T$, then a_s^A is defined and in fact equal to $a_s^{A_{|s|+1}}$. Suppose $s \notin T$. Let $t \subseteq s$ be maximal with $t \in T$. The above properties imply that $A \cap N_{a_t^A \hat{} s(|t|)} = \{c_{t \hat{} s(|t|)}\} = \{c_s\}$. Hence $a_{t \hat{} s(|t|)}^A$ is not defined and hence a_s^A is not defined. Thus $T = \Psi(A)$. This shows that $\Phi[\mathscr{P}^\omega(B)] = {}^\omega 2$. Φ is an ω-Jónsson function for ${}^\omega 2$. □

Question 4.2. Under $\mathsf{ZF} + \neg\mathsf{AC}_\omega^\mathbb{R}$, can there be a classical ω-Jónsson function for ${}^\omega 2$?

The first statement of Theorem 4.1 may not be true without $\mathrm{AC}_\omega^\mathbb{R}$: Let $\mathbb{C}_\omega$ denote the finite support product of Cohen forcing $\mathbb{C}$. Let $G \subseteq \mathbb{C}_\omega$ be $\mathbb{C}_\omega$-generic over L. For each $n \in \omega$, let c_n be the n^{th}-Cohen generic real naturally added by G. Let $A = \{c_n : n \in \omega\}$. Let $H = (\mathrm{HOD}(A \cup \{A\}))^{L[G]}$. H is called the Cohen-Halpern-Lévy model. In H, A has no countably infinite subsets. Hence the first statement of Theorem 4.1 cannot hold. However A is not in bijection with ${}^\omega 2$. This suggest the following natural question: In H, is there a classical ω-Jónsson function for ${}^\omega 2$?

5. The Structure of E_0

Definition 5.1. E_0 is the equivalence relation defined on ${}^\omega 2$ by $x \; E_0 \; y$ if and only if $(\exists n)(\forall k > n)(x(k) = y(k))$.

Definition 5.2. Let $\{s, v_n^i : i \in 2 \wedge n \in \omega\} \subseteq {}^{<\omega}2$ have the property that for all $n \in \omega$ and $i \in 2$, $\langle i \rangle \subseteq v_n^i$ and $|v_n^0| = |v_n^1|$.

Let $\varphi(\emptyset) = s$. If $\sigma \in {}^{<\omega}2$ and $|\sigma| > 0$, then let $\varphi(\sigma) = s \hat{} v_0^{\sigma(0)} \hat{} ... \hat{} v_{|\sigma|-1}^{\sigma(|\sigma|-1)}$.

A perfect tree p is an E_0-tree if and only if there is a sequence $\{s, v_n^i : i \in 2 \wedge n \in \omega\}$ with the above properties so that p is the $\subseteq$-downward closure of $\{\varphi(\sigma) : \sigma \in {}^{<\omega}2\}$.

Let $\mathbb{P}_{E_0}$ be the collection of all perfect E_0 trees. If $p, q \in \mathbb{P}_{E_0}$, then $p \leq_{\mathbb{P}_{E_0}} q$ if and only if $p \subseteq q$. Let $1_{\mathbb{P}_{E_0}} = {}^{<\omega}2$. $(\mathbb{P}_{E_0}, \leq_{\mathbb{P}_{E_0}}, 1_{\mathbb{P}_{E_0}})$ is forcing with perfect E_0-trees.

If $p \in \mathbb{P}_{E_0}$, then the notation s^p and $v_n^{i,p}$ will be used to denote the strings witnessing p is a perfect E_0-tree.

Let $\Phi : {}^\omega 2 \to [p]$ be defined by $\Phi(x) = \bigcup_{n \in \omega} \varphi(x \upharpoonright n)$, where φ is associated with the E_0-tree p as above. Φ is the canonical homeomorphism of ${}^\omega 2$ onto $[p]$, and Φ is a reduction witnessing $E_0 \leq_{\mathbf{\Delta}_1^1} E_0 \upharpoonright [p]$.

Fact 5.3. *Suppose B is a $\mathbf{\Sigma}_1^1$ set so that $E_0 \leq_{\mathbf{\Delta}_1^1} E_0 \upharpoonright B$. Then there is an E_0-tree p so that $[p] \subseteq B$.*

Proof. This is implicit in [6]. See [20] Lemma 2.3.29 and [12] Theorem 10.8.3. ☐

The weak Mycielski property for E_0 considers E_0-products of $\mathbf{\Delta}_1^1$ sets $A_0, ..., A_{n-1}$ so that $E_0 \equiv_{\mathbf{\Delta}_1^1} E_0 \upharpoonright A_i$ and $[A_i]_{E_0} = [A_j]_{E_0}$. Showing the failure of the weak Mycielski property requires finding some structure shared by all of the sets $A_0, ..., A_{n-1}$. For instance, are there perfect E_0-trees p_i so that $[p_i] \subseteq A_i$ and $[[p_i]]_{E_0} = [[p_j]]_{E_0}$? How similar can p_i and p_j be chosen to be?

A simpler solution using the σ-additivity of the E_0-ideal, which follows Fact 5.3, will be given first. A stronger result giving more information using effective methods will follow.

Fact 5.4. *For each $n \in \omega$, suppose $A_n \subseteq {}^\omega 2$ is $\mathbf{\Sigma}_1^1$ and there is no E_0-tree p so that $[p] \subseteq A_n$. Then there is no E_0 tree p so that $[p] \subseteq \bigcup_{n \in \omega} A_n$.*

Proof. Suppose there is some E_0-tree p so that $[p] \subseteq \bigcup_{n\in\omega} A_n$. Let $\Phi : {}^\omega 2 \to [p]$ be the canonical injective reduction witnessing $E_0 \leq_{\mathbf{\Delta}_1^1} E_0 \upharpoonright [p]$. For each $n \in \omega$, $\Phi^{-1}[A_n]$ is a $\mathbf{\Sigma}_1^1$ set, and ${}^\omega 2 = \bigcup_{n\in\omega} \Phi^{-1}[A_n]$. There is some $m \in \omega$ so that $\Phi^{-1}[A_m]$ is nonmeager. Therefore, there is some continuous injective function $\Psi : {}^\omega 2 \to \Phi^{-1}[A_m]$ which witnesses $E_0 \leq_{\mathbf{\Delta}_1^1} E_0 \upharpoonright \Phi^{-1}[A_m]$. $\Phi \circ \Psi$ witnesses $E_0 \leq_{\mathbf{\Delta}_1^1} E_0 \upharpoonright A_m$. This implies there is some E_0-tree q so that $[q] \subseteq A_m$. Contradiction. ☐

Definition 5.5. If $x \in {}^\omega 2$ and $n \in \omega$, let $x_{\geq n} \in {}^\omega 2$ be defined by $x_{\geq n}(k) = x(n + k)$.

If $A \subseteq {}^\omega 2$, then let $(A)_{\geq n} = \{z : (\exists x \in A)(z = x_{\geq n})\}$.

Definition 5.6. Let $s \in {}^{<\omega} 2$. Define $\mathsf{switch}_s : {}^\omega 2 \to {}^\omega 2$ by

$$\mathsf{switch}_s(x)(n) = \begin{cases} s(n) & n < |s| \\ x(n) & \text{otherwise} \end{cases}.$$

Also if $\sigma \in {}^{<\omega} 2$, $\mathsf{switch}_s(\sigma) \in {}^{|\sigma|} 2$ is defined as above just for $n < |\sigma|$.

Theorem 5.7. *Let $n \in \omega$. For $k < n$, let $A_k \subseteq {}^\omega 2$ be $\mathbf{\Sigma}_1^1$ so that $E_0 \leq_{\mathbf{\Delta}_1^1} E_0 \upharpoonright A_k$ and for all $k < n - 1$, $[A_k]_{E_0} \subseteq [A_{k+1}]_{E_0}$. Then there exists E_0-trees p_k so that $[p_k] \subseteq A_k$ and for all $a, b < n$, $|s^{p_a}| = |s^{p_b}|$, and $v_m^{i,p_a} = v_m^{i,p_b}$ for all $m \in \omega$ and $i \in 2$.*

Proof. For each $c \in \omega$, let

$$E_c = \left\{ (x_0, ..., x_{n-1}) \in \prod_{k<n} A_k : (\forall i, j < n)((x_i)_{\geq c} = (x_j)_{\geq c}) \right\}.$$

For each $c \in \omega$, E_c is a $\mathbf{\Sigma}_1^1$ set. For $k < n$, let $\pi_k : {}^n({}^\omega 2) \to {}^\omega 2$ be the projection map onto the k^{th} coordinate. $\pi_0[E_c]$ is $\mathbf{\Sigma}_1^1$ for each $c \in \omega$. Since $[A_i]_{E_0} \subseteq [A_{i+1}]_{E_0}$, $\bigcup_{c\in\omega} \pi_0[E_c] = A_0$. By Fact 5.4, there is some $m \in \omega$ so that $\pi_0[E_m]$ contains the body of an E_0-tree q_0. By choosing an appropriate subtree, one may assume that $|s^{q_0}| > m$. Let $s_0 = s^{q_0} \upharpoonright m$.

Fix $k < n - 1$. Suppose the E_0-tree q_k and $s_k \in {}^m 2$ have been constructed so that $s_k \subseteq s^{q_k}$ and $[q_k] \subseteq \pi_k[E_m]$. Then

$$\left[\bigcup_{s \in {}^m 2} \mathsf{switch}_s[[q_k]] \cap \pi_{k+1}[E_m] \right]_{E_0} = [[q_k]]_{E_0}.$$

By Fact 5.4, there is some $s_{k+1} \in {}^m 2$ so that $\mathsf{switch}_{s_{k+1}}[[q_k]] \cap \pi_{k+1}[E_m]$ contains an E_0-tree. Let q_{k+1} be such an E_0-tree. Note that $\mathsf{switch}_{s_k}[q_{k+1}]$ is an E_0-subtree of q_k.

For $i < n$, let $p_i = \mathsf{switch}_{s_i}[q_{n-1}]$. Note that $[p_i] \subseteq \pi_i[E_m] \subseteq A_i$. ☐

The rest of this section will prove a result that implies Theorem 5.7 using an effective definability condition. The methods from [12] Theorem 10.8.3 will be used to simultaneously produce E_0-trees, which are very similar to each other, through several sets.

Definition 5.8. Let $z \in {}^\omega 2$. Let $\mathbb{P}_z$ be the forcing of nonempty $\Sigma^1_1(z)$ sets ordered by inclusion with largest element $\mathbb{P}_z = {}^\omega 2$. $\mathbb{P}_z$ is z-Gandy-Harrington forcing.

Fact 5.9. *There is a z-recursive (in a suitable sense) collection $\mathcal{D} = \{D_n : n \in \omega\}$ of dense open subsets of $\mathbb{P}_z$ so that if $G \subseteq \mathbb{P}_z$ is generic for $\mathcal{D}$, then $\bigcap G \neq \emptyset$.*

Proof. See [12] Theorem 2.10.4. □

Fact 5.10. *Let $z \in {}^\omega 2$. There is a $\Pi^1_1(z)$ set $D \subseteq \omega$, $\Sigma^1_1(z)$ set $P \subseteq \omega \times {}^\omega 2$, and $\Pi^1_1(z)$ set $Q \subseteq \omega \times {}^\omega 2$ with the following properties:*

(i) For all $e \in D$, $P^e = Q^e$, where if $X \subseteq \omega \times {}^\omega 2$, then $X^e = \{x \in {}^\omega 2 : (e, x) \in X\}$.

(ii) If $X \subseteq {}^\omega 2$ is $\Delta^1_1(z)$, then there is some $e \in D$ so that $X = P^e = Q^e$.

Definition 5.11. Let $z \in {}^\omega 2$. Let S_z be the union of all $\Delta^1_1(z)$ sets C so that for all $x, y \in C$, $\neg(x\ E_0\ y)$.

Let $H_z = {}^\omega 2 \setminus S_z$.

Fact 5.12. *Let $z \in {}^\omega 2$. S_z is $\Pi^1_1(z)$. H_z is $\Sigma^1_1(z)$. If $X \cap H_z \neq \emptyset$ and X is $\Sigma^1_1(z)$, then there exists $x, y \in X$ with $x \neq y$ and $x\ E_0\ y$. H_z is E_0-saturated.*

Proof. Let D, P, and Q be the sets from Fact 5.10. Note that

$$x \in S_z \Leftrightarrow (\exists e)(e \in D \wedge x \in Q^e \wedge (\forall f, g)((f \neq g \wedge f, g \in P^e) \Rightarrow \neg(f\ E_0\ g))).$$

S_z is $\Pi^1_1(z)$. Hence H_z is $\Sigma^1_1(z)$.

Let $\mathcal{A}$ be the collection of all $\Sigma^1_1(z)$ subsets of ${}^\omega 2$ whose elements are pairwise E_0-inequivalent. Let $U \subseteq \omega \times {}^\omega 2$ be a universal $\Sigma^1_1(z)$ set.

$$\{e : U^e \in \mathcal{A}\} = \{e : (\forall f, g)((f, g \in U^e \wedge f \neq g) \Rightarrow \neg(f\ E_0\ g))\}$$

The above is a $\Pi^1_1(z)$ set. So $\mathcal{A}$ is a collection of $\Sigma^1_1(z)$ sets which is $\Pi^1_1(z)$ in the codes. By $\Sigma^1_1(z)$-reflection (see [12], Theorem 2.7.1), every $\Sigma^1_1(z)$ set X whose elements are E_0-inequivalent has a $\Delta^1_1(z)$ set C whose elements are E_0-inequivalent and $X \subseteq C$.

Suppose X is $\Sigma^1_1(z)$, $X \cap H_z \neq \emptyset$, and the elements of X are pairwise E_0-inequivalent. By the previous paragraph, there is some $\Delta^1_1(z)$ set C which also E_0-inequivalent and $X \subseteq C$. Then $X \subseteq S_z$. Contradiction.

Suppose $x \in H_z$, $y \notin H_z$, and $x\ E_0\ y$. Let $n \in \omega$ be so that $x_{\geq n} = y_{\geq n}$. $y \in S_z$ implies that there is some $\Delta^1_1(z)$ E_0-inequivalent set X so that $y \in X$. $\mathsf{switch}_{x \restriction n}[X]$ is a $\Delta^1_1(z)$ E_0-inequivalent set containing x. This contradicts $x \in H_z$. This shows H_z is E_0-saturated. □

Lemma 5.13. *Let $z \in {}^\omega 2$ and $n, \ell \in \omega$. Suppose $(\hat{B}_i : i < n)$ is a collection of nonempty $\Sigma^1_1(z)$ sets. Suppose for all $i, j < n$, $(\hat{B}_i)_{\geq \ell} = (\hat{B}_j)_{\geq \ell}$. Let $D \subseteq \mathbb{P}_z$ be a dense open subset of the forcing $\mathbb{P}_z$. Then there is a collection $(B_i : i < n)$ of nonempty $\Sigma^1_1(z)$ sets so that for all $i, j < n$, $B_i \in D$, $B_i \subseteq \hat{B}_i$, and $(B_i)_{\geq \ell} = (B_j)_{\geq \ell}$.*

Proof. For $k < n$, $\Sigma^1_1(z)$ sets $\{B^k_i : -1 \leq k < n \wedge 0 \leq i < n\}$ will be constructed with the properties that

(i) For all $i < n$, $B^i_i \in D$.

(ii) If $-1 \leq k < n - 1$ and $0 \leq i < n$, then $B^{k+1}_i \subseteq B^k_i$.

(iii) For all $-1 \leq k < n$ and $0 \leq i, j \leq n$, $(B^k_i)_{\geq \ell} = (B^k_j)_{\geq \ell}$.

Note that this implies that if $k \geq i$, then $B^k_i \in D$.

Let $B^{-1}_i = \hat{B}_i$. (iii) is satisfied.

Suppose for $-1 \leq k < n - 1$ and $0 \leq i < n$, B^k_i has been constructed with the desired properties. Since D is dense open, there is some nonempty $\Sigma^1_1(z)$ set, denoted B^{k+1}_{k+1}, so that $B^{k+1}_{k+1} \subseteq B^k_{k+1}$ and $B^{k+1}_{k+1} \in D$. For $0 \leq i < n$, let $B^{k+1}_i = \{x \in B^k_i : (\exists z)(z \in B^{k+1}_{k+1} \wedge x_{\geq \ell} = z_{\geq \ell})\}$. All the conditions are satisfied.

Finally, let $B_i = B^{n-1}_i$.

□

Theorem 5.14. *Let $z \in {}^\omega 2$ and $n \in \omega$. Let $(A_a : a < n)$ be a collection of $\Sigma^1_1(z)$ sets so that $\bigcap_{a<n} [A_a \cap H_z]_{E_0} \neq \emptyset$. Then there are E_0-trees $(p_a : a \in n)$ so that for all $a, b < n$, $k \in \omega$ and $i < 2$,*

(i) $|s^{p_a}| = |s^{p_b}|$ and $v^{i,p_a}_k = v^{i,p_b}_k$

(ii) $[p_a] \subseteq A_a$.

Proof. The following objects will be constructed: For each $a < n$, $k \in \omega$, $i \in 2$, and $t \in {}^{<\omega}2$,
(a) $w^a_t, s^a, v^i_k \in {}^{<\omega}2$
(b) $\ell_k \in \omega$ and for all $k \in \omega$, $\ell_k \leq m_k < \ell_{k+1}$
(c) $\Sigma^1_1(z)$ nonempty sets X^a_t
with the following properties
(i) For all $a < n$ and $t \in {}^{<\omega}2$, $|w^a_t| = \ell_{|t|}$. For all $a < n$ and $k \in \omega$, $|s^a| = |s^b|$ and $|v^0_k| = |v^1_k|$. For all $k \in \omega$ and $i \in 2$, $\langle i \rangle \subseteq v^k_i$. For all $a < n$ and $t \in {}^{<\omega}2$, if $|t| = 1$, then $s^a \subseteq w^a_t$ and if $t > 1$, then $s^a{}^\frown v_0^{t(0)}{}^\frown ...{}^\frown v_{|t|-2}^{t(|t|-2)} \subseteq w^a_t$.
(ii) If $a \in n$, $t \in {}^{<\omega}2$, $u \in {}^{<\omega}2$, and $t \subseteq u$, then $X^a_u \subseteq X^a_t$, $X^a_t \subseteq N_{w^a_t}$, and $X^a_\emptyset \subseteq A_a \cap H_z$.
(iii) Let $\mathcal{D} = (D_n : n \in \omega)$ be the collection of dense open subset of $\mathbb{P}_z$ from Fact 5.9. For all $t \in {}^{<\omega}2$, $X^a_t \in D_{|t|}$.
(iv) For all $k < \omega$, $\ell_k < \ell_{k+1}$. For all $k < \omega$, $t, u \in {}^k2$, and $a, b \in n$, $(X^a_t)_{\geq \ell_k} = (X^b_u)_{\geq \ell_k}$.

Suppose objects with these properties can be constructed. For $a < n$, let p^a be the E_0-tree given by $s^{p_a} = s^a$ and $v^{i,p_a}_k = v^i_k$. Let $\Phi^a : {}^\omega 2 \to [p_a]$ be the canonical map associated with the E_0-tree p_a. For each $x \in {}^\omega 2$, let G^a_x be the $\mathbb{P}_z$-filter generated by the upward closure of $\{X^a_{x \upharpoonright k} : k \in \omega\}$. $G^a_x \cap D_k \neq \emptyset$ since $X^a_{a \upharpoonright k} \in D_k$. G^a_x is a filter generic for $\mathcal{D}$. By Fact 5.9, $\bigcap G^a_x \neq \emptyset$. By (i) and (ii), $\bigcap G^a_x = \{\Phi^a(x)\}$. Thus $\Phi^a(x) \in X^a_\emptyset \subseteq A_a \cap H_z$. Therefore, $[p_a] \subseteq A_a$.

Next, the construction will be described. Since $\bigcap_{a<n} [A_a \cap H_z]_{E_0} \neq \emptyset$, let $(x_a : a < n)$ be elements of ${}^\omega 2$ so that for all $a, b < n$, $x_a \in A_a \cap H_z$ and $x_a \; E_0 \; x_b$. Choose $\ell_0 \in \omega$ so that for all $a, b < n$, $(x_a)_{\geq \ell_0} = (x_b)_{\geq \ell_0}$. Let $w^a_\emptyset = x_a \upharpoonright \ell_0$. Let $Z = \{r \in {}^\omega 2 : (\exists y_0, ..., y_{n-1})(\bigwedge_{a<n} y_a \in A_a \cap H_z \wedge w^a_\emptyset{}^\frown r = y_a)\}$. Z is a nonempty $\Sigma^1_1(z)$ set. Let $B_a = \{x \in A_a \cap H_z : (\exists r)(r \in Z \wedge x \supseteq w^a_\emptyset \wedge x_{\geq \ell_0} = r)\}$. For each $a < n$, B_a is a nonempty $\Sigma^1_1(z)$ set. Note that for all $a, b < n$, $(B_a)_{\geq \ell_0} = (B_b)_{\geq \ell_0}$. Applying Lemma 5.13, find sets $(X^a_\emptyset : a \in \omega)$ so that $X^a_\emptyset \subseteq B_a$ and $X^a_\emptyset \in D_0$.

Suppose X^a_t has been constructed for $a < n$ and $t \in {}^k2$. $X^0_{0^k} \subseteq H_z$ is nonempty and $\Sigma^1_1(z)$. By Fact 5.12, there are $x, y \in X^0_{0^k}$ so that $x \neq y$ and $x \; E_0 \; y$. By (i) and (ii), $x \upharpoonright \ell_k = y \upharpoonright \ell_k$. Therefore, there is some $\ell_{k+1} > \ell_k$ so that $x_{\geq \ell_{k+1}} = y_{\geq \ell_{k+1}}$. Let m_k be smallest m so that $\ell_k \leq m < \ell_{k+1}$ and $x(m) \neq y(m)$. Without loss of generality, suppose that $x(m_k) = 0$ and $y(m_k) = 1$. For $i \in 2$, let $w^0_{0^k{}^\frown 0} = x \upharpoonright \ell_{k+1}$ and $w^0_{0^k{}^\frown 1} = y \upharpoonright \ell_{k+1}$. For $a \in n$ and $t \in {}^{<\omega}2$, let $w^a_{t^\frown i} = \text{switch}_{w^a_t} w^0_{0^k{}^\frown i}$.

If $k = 0$, then let $s^a = w^a_{\langle 0 \rangle} \upharpoonright m_0$. If $k > 0$, then let $L_k = |s^0| + \sum_{0 \leq j \leq k-1} |v^0_j|$. Let v^i_k be the string of length $m_k - L_k$ defined by $v^i_k(j) = w^0_{0^k{}^\frown i}(L_k + j)$.

Let $Z = \{z \in {}^\omega 2 : (\exists x, y)(x, y \in X^0_{0^n} \wedge w_{0^k{}^\frown 0} \subseteq x \wedge w_{0^k{}^\frown 1} \subseteq y \wedge z = x_{\geq \ell_{k+1}} = y_{\geq \ell_{k+1}})\}$. Z is a nonempty $\Sigma^1_1(z)$. For $t \in {}^{k+1}2$ and $a < n$, let $B^a_t = \{x \in X_{t \upharpoonright k} : (\exists z)(z \in Z \wedge x = w^a_t{}^\frown z)\}$. B^a_t is a nonempty $\Sigma^1_1(z)$ set and $(B^a_t)_{\geq \ell_{k+1}} = (B^b_u)_{\geq \ell_{k+1}}$ for all $a, b \in n$ and $t, u \in {}^{k+1}2$. Apply Lemma 5.13 to get $X^a_t \subseteq B^a_t$ so that $X^a_t \in D_{k+1}$ and $(X^a_t)_{\geq \ell_{k+1}} = (X^b_u)_{\geq \ell_{k+1}}$. This completes the construction. ☐

6. ${}^\omega 2/E_0$ Has the 2-Jónsson Property

Theorem 6.1. *(Holshouser-Jackson) E_0 has the 2-Mycielski property.*

This can be proved by producing an E_0-tree p using an E_0-tree fusion argument to ensure that $[[p]]^2_{E_0}$ meets a fixed countable sequence of dense (topologically) open subsets of ${}^2({}^\omega 2)$. The fusion argument is quite similar to the fusion argument used in the forcing style proof of the 2-Jónsson property for ${}^\omega 2/E_0$ in this section.

By Theorem 5.7, if $(A_a : a < n)$ is a sequence of $\boldsymbol{\Sigma}^1_1$ sets so that $[A_a]_{E_0} = [A_b]_{E_0}$ for all $a, b < n$, then there is a sequence of E_0-trees $(p_a : a \in n)$ so that for all $a, b < n$ and $i < 2$, $|s^{p_a}| = |s^{p_b}|$ and $v^{p_a}_i = v^{p_b}_i$. This motivates the definition of the following forcings:

Definition 6.2. Let $n > 0$, let $\widehat{\mathbb{P}}^n_{E_0}$ be the collection of n-tuples of E_0-trees $(p_0, ..., p_{n-1})$ so that for all $a, b < n$ and $i < 2$, $|s^{p_a}| = |s^{p_b}|$ and $v^{p_a}_i = v^{p_b}_i$. Let $\leq_{\widehat{\mathbb{P}}^n_{E_0}}$ be coordinatewise $\leq_{\mathbb{P}_{E_0}}$. Let $1_{\widehat{\mathbb{P}}^n_{E_0}} = (1_{\mathbb{P}_{E_0}}, ..., 1_{\mathbb{P}_{E_0}})$. $(\widehat{\mathbb{P}}^n_{E_0}, \leq_{\widehat{\mathbb{P}}^n_{E_0}}, 1_{\widehat{\mathbb{P}}^n_{E_0}})$ is forcing with n E_0-trees with the same E_0-saturation.

Let x^0_{gen} and x^1_{gen} be the $\widehat{\mathbb{P}}^2_{E_0}$ names for the left and right generic real added by a generic filter for $\widehat{\mathbb{P}}^2_{E_0}$.

Definition 6.3. If $p, q \in \mathbb{P}_{E_0}$, then let $p \leq^n_{\mathbb{P}_{E_0}} q$ be defined in the same way as $p \leq^n_{\mathbb{S}} q$, when p and q are considered as conditions in Sacks forcing $\mathbb{S}$.

Let $\leq^n_{\widehat{\mathbb{P}}^2_{E_0}}$ be the coordinate-wise ordering using $\leq^n_{\mathbb{P}_{E_0}}$.

A sequence $\langle p_n : n \in \omega\rangle$ of conditions of $\mathbb{P}_{E_0}$ is a fusion sequence if and only if $p_{n+1} \leq^n_{\mathbb{P}_{E_0}} p_n$ for all $n \in \omega$. Similarly, a sequence $\langle (p_n, q_n) : n \in \omega\rangle$ of conditions in $\widehat{\mathbb{P}}^2_{E_0}$ is a fusion sequence if and only if $(p_{n+1}, q_{n+1}) \leq^n_{\widehat{\mathbb{P}}^2_{E_0}} (p_n, q_n)$ for all $n \in \omega$.

Suppose $p \in \mathbb{P}_{E_0}$. Let $n \in \omega$ with $n > 0$. Let $u, v \in {}^n2$ with $u(n-1) \neq v(n-1)$. Suppose $(p', q') \in \widehat{\mathbb{P}}^2_{E_0}$ with the property that $p' \leq_{\mathbb{P}_{E_0}} \Xi(p, u)$ and $q' \leq_{\mathbb{P}_{E_0}} \Xi(p, v)$. Let $A = \{s \in {}^n2 : s(n-1) = u(n-1)\}$ and $B = \{s \in {}^n2 : s(n-1) = v(n-1)\}$. Then define $\mathsf{prune}^{(u,v)}_{(p',q')}(p) \in \mathbb{P}_{E_0}$ by

$$\mathsf{prune}^{(u,v)}_{(p',q')}(p) = \bigcup_{t \in A} \mathsf{switch}_{s^{\Xi(p,t)}}(p') \cup \bigcup_{t \in B} \mathsf{switch}_{s^{\Xi(p,t)}}(q')$$

Suppose $(p, q) \in \widehat{\mathbb{P}}^2_{E_0}$. Let $n \in \omega$, $n > 0$, and $u, v \in {}^n2$ so that $u(n-1) \neq v(n-1)$. Let $(p', q') \leq_{\widehat{\mathbb{P}}^n_{E_0}} (p, q)$ so that $p' \leq_{\mathbb{P}_{E_0}} \Xi(p, u)$ and $q' \leq_{\mathbb{P}_{E_0}} \Xi(q, v)$. Define $2\mathsf{prune}^{(u,v)}_{(p',q')}(p, q)$ by

$$2\mathsf{prune}^{(u,v)}_{(p',q')}(p, q) = \Big(\mathsf{prune}^{(u,v)}_{(p',\mathsf{switch}_{s^p}(q'))}(p), \mathsf{prune}^{(u,v)}_{(\mathsf{switch}_{s^q}(p'),q')}(q)\Big)$$

Perhaps more concretely: Let $A = \{s \in {}^n2 : s(n-1) = u(n-1)\}$ and $B = \{s \in {}^n2 : s(n-1) = v(n-1)\}$.

$$2\mathsf{prune}^{(u,v)}_{(p',q')}(p, q) = \Big(\bigcup_{t \in A} \mathsf{switch}_{s^{\Xi(p,t)}}(p') \cup \bigcup_{t \in B} \mathsf{switch}_{s^{\Xi(p,t)}}(q'), \bigcup_{t \in A} \mathsf{switch}_{s^{\Xi(q,t)}}(p') \cup \bigcup_{t \in B} \mathsf{switch}_{s^{\Xi(q,t)}}(q')\Big)$$

Fact 6.4. *Suppose (p, q), (p', q'), n, u, and v are as in Definition 6.3. Then $2\mathsf{prune}^{(u,v)}_{(p',q')}(p, q) \in \widehat{\mathbb{P}}^2_{E_0}$ and if $2\mathsf{prune}^{(u,v)}_{(p',q')}(p, q) = (x, y)$, then $s^x = s^p$ and $s^y = s^q$.*

If $|u| = n$, then $2\mathsf{prune}^{(u,v)}_{(p',q')}(p, q) \leq^n_{\widehat{\mathbb{P}}^2_{E_0}} (p, q)$.

If $\langle p_n : n \in \omega\rangle$ is a fusion sequence of conditions in $\mathbb{P}_{E_0}$, then $p_\omega = \bigcap_{n\in\omega} p_n$ is a condition in $\mathbb{P}_{E_0}$ and is called the fusion of the fusion sequence.

Similarly, if $\langle (p_n, q_n) : n \in \omega\rangle$ is a fusion sequence of conditions in $\widehat{\mathbb{P}}^2_{E_0}$, then $(p_\omega, q_\omega) = (\bigcap_{n\in\omega} p_n, \bigcap_{n\in\omega} q_n)$ is a condition in $\widehat{\mathbb{P}}^2_{E_0}$ and is called the fusion of the fusion sequence.

Fact 6.5. *([13] Proposition 7.6) $\widehat{\mathbb{P}}^2_{E_0}$ is a proper forcing. In fact, for any countable model M and $(p, q) \in (\widehat{\mathbb{P}}^2_{E_0})^M$, there is a $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p, q)$ which is a $(M, \widehat{\mathbb{P}}^2_{E_0})$-master condition and $[p'] \times_{E_0} [q']$ consists of pairs of reals which are $\widehat{\mathbb{P}}^2_{E_0}$-generic over M.*

Moreover, if τ is a $\widehat{\mathbb{P}}^2_{E_0}$-name in M such that $(p, q) \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \tau \in {}^\omega2$, then one can even find (p', q') with the above properties and so that either
(i) there is some $z \in {}^\omega2 \cap M$ with $z\ E_0\ \tilde{0}$ so that $(p', q') \Vdash_{\widehat{\mathbb{P}}^2_{E_0}} \tau = \check{z}$
or
(ii) $(p', q') \Vdash_{\widehat{\mathbb{P}}^2_{E_0}} \neg(\tau\ E_0\ \tilde{0})$.

Proof. Let $(D_n : n \in \omega)$ be a decreasing sequence of dense open subsets of $\widehat{\mathbb{P}}^2_{E_0}$ in M with the property that if D is a dense open subset of $\widehat{\mathbb{P}}^2_{E_0}$ in M, then there is some $k \in \omega$ so that $D_k \subseteq D$.

There are two cases: (Note that $\tilde{0}$ can be replaced by any real in M in the following argument and hence as well in the statement of the fact.)

(Case I) There is some $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p, q)$ so that $(p', q') \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \tau\ E_0\ \tilde{0}$. Then there is some $z \in ({}^\omega2)^M$ and (p'', q'') so that $(p'', q'') \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \tau = \check{z}$. Since D_0 is dense open in $\widehat{\mathbb{P}}^2_{E_0}$, one may assume $(p'', q'') \in D_0$. Let $(p_0, q_0) = (p'', q'')$.

(Case II) $(p, q) \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \neg(\tau\ E_0\ \tilde{0})$. Let (p', q') be any condition below (p, q) so that $(p', q') \in D_0$. Let $(p_0, q_0) = (p', q')$.

In either case, a fusion sequence $((p_n, q_n) : n \in \omega)$ of conditions in $(\widehat{\mathbb{P}}^2_{E_0})^M$ will be constructed with the following property:
(i) For all $n \in \omega$, $(u, v) \in {}^n2 \times {}^n2$ so that $u(n-1) \neq v(n-1)$, $(\Xi(p_n, u), \Xi(q_n, v)) \in D_n$.

Suppose this can be done to produce a fusion sequence $\langle (p_n, q_n) : n \in \omega \rangle$. Let (p', q') be the fusion of this fusion sequence. First, it will be shown that (i) implies that (p', q') is a $(M, \widehat{\mathbb{P}}^2_{E_0})$-master condition with the property that $[p'] \times_{E_0} [q']$ consists entirely of pairs of reals which are $\widehat{\mathbb{P}}^2_{E_0}$-generic over M.

Let D be a dense open subset of $\widehat{\mathbb{P}}^2_{E_0}$ with $D \in M$. By the choice of $(D_n : n \in \omega)$, there is some $k \in \omega$ so that $D_k \subseteq D$. Let $G \subseteq \widehat{\mathbb{P}}^2_{E_0}$ be $\widehat{\mathbb{P}}^2_{E_0}$-generic over M containing (p', q'). Let $K = |\Lambda(p_k, 0^k)|$. Let E_K be the collection of $(p, q) \in \widehat{\mathbb{P}}^2_{E_0}$ so that $|s^p| > K$ and there is some j with $K < j < |s^p|$ so that $s^p(j) \neq s^q(j)$. E_K is dense open. Since G is generic, $G \cap E_K \neq \emptyset$. As G is generic and $(p', q') \in G$, one may assume that there is some $(p'', q'') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p', q')$ with $(p'', q'') \in G \cap E_k$. So there is some $J > k$ and $u, v \in {}^J2$ with $u(J-1) \neq v(J-1)$ so that $(p'', q'') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (\Xi(p', u), \Xi(q', v)) \leq_{\widehat{\mathbb{P}}^2_{E_0}} (\Xi(p_J, u), \Xi(q_J, v))$. Since $(p'', q'') \in G$ and G is a filter, $(\Xi(p_J, u), \Xi(q_J, v)) \in G$. However $(\Xi(p_J, u), \Xi(q_J, v)) \in D_J \subseteq D_k$ by (i). Note that $(\Xi(p_J, u), \Xi(q_J, v)) \in M$. Since G was an arbitrary generic containing (p', q'), it has been shown that $(p', q') \Vdash^V_{\widehat{\mathbb{P}}^2_{E_0}} \dot{G} \cap \check{M} \cap \check{D} \neq \emptyset$. (p', q') is a $(M, \widehat{\mathbb{P}}^2_{E_0})$-master condition. $\widehat{\mathbb{P}}^2_{E_0}$ is proper.

Now suppose $(a, b) \in [p'] \times_{E_0} [q']$. Let $G_{(a,b)} = \{(p, q) \in \widehat{\mathbb{P}}^2_{E_0} \cap M : (a, b) \in [p] \times [q]\}$. Let D be a dense open set. There is some k so that $D_k \subseteq D$. Since $\neg(a\ E_0\ b)$, there is some $j > k$ and some $u, v \in {}^j2$ with $u(j-1) \neq v(j-1)$ so that $\Lambda(p', u) \subseteq a$ and $\Lambda(q', v) \subseteq b$. Then $(a, b) \in [\Xi(p_j, u)] \times_{E_0} [\Xi(q_j, v)]$. Therefore $(\Xi(p_j, u), \Xi(q_j, v)) \in G_{(a,b)} \cap D_k \subseteq G_{(a,b)} \cap D$. $G_{(a,b)}$ is $\widehat{\mathbb{P}}^2_{E_0}$-generic over M. (a, b) is a $\widehat{\mathbb{P}}^2_{E_0}$-generic pair of reals over M.

It is clear using the forcing theorems that if Case I holds, then statement (i) of the fact holds and if Case II holds, then statement (ii) of the fact holds.

Now it remains to construct the fusion sequence:

(p_0, q_0) is already given depending on the case. The rest of the construction is the same for both cases.

Suppose (p_n, q_n) have been constructed with the desired properties. For some $J \in \omega$, let (u_0, v_0), ..., (u_{J-1}, v_{J-1}) enumerate all $(u, v) \in {}^n2 \times {}^n2$ with $u(n-1) \neq v(n-1)$.

A sequence of conditions in $\widehat{\mathbb{P}}^2_{E_0}$, (x_{-1}, y_{-1}), ..., (x_{J-1}, y_{J-1}) will be constructed:

Let $(x_{-1}, y_{-1}) = (p_n, q_n)$.

Suppose (x_k, y_k) has been constructed for $k < J-1$.

Since D_{n+1} is dense open, find some $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (\Xi(x_k, u_k), \Xi(y_k, v_k))$ so that $(p', q') \in D_{n+1}$. Let $(x_{k+1}, y_{k+1}) = \mathsf{2prune}^{(u_k, v_k)}_{(p', q')}(x_k, y_k)$. ∎

Lemma 6.6. *Let $f : [{}^\omega 2]^2_{E_0} \to {}^\omega 2$. Suppose there is a countable model M of some sufficiently large fragment of* ZF*, $(p, q) \in \widehat{\mathbb{P}}^2_{E_0} \cap M$, and $\tau \in M^{\widehat{\mathbb{P}}^2_{E_0}}$ so that $(p, q) \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \tau \in {}^\omega 2$ and whenever $G \subseteq \widehat{\mathbb{P}}^2_{E_0}$ is $\widehat{\mathbb{P}}^2_{E_0}$-generic over M with $(p, q) \in G$, then $\tau[G] = f(x^0_{\text{gen}}[G], x^1_{\text{gen}}[G])$. Then there is a $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p, q)$ so that $[f[[p'] \times_{E_0} [q']]]_{E_0} \neq {}^\omega 2$.*

Proof. Let $(p', q') \in \widehat{\mathbb{P}}^2_{E_0}$ be given by Fact 6.5. Then exactly one of the two happens:

(i) For all $G \subseteq \widehat{\mathbb{P}}^2_{E_0}$ which are generic over M, $M[G] \models \tau[G]\ E_0\ \tilde{0}$. By absoluteness, $\tau[G]\ E_0\ \tilde{0}$.

(ii) For all $G \subseteq \widehat{\mathbb{P}}^2_{E_0}$ which are generic over M, $M[G] \models \neg(\tau[G]\ E_0\ \tilde{0})$. By absoluteness, $\neg(\tau[G]\ E_0\ \tilde{0})$.

Let $(a, b) \in [p'] \times_{E_0} [q']$. By Fact 6.5, there is some $\widehat{\mathbb{P}}^2_{E_0}$-generic filter $G_{(a,b)}$ so that $x^0_{\text{gen}}[G_{(a,b)}] = a$ and $x^1_{\text{gen}}[G_{(a,b)}] = b$.

If (i) holds, then $f(a, b) = f(x^0_{\text{gen}}[G_{(a,b)}], x^1_{\text{gen}}[G_{(a,b)}]) = \tau[G_{(a,b)}]$ which is E_0 related to $\tilde{0}$. So $[f[[p'] \times_{E_0} [q']]]_{E_0} = [\tilde{0}]_{E_0} \neq {}^\omega 2$.

If (ii) holds, then $f(a, b) = f(x^0_{\text{gen}}[G_{(a,b)}], x^1_{\text{gen}}[G_{(a,b)}]) = \tau[G_{(a,b)}]$, but $\neg(\tau[G_{(a,b)}]\ E_0\ \tilde{0})$. So $\tilde{0} \notin [f[[p'] \times_{E_0} [q']]]_{E_0}$. ∎

Theorem 6.7. *(Holshouser-Jackson)* $(\mathsf{ZF} + \mathsf{DC} + \mathsf{AD})$ *${}^\omega 2/E_0$ has the 2-Jónsson property.*

Proof. Let $F : [{}^\omega 2/E_0]^2_= \to {}^\omega 2/E_0$. Define the relation $R \subseteq {}^\omega 2 \times {}^\omega 2 \times {}^\omega 2$ by

$$R(x, y, z) \Leftrightarrow z \in F([x]_{E_0}, [y]_{E_0})$$

AD can prove comeager uniformization (Fact 2.22): There is a comeager set $C \subseteq [{}^\omega 2]^2_{E_0}$ and a continuous function $f : C \to {}^\omega 2$ which uniformizes R on C. Since E_0 has the 2-Mycielski property, let p be an E_0-tree so that $[[p]]^2_{E_0} \subseteq C$. So $f \restriction [[p]]^2_{E_0}$ is a continuous function. By a similar argument as in Fact 3.4, one can find a name τ satisfying Lemma 6.6 using the condition $(p, p) \in \widehat{\mathbb{P}}^2_{E_0}$. Then Lemma 6.6 gives some $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p, p)$ so that $[f[[p'] \times_{E_0} [q']]]_{E_0}$ is either $[\tilde{0}]_{E_0}$ or does not contain $\tilde{0}$. Since $(p', q') \in \widehat{\mathbb{P}}^2_{E_0}$, $[p']$ and $[q']$ have the same E_0-saturation. Let $A = [[p']]_{E_0} = [[q']]_{E_0}$. Note that $E_0 \equiv_{\mathbf{\Delta}^1_1} E_0 \restriction A$. Let $B = A/E_0$. There is a bijection between B and ${}^\omega 2/E_0$. Moreover, $F[[B]^2_=]$ is either the singleton $\{[\tilde{0}]_{E_0}\}$ or is missing the element $[\tilde{0}]_{E_0}$. In either case, $F[[B]^2_=] \neq {}^\omega 2/E_0$. ☐

7. Partition Properties of ${}^\omega 2/E_0$ in Dimension 2

Definition 7.1. Let X and Y be sets. Let $n \in \omega$. Denote $X \to (X)^n_Y$ to mean that for any function $f : \mathscr{P}^n(X) \to Y$, there is some $Z \subseteq X$ with $Z \approx X$ and $|f[\mathscr{P}^n(Z)]| = 1$.

Denote $X \mapsto (X)^n_Y$ to mean that for any function $f : [X]^n_= \to Y$, there is some $Z \subseteq X$ with $Z \approx X$ and $|f[[Z]^n_=]| = 1$.

Fact 7.2. *(Galvin) Assuming* ZF *and all sets of reals have the Baire property,* ${}^\omega 2 \to ({}^\omega 2)^2_n$ *for all* $n \in \omega$.

Note that ${}^\omega 2 \mapsto ({}^\omega 2)^2_2$ is not true: If $x, y \in {}^\omega 2$ and $x \neq y$, then define $d(x, y) = \min\{n : x(n) \neq y(n)\}$. Define $f : [{}^\omega 2]^2_= \to 2$ by $f(x, y) = x(d(x, y))$. Note that $f(x, y) \neq f(y, x)$. It is impossible to find a homogeneous set for this coloring of $[{}^\omega 2]^2_=$.

However, under AD, ${}^\omega 2/E_0 \mapsto ({}^\omega 2/E_0)^2_n$ does hold:

Lemma 7.3. *Let $n > 1$. Let $F : [{}^\omega 2]^2_{E_0} \to n$ be a function. Suppose there is a countable model M of some sufficiently large fragment of* ZF*, $(p, q) \in \widehat{\mathbb{P}}^2_{E_0} \cap M$, and τ is a $\widehat{\mathbb{P}}^2_{E_0}$-name in M so that $(p, q) \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \tau \in \check{n}$ and whenever $G \subseteq \widehat{\mathbb{P}}^2_{E_0}$ is $\widehat{\mathbb{P}}^2_{E_0}$-generic over M with $(p, q) \in G$, $\tau[G] = F(x^0_{\text{gen}}[G], x^1_{\text{gen}}[G])$. Then there is a $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p, q)$ so that $|F[[p'] \times_{E_0} [q']]| = 1$.*

Proof. Since $(p, q) \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \tau \in \hat{n}$, there is some $(r, s) \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p, q)$ and some $k \in n$ so that $(r, s) \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \tau = \check{k}$. Using Fact 6.5, let $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (r, s)$ be a $(M, \widehat{\mathbb{P}}^2_{E_0})$-master condition so that $[p'] \times_{E_0} [q']$ consists of pairs of reals which are $\widehat{\mathbb{P}}^2_{E_0}$-generic over M. Using the forcing theorem and the assumptions, (p', q') works. ☐

Theorem 7.4. $(\mathsf{ZF} + \mathsf{DC} + \mathsf{AD})$ ${}^\omega 2/E_0 \mapsto ({}^\omega 2/E_0)^2_n$ *for all* $n \in \omega$.

Proof. Let $f : [{}^\omega 2/E_0]^2_= \to n$. Define a relation $R \subseteq {}^\omega 2 \times {}^\omega 2 \times n$ by

$$R(x, y, k) \Leftrightarrow k = f([x]_{E_0}, [y]_{E_0}).$$

AD can prove comeager uniformization (Fact 2.22) so there is some comeager $C \subseteq [{}^\omega 2]^2_{E_0}$ and a continuous function $F : C \to n$ which uniformizes R on C. Since E_0 has the 2-Mycielski property, let p be an E_0-tree so that $[[p]]^2_{E_0} \subseteq C$. $F \restriction [[p]]^2_{E_0}$ is a continuous function. As before, one can find a $\widehat{\mathbb{P}}^2_{E_0}$-name τ satisfying Lemma 7.3 using the the condition (p, p). Then using Lemma 7.3, there is a $(r, s) \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p, p)$ so that $|F[[r] \times_{E_0} [s]]| = 1$. Let k be the unique element in this set.

Let $A = [r]_{E_0} = [s]_{E_0}$. $A/E_0 \approx {}^\omega 2/E_0$. Suppose $x, y \in A/E_0$ and $x \neq y$. There is some $a, b \in A$ with $a \in [r]$, $b \in [s]$, $a \in x$, and $b \in y$. $F(a, b) = k$. Therefore, $R(a, b, k)$. By definition, $f(x, y) = k$. So $|f[[A]^2_=]| = 1$. ☐

Remark 7.5. Before, it was mentioned that ${}^\omega 2 \mapsto ({}^\omega 2)^2_2$ is not true. Using the notation from the above proof: Observe that the function F produced in the above proof is E_0-invariant in the sense that if $x \; E_0 \; x'$ and $y \; E_0 \; y'$ then $F(x, y) = F(x', y')$. Also since $(r, s) \in \widehat{\mathbb{P}}^2_{E_0}$, if $a \in A$, then there is some $a' \in [r]$ and $a'' \in [s]$ with $a \; E_0 \; a' \; E_0 \; a''$. These two facts are essential in proving ${}^\omega 2/E_0 \mapsto ({}^\omega 2/E_0)^2_n$.

Later it will be shown that the partition relation for ${}^\omega 2/E_0$ will fail in higher dimension. The counterexample is closely connected to the failure of the 3-Jónsson property.

8. E_0 Does Not Have the 3-Mycielski Property

An earlier section mentioned that Holshouser and Jackson proved E_0 has the 2-Mycielski property and ${}^{\omega}2/E_0$ has the 2-Jónsson property. The next few sections will show that dimension 2 is the best possible for these combinatorial properties. This section will show the failure of the 3-Mycielski property and the weak 3-Mycielski property for E_0.

Theorem 8.1. *Let $D \subseteq {}^3({}^{\omega}2)$ be defined by*

$$D = \{(x,y,z) \in {}^3({}^{\omega}2) : (\exists n)(x(n) \neq y(n) \wedge x(n) \neq z(n) \wedge y(n+1) \neq z(n+1))\}.$$

D is dense open in ${}^3({}^{\omega}2)$.

If p is an E_0-tree with associated objects $\{s, v_n^i : i \in 2 \wedge n \in \omega\}$, φ, and Φ as in Definition 5.2, then

$$(\Phi(\widetilde{010}), \Phi(\widetilde{110}), \Phi(\tilde{1})) \notin D.$$

E_0 does not have the 3-Mycielski property.

Proof. Suppose $(x,y,z) \in D$. Then there is an $n \in \omega$ so that $x(n) \neq y(n)$, $x(n) \neq z(n)$, and $y(n+1) \neq z(n+1)$. Let $\sigma = x \upharpoonright (n+2)$, $\tau = y \upharpoonright (n+2)$, and $\rho = z \upharpoonright (n+2)$. Then $N_{\sigma,\tau,\rho} \subseteq D$. D is open.

Suppose $\sigma, \tau, \rho \in {}^{<\omega}2$ and $|\sigma| = |\tau| = |\rho| = k$. Let $\sigma' = \sigma\hat{\ }00$, $\tau' = \tau\hat{\ }10$, and $\rho' = \rho\hat{\ }11$. Then $N_{\sigma',\tau',\rho'} \subseteq N_{\sigma,\tau,\rho}$ and $N_{\sigma',\tau',\rho'} \subseteq D$. D is dense open.

Let $L_{-1} = |s|$. For $n \in \omega$, let $L_n = |s| + \sum_{k \leq n} |v_k^0|$. Note that if $x(n) = y(n)$, then for all $L_{n-1} \leq k < L_n$, $\Phi(x)(k) = \Phi(y)(k)$. If $x(n) \neq y(n)$, then $\Phi(x)(L_{n-1}) \neq \Phi(y)(L_{n-1})$, and there may be other $L_{n-1} \leq k < L_n$ so that $\Phi(x)(k) \neq \Phi(y)(k)$.

Suppose $\Phi(\widetilde{010})(n) \neq \Phi(\widetilde{110})(n)$ and $\Phi(\widetilde{010})(n) \neq \Phi(\tilde{1})(n)$. Then there exists some k so that $L_{3k-1} \leq n < L_{3k}$. If $n \neq L_{3k} - 1$, then $n + 1 < L_{3k}$. Since $\widetilde{110}(3k) = 1 = \tilde{1}(3k)$, $\Phi(\widetilde{110})(n+1) = \Phi(\tilde{1})(n+1)$. Hence if $n \neq L_{3k} - 1$, then n cannot be used to witness that $(\Phi(\widetilde{101}), \Phi(\widetilde{110}), \Phi(\tilde{1})) \in D$. Suppose $n = L_{3k} - 1$. Since $\widetilde{110}(3k+1) = 1 = \tilde{1}(3k+1)$, one has that $\Phi(\widetilde{110})(n+1) = \Phi(\widetilde{110})(L_{3k}) = \Phi(\tilde{1})(L_{3k}) = \Phi(\tilde{1})(n+1)$. If $n = L_{3k} - 1$, then n does not witness membership in D. Hence $(\Phi(\widetilde{010}), \Phi(\widetilde{110}), \Phi(\tilde{1})) \notin D$.

Since $\neg(\widetilde{010}\ E_0\ \widetilde{110})$, $\neg(\widetilde{010}\ E_0\ \tilde{1})$, and $\neg(\widetilde{110}\ E_0\ \tilde{1})$, $(\Phi(\widetilde{010}), \Phi(\widetilde{110}), \Phi(\tilde{1})) \in [[p]]^3_{E_0}$. Hence $[[p]]^3_{E_0} \not\subseteq D$.

Suppose B is $\mathbf{\Delta}^1_1$ so that $E_0 \upharpoonright B \equiv_{\mathbf{\Delta}^1_1} E_0$. By Fact 5.3, there is some E_0-tree p so that $[p] \subseteq B$. By the above, $[B]^3_{E_0} \not\subseteq D$. E_0 does not have the 3-Mycielski property. □

The 3-Mycielski property asks for a single $\mathbf{\Delta}^1_1$ set A with $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A$ so that $[A]^3_{E_0} = A \times_{E_0} A \times_{E_0} A$ is contained inside a comeager set. If one is interested in combinatorial properties of the quotient ${}^{\omega}2/E_0$, such as the Jónsson property, then one is only concerned with three sets A, B, and C with the same E_0-saturation. With this consideration, the 3-Mycielski property seems unnecessarily restrictive. The weak 3-Mycielski property was defined to remove this demand.

One other curiosity of the 3-Mycielski property is that Theorem 8.1 allows a (topologically) dense open subset of ${}^3({}^{\omega}2)$ to be a counterexample to the 3-Mycielski property. Let $D \subseteq {}^3({}^{\omega}2)$ be any dense open set. There are three strings σ, τ, and γ of the same length so that $N_{\sigma,\gamma,\tau} \subseteq D$. Let p, q, r be any three perfect E_0-trees so that $s^p = \sigma$, $s^q = \tau$, $s^r = \gamma$, and for all $n \in \omega$ and $i \in 2$, $v_n^{i,p} = v_n^{i,q} = v_n^{i,r}$. Then $[p] \times_{E_0} [q] \times_{E_0} [r] \subseteq D$. Also $[[p]]_{E_0} = [[q]]_{E_0} = [[r]]_{E_0}$. So no dense open set can be a counterexample to the weak 3-Mycielski property.

Using the more informative structure theorem for E_0 proved above and the argument in Theorem 8.1, a comeager subset of ${}^3({}^{\omega}2)$ is used to show E_0 does not have the weak 3-Mycielski property.

Theorem 8.2. *For each $k \in \omega$, let*

$$D_k = \{(x,y,z) \in {}^3({}^{\omega}2) : (\exists n \geq k)(x(n) \neq y(n) \wedge x(n) \neq z(n) \wedge y(n+1) \neq z(n+1))\}.$$

Each D_k is dense open.

Let $C = \bigcap_{n \in \omega} D_n$. C is comeager.

For any $\mathbf{\Delta}^1_1$ sets A_0, A_1, and A_2 such that

(I) $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_0$, $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_1$, $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_2$

(II) $[A_0]_{E_0} = [A_1]_{E_0} = [A_2]_{E_0}$,

$A_0 \times_{E_0} A_1 \times_{E_0} A_2 \not\subseteq C$.

C does not have the weak 3-Mycielski property.

Proof. Let A_0, A_1, A_2 be any three $\mathbf{\Delta}^1_1$ sets so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_i$ and have the same E_0-saturation. By Theorem 5.7, there are E_0-trees, p, q, and r so that

(i) $|s^p| = |s^q| = |s^r| = k$

(ii) $v^{i,p}_n = v^{i,q}_n = v^{i,r}_n$ for all $n \in \omega$ and $i \in 2$

(iii) $[p] \subseteq A_0$, $[q] \subseteq A_1$, and $[r] \subseteq A_2$.

Note that the only differences among the three E_0-trees occurs in the stems. Hence by the same argument as in Theorem 8.1, $[p] \times_{E_0} [q] \times_{E_0} [r] \not\subseteq D_k$. Hence $A_0 \times_{E_0} A_1 \times_{E_0} A_2 \not\subseteq D_k$. So $A_0 \times_{E_0} A_1 \times_{E_0} A_2 \not\subseteq C$. $\square$

9. Surjectivity and Continuity Aspects of E_0

From Holshouser and Jackson's proof of the Jónsson property for ${}^\omega 2$, the Mycielski property was used primarily to show an arbitrary function f on ${}^n({}^\omega 2)$ could be continuous on $[[p]]^n_=$ for some perfect tree p. Using the continuity of $f \upharpoonright [[p]]^n_=$, they show that there is some perfect subtree $q \subseteq p$ so that $f[[[q]]^n_=] \neq {}^\omega 2$.

As the previous section shows that E_0 does not have the 3-Mycielski property, it is natural to ask if by some other means it is possible to find for any $f : {}^3({}^\omega 2) \to {}^\omega 2$, some $\mathbf{\Delta}^1_1$ set A so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A$ and $f \upharpoonright [A]^3_{E_0}$ is continuous. Also if the function $f \upharpoonright [A]^3_{E_0}$ is continuous, is it possible to find some $\mathbf{\Delta}^1_1$ $B \subseteq A$ with $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright B$ so that $f[[B]^3_{E_0}]$ does not meet all E_0 equivalence classes? This section will provide an example to show both of these properties can fail. This example will also be modified in the next section to show the failure of the 3-Jónsson property for E_0.

Fact 9.1. *Let $A = \{x \in {}^\omega 3 : (\forall n)(x(n) \neq x(n+1))\}$. There is a continuous function $P : [{}^\omega 2]^3_{E_0} \to A$ so that for any E_0-tree p, $P[[[p]]^3_{E_0}] = A$. Moreover, if p, q, and r are E_0-trees so that $|s^p| = |s^q| = |s^r|$ and for all $i \in 2$ and $n \in \omega$, $v^{i,p}_n = v^{i,q}_n = v^{i,r}_n$, then $P[[[p]] \times_{E_0} [[q]] \times_{E_0} [[r]]]$ meets all E_0-classes of A, where the latter E_0 is defined on ${}^\omega 3$.*

Proof. Let $(x, y, z) \in [{}^\omega 2]^3_{E_0}$. Let L_0 be the largest $N \in \omega$ so that $x \upharpoonright N = y \upharpoonright N = z \upharpoonright N$. Define

$$a_0 = \begin{cases} 0 & x(L_0) = y(L_0) \\ 1 & x(L_0) = z(L_0) \\ 2 & y(L_0) = z(L_0) \end{cases}.$$

Suppose L_n and a_n have been defined. Let L_{n+1} be the smallest $N > L_n$ so that $x(N) \neq y(N)$ if $a_n = 0$, $x(N) \neq z(N)$ if $a_n = 1$, and $y(N) \neq z(N)$ if $a_n = 2$. Define

$$a_{n+1} = \begin{cases} 0 & x(L_{n+1}) = y(L_{n+1}) \\ 1 & x(L_{n+1}) = z(L_{n+1}) \\ 2 & y(L_{n+1}) = z(L_{n+1}) \end{cases}.$$

Define $P(x, y, z) \in A$ by $P(x, y, z)(n) = a_n$. P is continuous.

Now let p be an E_0 tree. Let s and v^i_n, for $n \in \omega$ and $i \in 2$, be associated with the E_0-tree p. Let $\Phi : {}^\omega 2 \to [p]$ be the canonical homeomorphism.

Let $v \in A$. Let

$$(a_i, b_i) = \begin{cases} (0,1) & v(i) = 0 \\ (1,0) & v(i) = 1 \\ (1,1) & v(i) = 2 \end{cases}.$$

Let $a, b \in {}^\omega 2$ be defined by $a(n) = a_n$ and $b(n) = b_n$. Then $(\Phi(\tilde{0}), \Phi(a), \Phi(b)) \in [[p]]^3_{E_0}$ and $P((\Phi(\tilde{0}), \Phi(a), \Phi(b)))$ $= v$. Hence $P[[[p]]^3_{E_0}] = A$. The second statement is proved similarly after noting the three E_0-trees are the same after their stems. $\square$

Theorem 9.2. *There is a continuous $Q : [{}^\omega 2]^3_{E_0} \to {}^\omega 2$ so that for any E_0-tree p, $Q[[[p]]^3_{E_0}] = {}^\omega 2$.*

Proof. Let $t_0 = 00$, $t_1 = 01$, and $t_2 = 10$.

Let $Q' : {}^\omega 3 \to {}^\omega 2$ be defined by $Q'(x) = t_{x(0)} \hat{} t_{x(1)} \hat{} \ldots$. Q' is a continuous injection.

$Q'[A]$ is a perfect subset of ${}^\omega 2$. $Q'[A] = [T]$ for some perfect tree T. Let $Q'' : Q'[A] \to {}^\omega 2$ be the continuous bijection naturally induced by T.

Let $Q = Q'' \circ Q' \circ P$. Q has the desired property. $\square$

Corollary 9.3. *There is a $\mathbf{\Delta}^1_1$ function $K : {}^3({}^\omega 2) \to {}^\omega 2$ so that on any $\mathbf{\Sigma}^1_1$ set A so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A$, $K[[A]^3_{E_0}] = {}^\omega 2$. Moreover, for any $\mathbf{\Sigma}^1_1$ sets A_0, A_1, and A_2 so that for all $i < 3$, $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_i$ and $[A_0]_{E_0} = [A_1]_{E_0} = [A_2]_{E_0}$, $[K[\prod^{E_0}_{i<3} A_i]]_{E_0} = {}^\omega 2$.*

Fact 9.4. *There is a $\mathbf{\Delta}^1_1$ function $P' : {}^3({}^\omega 2) \to A$ so that for all E_0-tree p, $P' \upharpoonright [[p]]^3_{E_0}$ is not continuous.*

Proof. Let P be the function from Fact 9.1. Define P' by

$$P'(x, y, z) = \begin{cases} \widetilde{01} & (x, y, z) \notin [{}^\omega 2]^3_{E_0} \\ P(x, y, z) & (\forall k)(\exists n > k)(P(x, y, z)(n) = 2) \\ \widetilde{01} & (\exists k)(\forall n > k)(P(x, y, z)(n) < 2) \end{cases}.$$

Suppose P' is continuous on some $[[p]]^3_{E_0}$. Let $s \in {}^{<\omega}2$ be so that for all $n < |s| - 1$, $s(n) \neq s(n + 1)$ and there exists some $n < |s|$ so that $s(n) = 2$. By continuity, $(P')^{-1}[N_s] \cap [[p]]^3_{E_0}$ is open in $[[p]]^3_{E_0}$. There is some $u, v, w \in {}^{<\omega}2$ so that $|u| = |v| = |w|$ and $N_{\varphi(u),\varphi(v),\varphi(w)} \cap [[p]]^3_{E_0} \subseteq (P')^{-1}[N_s] \cap [[p]]^3_{E_0}$.

Let $x = u\hat{}\tilde{0}$, $y = v\hat{}\widetilde{01}$, and $z = w\hat{}\widetilde{10}$. Then $(\Phi(x), \Phi(y), \Phi(z)) \in (P')^{-1}[N_s] \cap [[p]]^3_{E_0}$. However, there is a k so that for all $n > k$, $P(\Phi(x), \Phi(y), \Phi(z))(n) < 2$. Therefore, $P'(\Phi(x), \Phi(y), \Phi(z)) = \widetilde{01}$. However, $\widetilde{01} \notin N_s$ since there is some n so that $s(n) = 2$. ☐

Theorem 9.5. *There is a $\mathbf{\Delta}^1_1$ function $K : {}^3({}^\omega 2) \to {}^\omega 2$ so that for all $\mathbf{\Sigma}^1_1$ sets A so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A$, $K \upharpoonright [A]^3_{E_0}$ is not continuous.*

Proof. Let Q' be the function from the proof of Theorem 9.2. Let P' be the function from Fact 9.4. $K = Q' \circ P'$ works. ☐

As a consequence, one has another proof of the failure of the 3-Mycielski property for E_0.

Corollary 9.6. *E_0 does not have the 3-Mycielski property.*

Proof. Let $C \subseteq {}^3({}^\omega 2)$ be any comeager set so that $K \upharpoonright C$ is a continuous function, where K is from Fact 9.5. Then C witnesses the failure of the 3-Mycielski property for E_0. ☐

10. ${}^\omega 2/E_0$ Does Not Have the 3-Jónsson Property

Definition 10.1. For $n \in \omega$, let E^n_{tail} be the equivalence relation defined on ${}^\omega n$ by $x \ E^n_{\text{tail}} \ y$ if and only if $(\exists r)(\exists s)(\forall a)(x(r + a) = y(s + a))$.

Fact 10.2. *The function $P : [{}^\omega 2]^3_{=} \to A$ from Fact 9.1 is E_0 to E^3_{tail} invariant, which means for all $(x, y, z), (x', y', z') \in [{}^\omega 2]^3_{E_0}$ such that $x \ E_0 \ x'$, $y \ E_0 \ y'$, and $z \ E_0 \ z'$, $P(x, y, z) \ E^3_{\text{tail}} \ P(x', y', z')$.*

Proof. Using the notation from Fact 9.1, let $(L_k : k \in \omega)$ and $(a_k : k \in \omega)$ be the L and a sequences for (x, y, z) and let $(J_k : k \in \omega)$ and $(b_k : k \in \omega)$ be the L and a sequences for (x', y', z').

Let $M \in \omega$ be so that $x_{\geq M} = x'_{\geq M}$, $y_{\geq M} = y'_{\geq M}$, and $z_{\geq M} = z'_{\geq M}$. Let $r \in \omega$ be largest so that $L_r < M$, and let $s \in \omega$ be largest so that $J_s < M$.

(Case I) Suppose $L_{r+1} = J_{s+1}$. Then $a_{r+1} = b_{s+1}$. Since for all $n \geq 1$, $L_{r+n}, J_{s+n} \geq M$, $L_{r+n} = J_{s+n}$ and $a_{r+n} = b_{s+n}$. Hence $P(x, y, z) \ E^3_{\text{tail}} \ P(x', y', z')$.

(Case II) Suppose $a_r = b_s$. Then one must have for all $n \in \omega$, $L_{r+n} = J_{s+n}$ and $a_{r+n} = b_{s+n}$. Hence $P(x, y, z) \ E^3_{\text{tail}} \ P(x', y', z')$.

(Case III) Suppose $a_r \neq b_s$ and $L_{r+1} \neq J_{s+1}$. Without loss of generality, $L_{r+1} < J_{s+1}$, $a_r = 0$, and $b_s = 2$. This implies that for all $M \leq k < L_{r+1}$, $x(k) = x'(k) = y(k) = y'(k) = z(k) = z'(k)$. However, $x(L_{r+1}) \neq y(L_{r+1})$ and $y(L_{r+1}) = y'(L_{r+1}) = z'(L_{r+1}) = z(L_{r+1})$ because $L_{r+1} < J_{s+1}$. Hence $a_{r+1} = 2$. For any k so that $L_{r+1} \leq k < J_{s+1}$, $y(k) = y'(k) = z(k) = z'(k)$. However, $y'(J_{s+1}) \neq z'(J_{s+1})$ hence $y(J_{s+1}) \neq z(J_{s+1})$ and J_{s+1} is the smallest $N > L_{r+1}$ for which this happens. Hence $L_{r+2} = J_{s+1}$. Also $a_{r+2} = b_{s+1}$. Hence for all $n \in \omega$, $a_{(r+2)+n} = b_{(s+1)+n}$. This implies $P(x, y, z) \ E^3_{\text{tail}} \ P(x', y', z')$. ☐

Fact 10.3. *$E^2_{\text{tail}} \leq_{\mathbf{\Delta}^1_1} E^3_{\text{tail}} \upharpoonright A$. Hence $E_0 \equiv_{\mathbf{\Delta}^1_1} E^3_{\text{tail}} \upharpoonright A$.*

Proof. Let $\Phi : {}^\omega 2 \to A$ be defined by $\Phi(x) = x \oplus \tilde{2}$, where

$$(x \oplus y)(n) = \begin{cases} x(k) & n = 2k \\ y(k) & n = 2k+1 \end{cases}$$

Suppose $x\ E^2_{\text{tail}}\ y$. Then there are some $a, b \in \omega$ so that for all n, $x(a+n) = y(b+n)$. For all $n \in \omega$, $\Phi(x)(2a+n) = \Phi(y)(2b+n)$.

Suppose $\neg(x\ E^2_{\text{tail}}\ y)$. Let $a, b \in \omega$. Suppose a is even and b is odd. Then $\Phi(x)(a+0) \in 2$ but $\Phi(y)(b+0) = 2$. The same argument works if a is odd and b is even. Suppose a and b are both even. Let $a = 2a'$ and $b = 2b'$. Since $\neg(x\ E^2_{\text{tail}}\ y)$, there is some k so that $x(a'+k) \neq y(b'+k)$. Then $\Phi(x)(a+2k) = x(a'+k) \neq y(b'+k) = \Phi(y)(b+2k)$. Suppose a and b are both odd. $a = 2a'+1$ and $b = 2b'+1$. Since $\neg(x\ E^2_{\text{tail}}\ y)$, there is some k so that $x((a'+1)+k) \neq y((b'+1)+k)$. Hence $\Phi(x)(a+(1+2k)) \neq \Phi(y)(b+(1+2k))$. This shows $\neg(\Phi(x)\ E^3_{\text{tail}}\ \Phi(y))$.

Since $E^2_{\text{tail}} \equiv_{\mathbf{\Delta}^1_1} E_0$ and $E^3_{\text{tail}} \upharpoonright A \leq_{\mathbf{\Delta}^1_1} E^3_{\text{tail}} \equiv_{\mathbf{\Delta}^1_1} E_0$, one has $E_0 \equiv_{\mathbf{\Delta}^1_1} E^3_{\text{tail}} \upharpoonright A$. ☐

Theorem 10.4. $(\mathsf{ZF} + \mathsf{AD})$ ${}^\omega 2/E_0$ *does not have the 3-Jónsson property.*

Proof. Let P be the function from Fact 10.2. Let $\bar{P} : [{}^\omega 2/E_0]^3_= \to A/E^3_{\text{tail}}$ be defined by $\bar{P}(a,b,c) = d$ if and only if

$$(\forall x, y, z)((x \in a \wedge y \in b \wedge z \in c) \Rightarrow P(x,y,z) \in d).$$

By Fact 10.2, $\bar{P}$ is a well defined function. Since $E_0 \equiv_{\mathbf{\Delta}^1_1} E^3_{\text{tail}} \upharpoonright A$ by Fact 10.3, let $U : A/E^3_{\text{tail}} \to {}^\omega 2/E_0$ be a bijection (given by Fact 2.5).

Let $F : [{}^\omega 2/E_0]^3_= \to {}^\omega 2/E_0$ be defined by $U \circ \bar{P}$.

Let $X \subseteq {}^\omega 2/E_0$ be such that there is a bijection $B : {}^\omega 2/E_0 \to X$. By Fact 2.22, AD implies B has a lift $B' : D \to \bigcup X$, where $D \subseteq {}^\omega 2$ is some comeager set. Since B was a bijection, B' is a reduction of $E_0 \upharpoonright D$ to $E_0 \upharpoonright \bigcup X$. Using AD, there is a comeager set $C \subseteq D$ so that $B' \upharpoonright C : C \to \bigcup X$ is a continuous reduction of $E_0 \upharpoonright C$ to $E_0 \upharpoonright \bigcup X$. There is a continuous function witnessing $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright C$. By composition, there is a continuous reduction witnessing $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright B'[C]$. There is an E_0-tree p so that $[p] \subseteq B'[C] \subseteq \bigcup X$. By Fact 9.1, $P[[[p]]^3_{E_0}] = A$. This implies that $\bar{P}[[X]^3_=] = A/E^3_{\text{tail}}$. Since U is a bijection, $U[\bar{P}[[X]^3_=]] = F[[X]^3_=] = {}^\omega 2/E_0$. F witnesses ${}^\omega 2/E_0$ does not have the 3-Jónsson property. ☐

As mentioned earlier, since this paper is often concerned with sets without well-orderings, the Jónsson property is defined using sets of tuples $[A]^n_=$. The usual definition of the Jónsson property (of cardinals) involve the n-elements subsets of A, $\mathscr{P}^n(A)$. This paper calls this the classical n-Jónsson property. With a slight modification, one can also obtain the failure of the classical 3-Jónsson property for ${}^\omega 2/E_0$.

Definition 10.5. Let S_3 be the permutation group on $3 = \{0,1,2\}$. S_3 acts on ${}^\omega 3$ in the natural way: if $p \in S_3$ and $x \in {}^\omega 3$, then $(p \cdot x)(n) = p(x(n))$.

Let F be an equivalence relation on ${}^\omega 3$ defined by $x\ F\ y$ if and only if $(\exists p \in S_3)(p \cdot x\ E^3_{\text{tail}}\ y)$.

Fact 10.6. *Let* $A = \{x \in {}^\omega 3 : (\forall n)(x(n) \neq x(n+1))\}$. $F \upharpoonright A \equiv_{\mathbf{\Delta}^1_1} E_0$.

Proof. Note that $E^3_{\text{tail}} \upharpoonright A$ is hyperfinite by Fact 10.3. Note that $E^3_{\text{tail}} \upharpoonright A \subseteq F \upharpoonright A$ and each $F \upharpoonright A$ equivalence class is a union of at most six $E^3_{\text{tail}} \upharpoonright A$ equivalence classes. By a result of Jackson, $F \upharpoonright A$ is hyperfinite. Hence, $F \upharpoonright A \leq_{\mathbf{\Delta}^1_1} E_0$.

Next a reduction $\Phi : {}^\omega 2 \to A$ will be produced witnessing $E^2_{\text{tail}} \leq_{\mathbf{\Delta}^1_1} F \upharpoonright A$:

$$\Phi(x) = x(0)\hat{\ }2012102\hat{\ }x(1)\hat{\ }2012102\hat{\ }x(2)...$$

If $x\ E^2_{\text{tail}}\ y$, then $\Phi(x)\ F\ \Phi(y)$.

Suppose $\Phi(x)\ F\ \Phi(y)$. This means there is some $g \in S_3$ so that $g \cdot \Phi(x)\ E^3_{\text{tail}}\ \Phi(y)$. Consider what happens for each $g \in S_3$: g will be presented in cycle notation.

$g = \text{id}$: It is clear that $g \cdot \Phi(x)\ E^3_{\text{tail}}\ \Phi(y)$ implies that $x\ E^2_{\text{tail}}\ y$.

$g = (0,1)$: Then a portion of $g \cdot \Phi(x)$ looks like

$$...\hat{\ }g(x(i))\hat{\ }2102012\hat{\ }g(x(i+1))\hat{\ }2102012\hat{\ }g(x(i+2))\hat{\ }...$$

$g = (0,2)$:

$$...\hat{\ }g(x(i))\hat{\ }0210120\hat{\ }g(x(i+1))\hat{\ }0210120\hat{\ }g(x(i+2))\hat{\ }...$$

$g = (0,1,2)$:

$$...\hat{\ } g(x(i))\hat{\ }0120210\hat{\ }g(x(i+1))\hat{\ }0120210\hat{\ }g(x(i+2))\hat{\ }...$$

$g = (0,2,1)$:

$$...\hat{\ } g(x(i))\hat{\ }1201021\hat{\ }g(x(i+1))\hat{\ }1201021\hat{\ }g(x(i+2))\hat{\ }...$$

In all these cases, $\Phi(y)$ will contain a block of 2012102, but $g \cdot \Phi(x)$ can not possibly contain such a block. So it is impossible that $g \cdot \Phi(x)\ E^3_{\text{tail}}\ \Phi(y)$.

$g = (1,2)$:

$$...\hat{\ } g(x(i))\hat{\ }1021201\hat{\ }g(x(i+1))\hat{\ }1021201\hat{\ }g(x(i+2))\hat{\ }...$$

The only way that some tail of $g \cdot \Phi(x)$ contains blocks of 2012102 is if $x\ E^2_{\text{tail}}\ \tilde{1}$. This however forces $g \cdot \Phi(x)\ E^3_{\text{tail}}\ \Phi(\tilde{1})$. This implies that both $x\ E^2_{\text{tail}}\ \tilde{1}$ and $y\ E^2_{\text{tail}}\ \tilde{1}$. $x\ E^2_{\text{tail}}\ y$.

This shows that Φ is a reduction of E^2_{tail} into $F \restriction A$.

This completes the proof that $E_0 \equiv_{\mathbf{\Delta}^1_1} F \restriction A$. □

Theorem 10.7. ($\mathsf{ZF} + \mathsf{AD}$) *${}^\omega 2/E_0$ does not have the classical 3-Jónsson property.*

Proof. Let $P : [{}^\omega 2]^3_{E_0} \to A$ be the function from Fact 9.1. Fact 10.2 shows that P is E_0 to E^3_{tail} invariant. Note that if $(x_0, x_1, x_2) \in [{}^\omega 2]^3_{E_0}$ and $g \in S_3$, then there is some other $h \in S_3$ so that $P(x_{g(0)}, x_{g(1)}, x_{g(2)}) = h \cdot P(x_0, x_1, x_2)$.

Define a function $\Psi : \mathscr{P}^3({}^\omega 2/E_0) \to A/F$ as follows: Let $D \in \mathscr{P}^3({}^\omega 2/E_0)$. Choose any $(x_0, x_1, x_2) \in [{}^\omega 2]^3_{E_0}$ so that $D = \{[x_0]_{E_0}, [x_1]_{E_0}, [x_2]_{E_0}\}$. Let $\Psi(A) = [P(x_0, x_1, x_2)]_F$.

By the above observations, Ψ is a well-defined surjection onto A/F. By Fact 10.6, there is a bijection $\Gamma : A/F \to {}^\omega 2/E_0$. Let $\Phi = \Gamma \circ \Psi$. By an argument similar to Theorem 10.4, Φ witnesses ${}^\omega 2/E_0$ does not have the classical 3-Jónsson property. □

11. Failure of Partition Properties of ${}^\omega 2/E_0$ in Dimension Higher Than 2

This section will use the failure of the classical 3-Jónsson property to show that the classical partition property in dimension three fails for $\mathbb{R}/E_0$. Note that for any Y, the failure of ${}^\omega 2/E_0 \to ({}^\omega 2/E_0)^3_Y$ implies the failure of ${}^\omega 2/E_0 \mapsto ({}^\omega 2/E_0)^3_Y$.

Theorem 11.1. ($\mathsf{ZF} + \mathsf{AD}$) *For any set Y with at least two elements, ${}^\omega 2/E_0 \to ({}^\omega 2/E_0)^3_Y$ fails.*

In fact, if Y is a set so that there is a partition of ${}^\omega 2/E_0$ by nonempty sets indexed by elements of Y, then there is map $f : \mathscr{P}^3({}^\omega 2\ /E_0) \to Y$ with the property that for all $C \subseteq {}^\omega 2/E_0$ with $C \approx {}^\omega 2/E_0$, $f[\mathscr{P}^3(C)] \approx Y$.

Proof. Let $a, b \in Y$. Partition ${}^\omega 2/E_0$ into two nonempty disjoint sets A and B. Let $\Lambda : {}^\omega 2/E_0 \to Y$ be defined by

$$\Lambda(x) = \begin{cases} a & x \in A \\ b & x \in B \end{cases}$$

Let Φ be the classical 3-Jónsson function from the proof of Theorem 10.7.

Define $f : \mathscr{P}^3({}^\omega 2/E_0) \to Y$ by $f = \Lambda \circ \Phi$.

Suppose $C \subseteq {}^\omega 2/E_0$ and $C \approx {}^\omega 2/E_0$. Suppose $a_0 \in A$ and $b_0 \in B$. Since Φ is a classical 3-Jónsson map, there are some $R, S \in \mathscr{P}^3(C)$ so that $\Phi(R) = a_0$ and $\Phi(S) = b_0$. Since $f(R) = a$ and $f(S) = b$. $|f[\mathscr{P}^3(C)]| = 2$.

For the second statement, suppose $(A_y : y \in Y)$ is a partition of ${}^\omega 2/E_0$ into nonempty sets. Define $\Lambda : {}^\omega 2/E_0 \to Y$ by $\Lambda(x) = y$ if and only if $x \in A_y$. Let $f = \Lambda \circ \Phi$. This maps works by an argument like above. □

Given a set X and $n \in \omega$, one can define $d_X(n)$ to be the smallest element of ω, if it exists, such that for every k and every function $f : \mathscr{P}^n(X) \to k$, there is some $S \subseteq k$ with $|S| \leq d_X(n)$ and $A \subseteq X$ with $A \approx X$ so that $f[\mathscr{P}^n(A)] \subseteq S$. Say that $d_X(n)$ is infinite if no such integer can be found.

[1] showed that assuming the appropriate sets have the Baire property, for every $n, k \in \omega$ and function $f : \mathscr{P}^n({}^\omega 2) \to k$, there is an $S \subseteq k$ with $|S| \leq (n-1)!$ and a set $A \subseteq {}^\omega 2$ with $A \approx {}^\omega 2$ so that $f[\mathscr{P}^n(A)] \subseteq S$. Hence for $n > 0$, $d_{{}^\omega 2}(n) \leq (n-1)!$ assuming AD.

Under AD^+, $d_{{}^\omega 2/E_0}(2)$ is finite and equal to 1, but for $n \geq 3$, $d_{{}^\omega 2/E_0}(n)$ is infinite.

12. $\widehat{\mathbb{P}}^3_{E_0}$ Is Proper

Fact 6.5 shows that $\widehat{\mathbb{P}}^2_{E_0}$ is proper by having a very flexible fusion argument. Moreover, below any condition $(p,q) \in \widehat{\mathbb{P}}^2_{E_0}$ and countable elementary submodels M, one can find a $(M, \widehat{\mathbb{P}}^2_{E_0})$-master condition (p',q') so that every element of $[p'] \times_{E_0} [q']$ is a $\widehat{\mathbb{P}}^2_{E_0}$-generic real over M. This fusion argument for $\widehat{\mathbb{P}}^2_{E_0}$ is also used to prove numerous combinatorial properties in dimension 2. The analog of most of these properties in dimension 3 fails. No fusion with the type of property that $\widehat{\mathbb{P}}^2_{E_0}$ has can exist for $\widehat{\mathbb{P}}^3_{E_0}$. The natural question to ask would be whether $\widehat{\mathbb{P}}^3_{E_0}$ is proper at all.

This section will show that $\widehat{\mathbb{P}}^3_{E_0}$ is proper via a fusion argument. However, one loses control of when exactly dense sets are met.

Definition 12.1. Suppose $(p,q,r) \in \widehat{\mathbb{P}}^3_{E_0}$. Let (u,v,z) be a triple of strings in ${}^{<\omega}2$ of the same length $n+1$ so that $\{0,1\} = \{u(n), v(n), z(n)\}$. Suppose $(p',q',r') \leq_{\widehat{\mathbb{P}}^3_{E_0}} (\Xi(p,u), \Xi(q,v), \Xi(r,z))$. Let $\mathsf{3prune}^{(u,v,z)}_{(p',q',r')}(p,q,r)$ be the unique condition $(a,b,c) \leq^{n+1}_{\widehat{\mathbb{P}}^3_{E_0}} (p,q,r)$ so that $\Xi(a,u) = p'$, $\Xi(b,v) = q'$ and $\Xi(c,z) = r'$.

In the above, the relation $\leq^n_{\widehat{\mathbb{P}}^3_{E_0}}$ is defined as coordinate-wise $\leq^n_{\mathbb{P}_{E_0}}$. $\mathsf{3prune}^{(u,v,z)}_{(p',q',r')}(p,q,r)$ has an explicit definition that is obtained by copying p', q' and r' below the appriopriate part of (p,q,r) like in Definition 6.3.

Fact 12.2. *(With Zapletal) $\widehat{\mathbb{P}}^3_{E_0}$ is a proper forcing.*

Proof. Let $(p,q,r) \in \widehat{\mathbb{P}}^3_{E_0}$. Let Ξ be some large regular cardinal. Let $M \prec V_\Xi$ be a countable elementary substructure containing (p,q,r). Let $(D_n : n \in \omega)$ enumerate all the dense open subsets of $\widehat{\mathbb{P}}^3_{E_0}$ that belong to M. One may assume that $D_{n+1} \subseteq D_n$ for all $n \in \omega$.

If there exists some condition $(p',q',r') \leq^0_{\widehat{\mathbb{P}}^3_{E_0}} (p,q,r)$ with $(p',q',r') \in D_0$, then by elementarity there is such a condition in M. Let (p_0,q_0,r_0) be such a condition in M. Otherwise, let $(p_0,q_0,r_0) = (p,q,r)$.

Suppose (p_n,q_n,r_n) have been defined. Let $\{(u_i,v_i,z_i) : i < K\}$ enumerate all the strings in ${}^{<\omega}2$ with length $n+1$ so that $\{u(n), v(n), z(n)\} = \{0,1\}$.

Let $(a_{-1}, b_{-1}, c_{-1}) = (p_n,q_n,r_n)$. For i with $-1 \leq i < K-1$, suppose (a_i,b_i,c_i) has been defined.

Let $(a^{-1}_{i+1}, b^{-1}_{i+1}, c^{-1}_{i+1}) = (a_i,b_i,c_i)$. Suppose for j with $-1 \leq j < n+1$, $(a^j_{i+1}, b^j_{i+1}, c^j_{i+1})$ has been defined. If there exists some condition below $(\Xi(a^j_{i+1}, u_{i+1}), \Xi(b^j_{i+1}, v_{i+1}), \Xi(c^j_{i+1}, z_{i+1}))$ that belongs to D_{j+1}, then choose, by elementarity, such a condition $(a',b',c') \in M \cap D_{j+1}$. Let $(a^{j+1}_{i+1}, b^{j+1}_{i+1}, c^{j+1}_{i+1}) = \mathsf{3prune}^{(u_{i+1},v_{i+1},z_{i+1})}_{(a',b',c')}(a^j_{i+1}, b^j_{i+1}, c^j_{i+1})$. If no such condition exists, then let $(a^{j+1}_{i+1}, b^{j+1}_{i+1}, c^{j+1}_{i+1}) = (a^j_{i+1}, b^j_{i+1}, c^j_{i+1})$.

Let $(a_{i+1}, b_{i+1}, c_{i+1}) = (a^{n+1}_{i+1}, b^{n+1}_{i+1}, c^{n+1}_{i+1})$. Let $(p_{n+1}, q_{n+1}, r_{n+1}) = (a_{K-1}, b_{K-1}, c_{K-1})$. Note that $(p_{n+1}, q_{n+1}, r_{n+1}) \leq^{n+1}_{\widehat{\mathbb{P}}^3_{E_0}} (p_n,q_n,r_n)$.

$\langle (p_n,q_n,r_n) : n \in \omega \rangle$ forms a fusion sequence in $\widehat{\mathbb{P}}^3_{E_0}$. Let $(p_\omega, q_\omega, r_\omega)$ be the fusion of this fusion sequence. The claim is that this is a $(M, \widehat{\mathbb{P}}^3_{E_0})$-master condition below (p,q,r).

It needs to be shown for each n that $(p_\omega,q_\omega,r_\omega) \Vdash_{\widehat{\mathbb{P}}^3_{E_0}} \check{M} \cap \check{D}_n \cap \dot{G} \neq \emptyset$. Let G be any $\widehat{\mathbb{P}}^3_{E_0}$-generic over M containing $(p_\omega,q_\omega,r_\omega)$. There is some $(p',q',r') \in G \cap D_n$. Since G is a filter, there is some $(p'',q'',r'') \leq_{\widehat{\mathbb{P}}^3_{E_0}} (p_\omega,q_\omega,r_\omega)$ so that $(p'',q'',r'') \in G \cap D_n$. By genericity, one may assume there is some $m > n$ and some $(u,v,z) \in {}^m2$ so that $(p'',q'',r'') \leq_{\widehat{\mathbb{P}}^3_{E_0}} (\Xi(p_\omega,u), \Xi(q_\omega,v), \Xi(r_\omega,z))$. During the construction while producing (p_m,q_m,r_m), the strings $(u,v,z) = (u_i,v_i,z_i)$ for some i in the chosen enumeration of strings. Note that $(p'',q'',r'') \leq_{\widehat{\mathbb{P}}^3_{E_0}} (\Xi(a^{n-1}_i, u_i), \Xi(b^{n-1}_i, v_i), \Xi(c^{n-1}_i, z_i))$ and $(p'',q'',r'') \in D_n$. At this stage, one would have chosen some $(a',b',c') \in M \cap D_n$ below $(\Xi(a^{n-1}_i, u_i), \Xi(b^{n-1}_i, v_i), \Xi(c^{n-1}_i, z_i))$ and set $(a^n_i, b^n_i, c^n_i) = \mathsf{3prune}^{(u_i,v_i,z_i)}_{(a',b',c')}(a^{n-1}_i, b^{n-1}_i, c^{n-1}_i)$. Note that $(p'',q'',r'') \leq_{\widehat{\mathbb{P}}^3_{E_0}} (\Xi(p_\omega,u), \Xi(q_\omega,v), \Xi(r_\omega,z)) \leq_{\widehat{\mathbb{P}}^3_{E_0}} (a',b',c')$. Since G is a filter and $(p'',q'',r'') \in G$, $(a',b',c') \in G \cap M \cap D_n$. This shows that $\widehat{\mathbb{P}}^3_{E_0}$ is a proper forcing. $\square$

In the proof, one extends a portion of the three trees to get into a dense set D only if it was possible and otherwise ignored D. Because of this, one cannot prove that $[p_\omega] \times_{E_0} \times [q_\omega] \times_{E_0} [r_\omega]$ consists entirely of reals which are $\widehat{\mathbb{P}}^3_{E_0}$-generic over M.

13. E_1 Does Not Have the 2-Mycielski Property

This section will give an example to show E_1 does not have the 2-Mycielski property. The notation of Definition 2.3 will be used in the following.

As in earlier sections, an understanding of the structure theorem of E_1-big $\mathbf{\Sigma}^1_1$ sets is essential:

Definition 13.1. E_1 is the equivalence relation on ${}^\omega({}^\omega 2)$ defined by $x\ E_1\ y$ if and only if there exists a k so that for all $n \geq k$, $x(n) = y(n)$.

Definition 13.2. [13] Let s be an infinite subset of ω. Let $\pi_s : \omega \to s$ be the unique increasing enumeration of s. A homeomorphism $\Phi : {}^\omega({}^\omega 2) \to {}^\omega({}^\omega 2)$ is an s-keeping homeomorphism if and only if the following hold:

1. For all $n \in \omega$, if $x(n) \neq y(n)$, then $\Phi(x)(\pi_s(m)) \neq \Phi(y)(\pi_s(m))$ for all $m \leq n$.
2. For all $n \in \omega$, if for all $m > n$, $x(m) = y(m)$, then for all $k > \pi_s(n)$, $\Phi(x)(k) = \Phi(y)(k)$.

Fact 13.3. *Let $B \subseteq {}^\omega({}^\omega 2)$ be $\mathbf{\Sigma}^1_1$. $E_1 \upharpoonright B \equiv_{\mathbf{\Delta}^1_1} E_1$ if and only if there is some infinite $s \subseteq \omega$ and s-keeping homeomorphism Φ so that $\Phi[{}^\omega({}^\omega 2)] \subseteq B$.*

Proof. This result is implicit in [15]. See [13], Section 7.2.1. ☐

Theorem 13.4. *Let $D = \{(x, y) \in {}^2({}^\omega({}^\omega 2)) : (\exists n)(x(n)(0) \neq y(n)(0))\}$. D is dense open and for all $\mathbf{\Delta}^1_1$ B such that $E_1 \upharpoonright B \equiv_{\mathbf{\Delta}^1_1} E_1$, $[B]^2_{E_1} \not\subseteq D$.*

E_1 does not have the 2-Mycielski property.

Proof. Suppose $(x, y) \in D$. There is some $n \in \omega$ so that $x(n)(0) \neq y(n)(0)$. Let $\sigma, \tau : (n + 1) \to {}^1 2$ be defined by $\sigma(k) = x(k) \upharpoonright 1$ and $\tau(k) = y(k) \upharpoonright 1$. Then $(x, y) \in N_{\sigma,\tau} \subseteq D$. D is open.

Let $\sigma, \tau : m \to {}^{<\omega}2$. Define $\sigma', \tau' : (m + 1) \to {}^{<\omega}2$ by

$$\sigma'(k) = \begin{cases} \sigma(k) & k < m \\ \langle 0 \rangle & k = m \end{cases} \quad \text{and} \quad \tau'(k) = \begin{cases} \tau(k) & k < m \\ \langle 1 \rangle & k = m \end{cases}.$$

$N_{\sigma',\tau'} \subseteq N_{\sigma,\tau}$ and $N_{\sigma',\tau'} \subseteq D$. D is dense open.

Let $s \subseteq \omega$ be infinite. Let $\pi_s : \omega \to s$ be the unique increasing enumeration of s. Let $\Phi : {}^\omega({}^\omega 2) \to {}^\omega({}^\omega 2)$ be an s-keeping homeomorphism.

Let $\delta_n : (\pi_s(n) + 1) \to {}^1 2$ be defined by $\delta_n(k) = \Phi(\bar{0})(k) \upharpoonright 1$. A strictly increasing sequence $\langle m_n : n \in \omega \rangle$ of natural numbers and functions $\sigma_n : m_n \to {}^{m_n}2$ satisfying the following for all $n \in \omega$ will be defined:

1. $\sigma_n(k) \subseteq \sigma_{n+1}(k)$ for each $k < m_n$.
2. $\Phi[N_{\sigma_n}] \subseteq N_{\delta_n}$.
3. There exists a $j < m_n$ such that $\sigma_n(n)(j) = 1$ and for all $k > n$ and $i < m_n$, $\sigma_n(k)(i) = 0$.

Let $m_{-1} = 0$ and $\sigma_{-1} = \delta_{-1} = \emptyset$.

Suppose m_n and σ_n have been defined and satisfy conditions 2 and 3 if $n \geq 0$. Define $y \in N_{\sigma_n}$ by $y(i)(j) = 0$ if $(i, j) \notin m_n \times m_n$. Then $\Phi(y) \in N_{\delta_n}$. Since $y(k) = \bar{0}(k)$ for all $k > n$ and Φ is an s-keeping homeomorphism, $\Phi(y)(k) = \Phi(\bar{0})(k)$ for all $k > \pi_s(n)$. Thus $\Phi(y) \in N_{\delta_{n+1}}$. By continuity of Φ, there is some $M \geq m_n$ so that if $\tau : M \to {}^M 2$ is defined by $\tau(i) = y(i) \upharpoonright M$, then $\Phi(N_\tau) \subseteq N_{\delta_{n+1}}$. Let $m_{n+1} = M + 1$ and define

$$\sigma_{n+1}(i)(j) = \begin{cases} 1 & i = n + 1 \wedge j = M \\ y(i)(j) & \text{otherwise} \end{cases}.$$

m_{n+1} and σ_{n+1} satisfy conditions 1, 2, and 3.

Let $x \in {}^\omega({}^\omega 2)$ be so that $\{x\} = \bigcap_{n\in\omega} N_{\sigma_n}$. $\neg(\bar{0}\ E_1\ x)$ since for all n, there exists a j so that $x(n)(j) = 1$ by condition 3. However since $\Phi(x) \in N_{\delta_n}$ for all n, $(\Phi(\bar{0}), \Phi(x)) \notin D$. From Definition 13.2, Φ is a E_1 reduction so $\neg(\Phi(\bar{0})\ E_1\ \Phi(x))$. Hence $[\Phi({}^\omega({}^\omega 2))]^2_{E_1} \not\subseteq D$.

So it has been shown that for all infinite $s \subseteq \omega$ and s-keeping homeomorphisms Φ, $[\Phi[{}^\omega({}^\omega 2)]]^2_{E_1} \not\subseteq D$. By Fact 13.3, every $\mathbf{\Delta}^1_1$ set B so that $E_1 \upharpoonright B \equiv_{\mathbf{\Delta}^1_1} E_1$ contains $\Phi[{}^\omega({}^\omega 2)]$ for some s and some s-keeping homeomorphism Φ. Therefore, $[B]^2_{E_1} \not\subseteq D$ for all such B. E_1 does not have the 2-Mycielski property. ☐

14. The Structure of E_2

This section will give a proof of a result about the structure of E_2-big sets necessary for analyzing the weak-Mycielski property for E_2. The proof is similar to but a bit a more technical than the argument of [12] Theorem 15.4.1. Some of the notation and terminology come from [12].

Definition 14.1. Suppose $x, y \in {}^\omega 2$. Define

$$\delta(x,y) = \sum_{k \in x \triangle y} \frac{1}{k+1}.$$

Suppose $m < n \leq \omega$. Suppose $x, y \in {}^N 2$ where $n \leq N \leq \omega$. Define

$$\delta_m^n(x,y) = \sum \left\{ \frac{1}{k+1} : (m \leq k < n) \wedge (k \in x \triangle y) \right\}.$$

Let $A, B \subseteq {}^\omega 2$. Let $m < n \leq \omega$. Let $\epsilon > 0$. Define $\delta_m^n(A,B) < \epsilon$ if and only if for all $x \in A$, there is some $y \in B$ so that $\delta_m^n(x,y) < \epsilon$ and for all $y \in B$, there exists some $x \in A$ so that $\delta_m^n(x,y) < \epsilon$.

Definition 14.2. E_2 is the equivalence relation on ${}^\omega 2$ defined $x \; E_2 \; y$ if and only if $\delta(x,y) < \infty$.

Lemma 14.3. *Let $m < n \leq \omega$. Fix N so that $n \leq N \leq \omega$. If $n < \omega$, then δ_m^n is a pseudo-metric on ${}^N 2$. If $n = \omega$, then δ_m^n is a pseudo-metric on any E_2 equivalence class.*

Lemma 14.4. *Let $m, p \in \omega$. Let $q \in \mathbb{Q}^+$. Let $(\hat{X}_i : i < p)$ be a sequence of $\Sigma_1^1(z)$ subsets of ${}^\omega 2$. Let $(x_i : i < p)$ be a sequence in ${}^\omega 2$ with the property that $x_i \in \hat{X}_i$ and $\delta_m^\omega(x_0, x_i) < q$. Then there exists a sequence $(X_i : i < p)$ of $\Sigma_1^1(z)$ sets with $x_i \in X_i$ and $\delta_m^\omega(X_0, X_i) < q$.*

Proof. Let

$$X_0 = \left\{ x \in \hat{X}_0 : (\exists z_1, ..., z_{p-1}) \left(\bigwedge_{1 \leq i < p} z_i \in \hat{X}_i \wedge \delta_m^\omega(x, z_i) < q \right) \right\}.$$

For $1 \leq i < p$, define

$$X_i = \left\{ x \in \hat{X}_i : (\exists z)(x \in X_0 \wedge \delta_m^\omega(x,z) < q) \right\}.$$

□

Lemma 14.5. *Let $m, p \in \omega$. Let $q \in \mathbb{Q}^+$. Let $(\hat{X}_i : i < p)$ be a sequence of $\Sigma_1^1(z)$ sets with $\delta_m^\omega(\hat{X}_0, \hat{X}_i) < q$ for all $i < p$. Let $j < p$. Suppose $A \subseteq \hat{X}_j$ is a $\Sigma_1^1(z)$ set. Then there exists a sequence $(X_i : i < p)$ of $\Sigma_1^1(z)$ sets with the property that for all $i < p$, $X_i \subseteq \hat{X}_i$, $X_j = A$, and $\delta_m^\omega(X_0, X_i) < q$.*

Proof. Let

$$X_0 = \{x \in \hat{X}_0 : (\exists z)(z \in A \wedge \delta_m^\omega(x,z) < q)\}.$$

For all $i < p$ and $i \neq j$, let

$$X_i = \{x \in \hat{X}_i : (\exists z)(z \in X_0 \wedge \delta_m^\omega(x,z) < q)\}$$

Let $X_j = A$. □

Lemma 14.6. *Let $m, p \in \omega$. Let $q \in \mathbb{Q}^+$. Let $(\hat{X}_i : i < p)$ be a sequence of $\Sigma_1^1(z)$ sets with $\delta_m^\omega(\hat{X}_0, \hat{X}_i) < q$ for all $i < p$. Let D be a dense open subset of $\mathbb{P}_z$. Then there exists a sequence $(X_i : i < p)$ of $\Sigma_1^1(z)$ sets with $X_i \subseteq \hat{X}_i$, $X_i \in D$, and $\delta_m^\omega(X_0, X_i) < q$ for all $i < p$.*

Proof. Let $Y_i^{-1} = \hat{X}_i$.

One seeks to define $\Sigma_1^1(z)$ sets Y_i^j for all $-1 \leq j < p$ with the property that if $-1 \leq j < p-1$, then $Y_i^{j+1} \subseteq Y_i^j$, and for any $-1 \leq j < p$ and $0 \leq i < p$, $Y_j^j \in D$ and $\delta_m^\omega(Y_0^j, Y_i^j) < q$.

Suppose Y_i^j has been defined with the desired properties for $j < p-1$ and all $i < p$. Since D is dense open in $\mathbb{P}_z$, pick some $A \subseteq Y_{j+1}^j$ so that $A \in D$. Use Lemma 14.5 with $\{Y_i^j : i < p\}$ and $A \subseteq Y_i^j$ to obtain $\{Y_i^{j+1} : i < p\}$ with the desired properties.

Let $X_i = Y_i^{p-1}$. □

In the previous three lemmas, the first set was distinguished. In the following argument, sets may be indexed by strings and so in applications of the three lemmas, one will need to indicate what this distinguished set is.

Lemma 14.7. *Let $z \in {}^{\omega}2$. Let $k, m, p \in \omega$. Let $r, v \in \mathbb{Q}^+$. Let $(B_s^i : s \in {}^k2 \wedge i < p)$ be a sequence of $\Sigma_1^1(z)$ sets. Let $(b_s^i : s \in {}^k2 \wedge i < p)$ be a sequence in ${}^{\omega}2$ with $b_s^i \in B_s^i$. Suppose for all $i < p$, $\delta_m^{\omega}(b_{0^k}^0, b_{0^k}^i) < r$. Suppose for each $i < p$ and for all $s \in {}^k2$, $\delta_m^{\omega}(b_{0^k}^i, b_s^i) < v$. Let D be a dense open subset of $\mathbb{P}_z$.*

Then there is a sequence $(C_s^i : s \in {}^k2 \wedge i < p\}$ of $\Sigma_1^1(z)$ sets so that for all $i < p$ and $s \in {}^k2$, $\delta_m^{\omega}(C_{0^k}^0, C_{0^k}^i) < r$, $\delta_m^{\omega}(C_{0^k}^i, C_s^i) < v$, and $C_s^i \in D$.

Proof. For each $i < p$, apply Lemma 14.4 to $\{B_s^i : s \in {}^k2\}$ and $\{b_s^i : s \in {}^k2\}$ using 0^k as the distinguished index to obtain a sequence of $\Sigma_1^1(z)$ sets $\{E_s^i : s \in {}^k2\}$ with the property that $E_s^i \subseteq B_s^i$, $b_s^i \in E_s^i$, and $\delta_m^{\omega}(E_{0^k}^i, E_s^i) < v$.

Now apply Lemma 14.4 to $\{E_{0^k}^i : i < p\}$ and $\{b_{0^k}^i : i < p\}$ with 0 as the distinguished index to obtain $\Sigma_1^1(z)$ sets $A_i \subseteq E_{0^k}^i$ so that $\delta_m^{\omega}(A_0, A_i) < r$.

For each $i < p$, apply Lemma 14.5 to $\{E_s^i : s \in {}^k2\}$ with 0^k as the distinguished index and $A_i \subseteq E_{0^k}^i$ to obtain $\Sigma_1^1(z)$ sets $G_s^{i,-1}$ with the properties that $G_{0^k}^{i,-1} = A_i$ and $\delta_m^{\omega}(G_{0^k}^{i,-1}, G_s^{i,-1}) < v$.

Note that since $G_{0^k}^{i,-1} = A_i$, the sequence $\{G_s^{i,-1} : i < p \wedge s \in {}^k2\}$ has the property that for all $i < p$, $\delta_m^{\omega}(G_{0^k}^{0,-1}, G_{0^k}^{i,-1}) < r$ and for each $i < p$ and $s \in {}^k2$, $\delta_m^{\omega}(G_{0^k}^{i,-1}, G_s^{i,-1}) < v$.

One wants to create $G_s^{i,j}$ for $-1 \leq j < p$, $i < p$, and $s \in {}^k2$ so that

(i) For each $i \in \omega$, $s \in {}^k2$, and $-1 \leq l \leq j < p$, $G_s^{i,j} \subseteq G_s^{i,l}$.

(ii) For all $i < p$, $\delta_m^{\omega}(G_{0^k}^{0,j}, G_{0^k}^{i,j}) < r$.

(iii) For each $i < p$ and $s \in {}^k2$, $\delta_m^{\omega}(G_{0^k}^{i,j}, G_s^{i,j}) < v$.

(iv) If $j \geq 1$ and $0 \leq l \leq j$, then $G_s^{l,j} \in D$.

This already holds for $j = -1$. Suppose the construction worked up to stage $j < p - 1$ producing objects with the above properties. Apply Lemma 14.6 to $\{G_s^{j+1,j} : s \in {}^k2\}$ to get sets $\{G_s^{j+1,j+1} : s \in {}^k2\}$ each in D and $\delta_m^{\omega}(G_{0^k}^{j+1,j+1}, G_s^{j+1,j+1}) < v$.

Next apply Lemma 14.5 to $\{G_{0^k}^{i,j} : i < p\}$ with 0 as the distinguished index and $G_{0^k}^{j+1,j+1} \subseteq G_{0^k}^{j+1,j}$ to obtain sets $G_{0^k}^{i,j+1} \subseteq G_{0^k}^{i,j}$ with $\delta_m^{\omega}(G_{0^k}^{0,j+1}, G_{0^k}^{i,j+1}) < r$. (Note it is acceptable to use the notation $G_{0^k}^{j+1,j+1}$ since it is the same set as before by the statement of Lemma 14.5.)

For each $i \neq j + 1$, apply Lemma 14.5 on $\{G_s^{i,j} : s \in {}^k2\}$ with 0^k as the distinguished index and $G_{0^k}^{i,j+1} \subseteq G_{0^k}^{i,j}$ to obtain sets $\{G_s^{i,j+1} : s \in {}^k2\}$ with the property that $\delta_m^{\omega}(G_{0^k}^{i,j+1}, G_s^{i,j+1}) < v$. This completes the construction at stage $j + 1$.

Let $C_s^i = G_s^{i,p-1}$. ☐

Definition 14.8. ([12] Definition 15.2.2) Let $q > 0$ be a rational number. Let $A \subseteq {}^{\omega}2$. Let $a \in A$. The q-galaxy of a in A, denoted $\mathrm{Gal}_A^q(a)$, is the set of all $b \in A$ so that there exists $a_0, ..., a_l \in A$ with $a = a_0$, $b = a_l$, and $\delta(a_i, a_{i+1}) < q$ for all $0 \leq i < l - 1$.

$A \subseteq {}^{\omega}2$ is q-grainy if and only if for all $a \in A$ and $b \in \mathrm{Gal}_A^q(a)$, $\delta(a, b) < 1$. A is grainy if and only if A is q-grainy for some positive rational number q.

Fact 14.9. *Let $z \in {}^{\omega}2$ and rational $q > 0$. If A is a $\Sigma_1^1(z)$ q-grainy set, then there is some $B \supseteq A$ which is $\Delta_1^1(z)$ q-grainy.*

Proof. (See [12], Claim 15.2.4 for a more constructive proof.)

Let $U \subseteq \omega \times {}^{\omega}2$ be a universal $\Sigma_1^1(z)$ set. The relation in variables e, a, and b expressing $b \in \mathrm{Gal}_{U^e}^q(a)$ is $\Sigma_1^1(z)$.

Let $\mathcal{A}$ be the collection of all $\Sigma_1^1(z)$ q-grainy subsets of ${}^{\omega}2$.

$$\{e : U^e \in \mathcal{A}\} = \{e : (\forall a)(\forall b)(b \in \mathrm{Gal}_{U^e}^q(a) \Rightarrow \delta(a, b) < 1)\}$$

This shows that $\mathcal{A}$ is a collection of $\Sigma_1^1(z)$ sets which is $\Pi_1^1(z)$ in the code. By $\Sigma_1^1(z)$ reflection, every $\Sigma_1^1(z)$ q-grainy set is contained inside of a $\Delta_1^1(z)$ q-grainy set. ☐

Definition 14.10. Let $z \in {}^{\omega}2$. Let S_z be the union of all $\Delta_1^1(z)$ grainy sets. Let $H_z = {}^{\omega}2 \setminus S_z$.

Fact 14.11. *Let $z \in {}^\omega 2$. S_z is $\Pi^1_1(z)$. Hence H_z is $\Sigma^1_1(z)$.*
Every nonempty $\Sigma^1_1(z)$ subset of H_z is not grainy.

Proof. Similar to Fact 5.12. □

Theorem 14.12. *Let $z \in {}^\omega 2$. Let $p \in \omega$. Suppose $(X_i : i < p)$ is a collection of $\Sigma^1_1(z)$ subsets of ${}^\omega 2$ with the property that $\bigcap_{i<p}[X_i \cap H_z]_{E_2} \neq \emptyset$. Then there is a strictly increasing sequence $(m_k : k \in \omega)$ and functions $g^i : {}^{<\omega}2 \to {}^{<\omega}2$ for each $i < p$ with the following properties:*
1. *If $|s| = k$, then for all $i < p$, $g^i(s) \in {}^{m_k}2$.*
2. *If $s \subseteq t$, then for all $i < p$, $g^i(s) \subseteq g^i(t)$.*
3. *If $|s| = |t| = k > 0$ and $s(k-1) = t(k-1)$, then $\delta^{m_k}_{m_{k-1}}(g^i(s), g^i(t)) < 2^{-(k+1)}$,*
4. *If $|s| = |t| = k > 0$ and $s(k-1) \neq t(k-1)$, then $|\delta^{m_k}_{m_{k-1}}(g^i(s), g^i(t)) - \frac{1}{k}| < 2^{-(k+1)}$.*
5. *For $i, j < p$ and $s \in {}^k 2$ with $k > 0$, $\delta^{m_k}_{m_{k-1}}(g^i(s), g^j(s)) < 2^{-(k+1)}$.*
6. *Define $\Phi^i(x) = \bigcup_{n\in\omega} g^i(x \upharpoonright n)$. $\Phi^i : {}^\omega 2 \to X_i \cap H_z$ is a reduction witnessing $E_2 \leq_{\mathbf{\Delta}^1_1} E_2 \upharpoonright X_i \cap H_z$. Moreover, for $i, j < p$, $[\Phi^i[{}^\omega 2]]_{E_2} = [\Phi^j[{}^\omega 2]]_{E_2}$.*

Proof. During the construction, one will seek to create
(i) a strictly increasing sequence $(m_k : k \in \omega)$,
(ii) for each $i < p$ and $s \in {}^{<\omega}2$, $\Sigma^1_1(z)$ sets A^i_s,
(iii) and for each $i < p$, $g^i(s) \in {}^{<\omega}2$.

These objects will satisfy the following properties:
(I) If $|s| = k$, then $|g^i(s)| = m_k$. $s \subseteq t$ implies $g^i(s) \subseteq g^i(t)$.
(II) $\emptyset \neq A^i_s \subseteq X^i \cap H_z \cap N_{g^i(s)}$. $s \subseteq t$ implies $A^i_t \subseteq A^i_s$.
(III) If $k > 0$, $|s| = k$, then $\delta^\omega_{m_k}(A^i_{0^k}, A^i_s) \leq 2^{-(k+4)}$, where $0^k : k \to 2$ is the constant 0 function.
(IV) If $k > 0$, $|s| = |t| = k$, and $s(k-1) = t(k-1)$, then $\delta^{m_k}_{m_{k-1}}(g^i(s), g^i(t)) \leq 2^{-(k+1)}$.
(V) If $k > 0$, $|s| = |t| = k$, $s(k-1) \neq t(k-1)$, then $|\delta^{m_k}_{m_{k-1}}(g^i(s), g^i(t)) - \frac{1}{k}| < 2^{-(k+1)}$.
(VI) If $|s| = k$ and $i < p$, $\delta^\omega_{m_k}(A^0_s, A^i_s) < 2^{-(k+6)}$.
(VII) If $|s| = k$ and $k > 0$, then $\delta^{m_k}_{m_{k-1}}(g^i(s), g^j(s)) < 2^{-(k+1)}$.
(VIII) Let $\mathcal{D} = (D_n : n \in \omega)$ be the countable collection of dense open subsets of $\mathbb{P}_z$ from Fact 5.9. For all $s \in {}^{<\omega}2$ and $i < p$, $A^i_s \in D_{|s|}$.

Suppose these objects having the above properties could be constructed. It only remains to verify 6: $\{\Phi^i(x)\} = \bigcap_{k\in\omega} A_{x\upharpoonright k}$ by (II) and (VIII). Hence Φ^i maps into $X_i \cap H_z$ by (II). Note that $\delta(\Phi^i(x), \Phi^j(y)) = \lim_{k\to\infty} \delta^{m_k}_0(g^i(x \upharpoonright k), g^j(y \upharpoonright k))$. Also using (IV) and (V), for any k

$$|\delta^{m_k}_0(g^i(x \upharpoonright k), g^i(y \upharpoonright k)) - \delta^k_0(x \upharpoonright k, y \upharpoonright k)| < \sum_{j<k} 2^{-(j+1)} < 1$$

Hence $\Phi^i(x)\ E_2\ \Phi^i(y) \Leftrightarrow \delta(\Phi^i(x), \Phi^i(y)) < \infty \Leftrightarrow \delta(x, y) < \infty \Leftrightarrow x\ E_2\ y$. This shows each Φ^i witnesses $E_2 \leq_{\mathbf{\Delta}^1_1} E_2 \upharpoonright X_i \cap H_z$. Using (VII), for each $i, j < p$ and $x \in {}^\omega 2$,

$$\delta^{m_k}_{m_0}(g^i(x \upharpoonright k), g^j(x \upharpoonright j)) < \sum_{j<k} 2^{-(j+1)} < 1$$

Hence $\Phi^i(x)\ E_2\ \Phi^j(x)$. Hence $[\Phi^i[{}^\omega 2]]_{E_2} = [\Phi^j[{}^\omega 2]]_{E_2}$.

Next the construction: Since $\bigcap_{i<p}[X_i \cap H_z]_{E_2} \neq \emptyset$, let $(b^i_\emptyset : i < p)$ be such that for all $i, j < p$, $b^i_\emptyset\ E_2\ b^j_\emptyset$ and $b^i_\emptyset \in X_i \cap H_z$. Therefore, choose $m_0 \in \omega$ so that for all $i < p$, $\delta^\omega_{m_0}(b^0_\emptyset, b^i_\emptyset) < 2^{-6}$. For each $i < p$, let $g^i(\emptyset) = b^i_\emptyset \upharpoonright m_0$.

Let $B^i_\emptyset = X_i \cap H_z \cap N_{g^i(\emptyset)}$. Apply Lemma 14.7 to $\{B^i_\emptyset : i \in p\}$, $\{b^i_\emptyset : i < p\}$, and the dense open (in $\mathbb{P}_z$) set D_0 (where $r = 2^{-6}$ and $v = 1$) to obtain sets $A^i_\emptyset$ with the desired properties.

Suppose the objects from stage k have been constructed with the desired properties.

As $A^0_{0^k} \subseteq H_z$, Fact 14.11 implies that $A^0_{0^k}$ is not $2^{-(k+5)}$-grainy. Hence there is a sequence $a_0, ..., a_M$ of points in $A^0_{0^k}$ so that for each $0 \leq j < M - 1$, $\delta(a_j, a_{j+1}) < 2^{-(k+5)}$ but $\delta(a_0, a_M) > 1$. Hence there is some I so that $\delta(a_0, a_I) > \frac{1}{k+1}$ and $\delta(a_0, a_I) - \frac{1}{k+1} < 2^{-(k+5)}$.

Let $b^0_{0^k\frown 0} = a_0$ and $b^0_{0^k\frown 1} = a_I$. Since $\delta^\omega_{m_k}(A^0_s, A^i_s) < 2^{-(k+6)}$ by (VI), find $b^i_{0^{k+1}}, b^i_{0^k\frown 1} \in A^i_{0^k}$ so that $\delta(b^0_{0^{k+1}}, b^i_{0^{k+1}}) < 2^{-(k+6)}$ and $\delta(b^0_{0^k\frown 1}, b^i_{0^k\frown 1}) < 2^{-(k+6)}$.

The claim is that for all $i < p$, $|\delta^\omega_{m_k}(b^i_{0^{k+1}}, b^i_{0^k\frown 1}) - \frac{1}{k+1}| < 2^{-(k+4)}$: To see this,

$$\begin{aligned} &|\delta^\omega_{m_k}(b^i_{0^{k+1}}, b^i_{0^k\frown 1}) - \frac{1}{k+1}| \\ &\leq |\delta^\omega_{m_k}(b^i_{0^{k+1}}, b^i_{0^k\frown 1}) - \delta^\omega_{m_k}(b^0_{0^{k+1}}, b^0_{0^k\frown 1})| + |\delta^\omega_{m_k}(b^0_{0^{k+1}}, b^0_{0^k\frown 1}) - \frac{1}{k+1}| \\ &< |\delta^\omega_{m_k}(b^i_{0^{k+1}}, b^i_{0^k\frown 1}) - \delta^\omega_{m_k}(b^0_{0^{k+1}}, b^i_{0^k\frown 1})| + |\delta^\omega_{m_k}(b^0_{0^{k+1}}, b^i_{0^k\frown 1}) - \delta^\omega_{m_k}(b^0_{0^{k+1}}, b^0_{0^k\frown 1})| + 2^{-(k+5)} \\ &\leq \delta^\omega_{m_k}(b^i_{0^{k+1}}, b^0_{0^{k+1}}) + \delta^\omega_{m_k}(b^i_{0^k\frown 1}, b^0_{0^k\frown 1}) + 2^{-(k+5)} \\ &\leq 2^{-(k+6)} + 2^{-(k+6)} + 2^{-(k+5)} = 2^{-(k+4)} \end{aligned}$$

This proves the claim.

Now fix a $i < p$. By (III), $\delta^\omega_{m_k}(A^i_{0^k}, A^i_t) < 2^{-(k+4)}$ for each $t \in {}^k2$. For each $s \in {}^{k+1}2$, let $b^i_s \in A^i_{s\restriction k}$ be such that $\delta^\omega_{m_k}(b^i_{0^k\frown s(k)}, b^i_s) < 2^{-(k+4)}$.

Suppose $s \in {}^{k+1}2$ and $s(k) = 1$.

$$\begin{aligned} &|\delta^\omega_{m_k}(b^i_{0^{k+1}}, b^i_s) - \frac{1}{k+1}| \\ &\leq |\delta^\omega_{m_k}(b^i_{0^{k+1}}, b^i_s) - \delta^\omega_{m_k}(b^i_{0^{k+1}}, b^i_{0^k\frown 1})| + |\delta^\omega_{m_k}(b^i_{0^{k+1}}, b^i_{0^k\frown 1}) - \frac{1}{k+1}| \\ &< \delta(b^i_s, b^i_{0^k\frown 1}) + 2^{-(k+4)} \leq 2^{-(k+4)} + 2^{-(k+4)} = 2^{-(k+3)} \end{aligned}$$

By (I), (II), and the fact that $b^i_{0^{k+1}}, b^i_{0^k\frown 1} \in A^i_{0^k}$, there exists some $m_{k+1} > m_k$ so that

(i) $|\delta^{m_{k+1}}_{m_k}(b^i_{0^{k+1}}, b^i_s) - \frac{1}{k+1}| < 2^{-(k+3)}$ for all $s \in {}^{k+1}2$ with $s(k) = 1$.

(ii) $\delta^\omega_{m_{k+1}}(b^0_{0^{k+1}}, b^i_{0^{k+1}}) < 2^{-(k+7)}$ for all $i < p$.

(iii) $\delta^\omega_{m_{k+1}}(b^i_{0^{k+1}}, b^i_s) < 2^{-(k+5)}$ for all $s \in {}^{k+1}2$.

Let $g^i(s) = b^i_s \restriction m_{k+1}$. Suppose $s(k) = t(k)$. Without loss of generality, suppose $s(k) = t(k) = 1$. Then

$$\delta^{m_{k+1}}_{m_k}(g^i(s), g^i(t)) \leq \delta^{m_{k+1}}_{m_k}(b^i_s, b^i_{0^k\frown 1}) + \delta^{m_{k+1}}_{m_k}(b^i_{0^k\frown 1}, b^i_t) \leq 2^{-(k+4)} + 2^{-(k+4)} = 2^{-(k+3)} < 2^{-(k+2)}$$

This establishes (IV).

Suppose $s(k) \neq t(k)$. Without loss of generality, suppose $s(k) = 1$. Hence $t(k) = 0$.

$$\begin{aligned} &|\delta^{m_{k+1}}_{m_k}(g^i(s), g^i(t)) - \frac{1}{k+1}| \\ &\leq |\delta^{m_{k+1}}_{m_k}(b^i_s, b^i_t) - \delta^{m_{k+1}}_{m_k}(b^i_s, b^i_{0^{k+1}})| + |\delta^{m_{k+1}}_{m_k}(b^i_s, b^i_{0^{k+1}}) - \frac{1}{k+1}| \\ &< \delta^{m_{k+1}}_{m_k}(b^i_t, b^i_{0^{k+1}}) + 2^{-(k+3)} < 2^{-(k+4)} + 2^{-(k+3)} < 2^{-(k+2)} \end{aligned}$$

This establishes (V).

Let $s \in {}^{k+1}2$. Without loss of generality suppose $s(k) = 0$. Suppose $i, j < p$. Observe:

$$\begin{aligned} \delta^{m_{k+1}}_{m_k}(g^i(s), g^j(s)) &\leq \delta^\omega_{m_k}(b^i_s, b^j_s) \leq \delta^\omega_{m_k}(b^i_s, b^i_{0^{k+1}}) + \delta^\omega_{m_k}(b^i_{0^{k+1}}, b^j_s) \\ &\leq \delta^\omega_{m_k}(b^i_s, b^i_{0^{k+1}}) + \delta^\omega_{m_k}(b^i_{0^{k+1}}, b^j_{0^{k+1}}) + \delta^\omega_{m_k}(b^j_{0^{k+1}}, b^j_s) \\ &\leq \delta^\omega_{m_k}(b^i_s, b^i_{0^{k+1}}) + \delta^\omega_{m_k}(b^i_{0^{k+1}}, b^0_{0^{k+1}}) + \delta^\omega_{m_k}(b^0_{0^{k+1}}, b^j_{0^{k+1}}) + \delta^\omega_{m_k}(b^j_{0^{k+1}}, b^j_s) \\ &\leq 2^{-(k+4)} + 2^{-(k+6)} + 2^{-(k+6)} + 2^{-(k+4)} \leq 2^{-(k+3)} + 2^{-(k+5)} < 2^{-(k+2)} \end{aligned}$$

This establishes (VII).

For $s \in {}^{k+1}2$, let $B^i_s = A^i_{s\restriction k} \cap N_{g^i(s)}$. Apply Lemma 14.7 on $\{B^i_s : i < p \wedge s \in {}^{k+1}2\}$, $\{b^i_s : i < p \wedge s \in {}^{k+1}2\}$, $r = 2^{-(k+7)}$, $v = 2^{-(k+5)}$, and D_{k+1} to obtain the desired objects $(A^i_s : i < p \wedge s \in {}^{k+1}2)$ which satisfy the remaining conditions.

This completes the proof. □

By relativizing to the appropriate parameter, one can obtain the following result.

Corollary 14.13. *Let $p \in \omega$. Suppose $(X_i : i < p)$ is a collection of $\mathbf{\Sigma}_1^1$ subsets of ${}^\omega 2$ with the property that for all $i < p$, $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \upharpoonright X_i$ and for all $i, j < p$, $[X_i]_{E_2} = [X_j]_{E_2}$. Then there exists a sequence of strictly increasing integers $(m_k : k \in \omega)$ and maps $g^i : {}^{<\omega}2 \to {}^{<\omega}2$ satisfying conditions 1 - 5 of Theorem 14.12.*

Fact 14.14. *Let $B \subseteq {}^\omega 2$ be a $\mathbf{\Sigma}_1^1$ so that $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \upharpoonright B$. There there exists a strictly increasing sequence $(m_k : k \in \omega)$ with $m_0 = 0$ and function $g : {}^{<\omega}2 \to {}^{<\omega}2$ with the following properties:*
1. If $|s| = k$, then $|g(s)| = m_k$.
2. If $s \subseteq t$, then $g(s) \subseteq g(t)$.
3. If $|s| = |t| = k > 0$ and $s(k-1) = t(k-1)$, then $\delta_{m_{k-1}}^{m_k}(g(s), g(t)) < 2^{-(k+1)}$.
4. If $|s| = |t| = k > 0$ and $s(k-1) \neq t(k-1)$, then $|\delta_{m_{k-1}}^{m_k}(g(s), g(t)) - \frac{1}{k}| < 2^{-(k+1)}$.
5. Let $\Phi : {}^\omega 2 \to {}^\omega 2$ be defined by $\Phi(x) = \bigcup_{n\in\omega} g(x \upharpoonright n)$. Then Φ is a $\mathbf{\Delta}_1^1$ function such that $\Phi[{}^\omega 2] \subseteq B$ and Φ witnesses $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \upharpoonright B$.

Proof. This is implicit in [7]. Also see [12] Theorem 15.4.1 and [13] Theorem 7.43. The proof is quite similar to Theorem 14.12. ☐

15. E_2 Does Not Have the 2-Mycielski Property

Theorem 15.1. *Let $D \subseteq {}^2({}^\omega 2)$ be defined by*

$$D = \{(x, y) \in {}^2({}^\omega 2) : (\exists i < j)(\delta_i^j(x, y) > 2 \wedge (\forall n)(i \leq n < j \Rightarrow x(n) \neq y(n)))\}$$

D is dense open.

Let $(m_k : k \in \omega)$, g, and Φ be as in Fact 14.14. $(\Phi(\tilde{0}), \Phi(\widetilde{01})) \notin D$.

For any $\mathbf{\Delta}_1^1$ set B so that $E_2 \upharpoonright B \equiv_{\mathbf{\Delta}_1^1} E_2$, $[B]_{E_2}^2 \not\subseteq D$.

E_2 does not have the 2-Mycielski property.

Proof. Let $(x, y) \in D$. There is some $i < j$ so that for all n with $i \leq n < j$, $x(n) \neq y(n)$ and $\delta_i^j(x, y) > 2$. Let $\sigma = x \upharpoonright (j+1)$ and let $\tau = y \upharpoonright (j+1)$. Then $(x, y) \in N_{\sigma,\tau} \subseteq D$. D is open.

Let $\sigma, \tau \in {}^{<\omega}2$ with $|\sigma| = |\tau|$. Let $i = |\sigma|$. Find a $j > i$ so that $\sum_{i \leq n < j} \frac{1}{n+1} > 2$. Let $\sigma', \tau' \in {}^{j+1}2$ be defined by

$$\sigma'(k) = \begin{cases} \sigma(k) & k < i \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \tau'(k) = \begin{cases} \tau(k) & k < i \\ 1 & \text{otherwise} \end{cases}.$$

$N_{\sigma',\tau'} \subseteq N_{\sigma,\tau}$ and $N_{\sigma',\tau'} \subseteq D$. D is dense open.

Note that one must have

$$\sum_{m_{k-1} \leq m < m_k} \frac{1}{m+1} \geq 2^{-k-1}$$

because $\frac{1}{k} - 2^{-k-1} \geq 2^{-k} - 2^{-k-1} = 2^{-k-1}$ and so otherwise, condition 4 could not hold for any $s, t \in {}^k 2$ with $s(k-1) \neq t(k-1)$.

This implies that for any s and t so that $s(k-1) = t(k-1)$, there must be some m with $m_{k-1} \leq m < m_k$ so that $g(s)(m) = g(t)(m)$.

Note that for all $s, t \in {}^{k+1}2$.

$$\delta_{m_{k-1}}^{m_{k+1}}(g(s), g(t)) = \delta_{m_{k-1}}^{m_k}(g(s), g(t)) + \delta_{m_k}^{m_{k+1}}(g(s), g(t)) \leq \frac{1}{k} + 2^{-k-1} + \frac{1}{k+1} + 2^{-k-2} < 2.$$

Hence for any s, t, if there exists $i < j$ so that $\delta_i^j(g(s), g(t)) > 2$, then there is some $k \geq 1$ so that $i \leq m_{k-1} < m_k < m_{k+1} \leq j$.

Now suppose that there is some $i < j$ so that $\delta_i^j(\Phi(\tilde{0}), \Phi(\widetilde{01})) > 2$. There is some $k \geq 1$ so that $i \leq m_{k-1} < m_k < m_{k+1} \leq j$. Without loss of generality, suppose k is even. $\tilde{0}(k) = 0 = \widetilde{01}(k)$. By the above, there is some l with $m_k \leq l < m_{k+1}$ so that $\Phi(\tilde{0})(l) = \Phi(\widetilde{01})(l)$. This shows $(\Phi(\tilde{0}), \Phi(\widetilde{01})) \notin D$. $\neg(\tilde{0}\ E_2\ \widetilde{01})$ so $\neg(\Phi(\tilde{0})\ E_2\ \Phi(\widetilde{01}))$. Hence $[\Phi[{}^\omega 2]]_{E_2}^2 \not\subseteq D$.

It has been shown that for all $(m_k : k \in \omega)$, g, and associated Φ, $[\Phi[{}^\omega 2]]_{E_2}^2 \not\subseteq D$. If $B \subseteq {}^\omega 2$ is $\mathbf{\Delta}_1^1$ with the property that $E_2 \upharpoonright B \equiv_{\mathbf{\Delta}_1^1} E_2$, then Fact 14.14 implies that there is some $(m_k : k \in \omega)$ and g so that $\Phi[{}^\omega 2] \subseteq B$. This shows that for all such B, $[B]_{E_2}^2 \not\subseteq D$. E_2 does not have the 2-Mycielski property. ☐

Theorem 15.2. *For each $n \in \omega$, let*

$$D_n = \{(x,y) \in {}^2({}^\omega 2) : (\exists i < j)(n \leq i < j \wedge \delta_i^j(x,y) > 3 \wedge (\forall m)(i \leq m < j \Rightarrow x(m) \neq y(m)))\}.$$

Each D_n is a dense open subset of ${}^2({}^\omega 2)$. Hence $C = \bigcap_{n \in \omega} D_n$ is a comeager subset of ${}^2({}^\omega 2)$.

Suppose $(m_k : k \in \omega)$, g^0, g^1, Φ^0, and Φ^1 have properties 1 - 6 from Theorem 14.12. Then $(\Phi^0(\tilde{0}), \Phi^1(\widetilde{01})) \notin C$.

For any $\mathbf{\Delta}_1^1$ sets B_0 and B_1 with $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \upharpoonright B_0$, $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \upharpoonright B_1$, and $[B_0]_{E_2} = [B_1]_{E_2}$, $B_0 \times_{E_2} B_1 \not\subseteq C$.

E_2 does not have the weak 2-Mycielski property.

Proof. Using Theorem 14.12, if $|s| = |t| = k > 0$ and $s(k-1) \neq t(k-1)$, then $|\delta_{m_{k-1}}^{m_k}(g^0(s), g^1(t)) - \frac{1}{k}| < 2^{-k}$. To see this:

$$|\delta_{m_{k-1}}^{m_k}(g^0(s), g^1(t)) - \frac{1}{k}|$$

$$\leq |\delta_{m_{k-1}}^{m_k}(g^0(s), g^1(t)) - \delta_{m_{k-1}}^{m_k}(g^1(s), g^1(t))| + |\delta_{m_{k-1}}^{m_k}(g^1(s), g^1(t)) - \frac{1}{k}|$$

$$\leq \delta_{m_{k-1}}^{m_k}(g^0(s), g^1(s)) + 2^{-(k+1)} \leq 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}$$

Also if $|s| = |t| = k > 0$ and $s(k-1) = t(k-1)$, then $\delta_{m_{k-1}}^{m_k}(g^0(s), g^1(t)) < 2^{-k}$. To see this:

$$\delta_{m_{k-1}}^{m_k}(g^0(s), g^1(t)) \leq \delta_{m_{k-1}}^{m_k}(g^0(s), g^1(s)) + \delta_{m_{k-1}}^{m_k}(g^1(s), g^1(t)) = 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}$$

D_n is dense open by the same argument as in Theorem 15.2.

Note that if $k > 0$, then

$$\sum_{m_{k-1} \leq m < m_k} \frac{1}{m+1} \geq 2^{-k}$$

because $\frac{1}{k} - 2^{-k} \geq 2^{-(k-1)} - 2^{-k} = 2^{-k}$ and so otherwise $|\delta_{m_{k-1}}^{m_k}(g^0(s), g^1(s)) - \frac{1}{k}| < 2^{-k}$ could not hold.

Therefore if $|s| = |t| > 0$, and $s(k-1) = t(k-1)$, then there must be some m with $m_{k-1} \leq m < m_k$ so that $g^0(s)(m) = g^1(t)(m)$.

Note that for all $s, t \in {}^{k+1}2$ with $k > 0$,

$$\delta_{m_{k-1}}^{m_{k+1}}(g(s), g(t)) = \delta_{m_{k-1}}^{m_k}(g(s), g(t)) + \delta_{m_k}^{m_{k+1}}(g(s), g(t)) \leq \frac{1}{k} + 2^{-k} + \frac{1}{k+1} + 2^{-(k+1)} < 3.$$

Hence for any s, t, if there exists $i < j$ so that $\delta_i^j(g^0(s), g^1(t)) > 3$, then there is some $k \geq 1$ so that $i \leq m_{k-1} < m_k < m_{k+1} \leq j$.

Now by essentially the same argument as in Theorem 15.1, $(\Phi^0(\tilde{0}), \Phi^1(\widetilde{01})) \notin D_{m_0}$. Hence $(\Phi^0(\tilde{0}), \Phi^1(\widetilde{01})) \notin C$.

Now suppose that B_0 and B_1 are some $\mathbf{\Delta}_1^1$ sets so that $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \upharpoonright B_0$, $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \upharpoonright B_1$, and $[B_0]_{E_2} = [B_1]_{E_2}$. By Corollary 14.13, there is a sequence $(m_k : k \in \omega)$, g^0, g^1, Φ^0 and Φ^1 as above so that $\Phi^i[{}^\omega 2] \subseteq B_i$. By the earlier argument, $B_0 \times_{E_2} B_1 \not\subseteq C$. Hence E_2 does not have the weak 2-Mycielski property. □

16. Surjectivity and Continuity Aspects of E_2

Fact 14.14 states that every $\mathbf{\Sigma}_1^1$ set $B \subseteq {}^\omega 2$ so that $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \upharpoonright B$ has a closed set $C \subseteq B$ so that $E_2 \equiv_{\mathbf{\Delta}_1^1} E_2 \upharpoonright C$. Fact 14.14 even asserts that C is the body of a tree on 2 with a specific structure:

Definition 16.1. A tree $p \subseteq {}^{<\omega}2$ is an E_2-tree if and only if there is some sequence $(m_k : k \in \omega)$ and map $g : {}^{<\omega}2 \to {}^{<\omega}2$ satisfying the conditions of Fact 14.14 so that p is the downward closure of $g[{}^{<\omega}2]$. Note that if Φ is the map associated with $(m_k : k \in \omega)$ and g, then $[p] = \Phi[{}^\omega 2]$.

The following notation is used to avoid some very tedious superscripts and subscripts in the following results:

Definition 16.2. If $x, y \in {}^\omega 2$ and $m, n \in \omega$ with $m \leq n$, then let $\varsigma(m, n, x, y) = \delta_m^n(x, y)$.

Fact 16.3. *There is a continuous function $P : [{}^\omega 2]_{E_2}^3 \to {}^\omega 3$ so that for any E_2-tree p, $P[[[p]]_{E_2}^3] = {}^\omega 3$.*

Proof. For $(x,y) \in [{}^\omega 2]^2_{E_2}$ and any $n,m \in \omega$, define

$$S_{n,m}(x,y) = \min\{k \in \omega : \delta^k_n(x,y) > 3^{m+2}\}$$

Each $S_{n,m}$ is continuous on $[{}^\omega 2]^3_{E_2}$.

If $(x,y,z) \in [{}^\omega 2]^3_{E_2}$, then define a strictly increasing sequence of integers $(L_n : n \in \omega)$ by recursion as follows: Let $L_0 = 0$. Given L_n, let

$$L_{n+1} = \min\{S_{L_n,n}(x,y), S_{L_n,n}(x,z), S_{L_n,n}(y,z)\}$$

(It is implicit that L_n depends on the triple (x,y,z).) By induction, it can be shown that each L_n as a function of (x,y,z) is continuous on $[{}^\omega 2]^3_{E_2}$.

Define

$$P(x,y,z)(n) = \begin{cases} 0 & S_{L_n,n}(x,y) \leq S_{L_n,n}(x,z) \text{ and } S_{L_n,n}(x,y) \leq S_{L_n,n}(y,z) \\ 1 & S_{L_n,n}(x,z) < S_{L_n,n}(x,y) \text{ and } S_{L_n,n}(x,z) \leq S_{L_n,n}(y,z) \\ 2 & S_{L_n,n}(y,z) < S_{L_n,n}(x,y) \text{ and } S_{L_n,n}(y,z) < S_{L_n,n}(x,z) \end{cases}$$

P is continuous on $[{}^\omega 2]^3_{E_2}$.

Also define the sequence of integers $(N_n : n \in \omega)$ by recursion as follows: Let $N_0 = 0$ and if N_n has been defined, then let

$$N_{n+1} = \min\left\{k \in \omega : \sum_{N_i \leq i < k} \left(\frac{1}{i+1} - 2^{-(i+2)}\right) > 3^{n+2}\right\}$$

Note that $N_{n+1} > N_n + 2$ for each $n \in \omega$. By the definition of N_{n+1}, one has that

$$\sum_{N_n \leq i < N_{n+1}-1} \left(\frac{1}{i+1} - 2^{-i-2}\right) \leq 3^{n+2}.$$

These two facts imply

(1) $$\sum_{N_n \leq i < N_{n+1}} \frac{1}{i+1} \leq 3^{n+2} + \frac{1}{N_{n+1}} + \sum_{N_n \leq i < N_{n+1}-1} 2^{-i-2} < 3^{n+2} + 1$$

Let $k_n = N_{n+1} - N_n$. Fix a $v \in {}^\omega 3$. Define $\sigma_n, \tau_n \in {}^{k_n}2$ by

$$\sigma_n = \begin{cases} \tilde{1} \restriction k_n & v(n) = 0 \\ \widetilde{01} \restriction k_n & \text{otherwise} \end{cases} \qquad \tau_n = \begin{cases} \tilde{1} \restriction k_n & v(n) = 1 \\ \widetilde{10} \restriction k_n & \text{otherwise} \end{cases}$$

Let $x = \tilde{0}$, $y = \sigma_0\hat{\ }\sigma_1\hat{\ }\sigma_2\hat{\ }...$, and $z = \tau_0\hat{\ }\tau_1\hat{\ }\tau_2....$ Note $(x,y,z) \in [{}^\omega 2]^3_{E_2}$.

Fix an E_2-tree p. Let $(m_k : k \in \omega)$, $g : {}^{<\omega}2 \to {}^{<\omega}2$, and $\Phi : {}^\omega 2 \to {}^\omega 2$ be the associated objects of p coming from the definition of an E_2-tree.

Suppose $v(n) = 0$, then

(2)
$$\varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(y)) = \sum_{N_n \leq i < N_{n+1}} \varsigma(m_i, m_{i+1}, \Phi(x), \Phi(y)) > \sum_{N_n \leq i < n_{n+1}} \left(\frac{1}{i+1} - 2^{-i-2}\right) > 3^{n+2}$$

using the definition of N_{n+1}. Also

(3) $$\varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(y)) < \sum_{N_n \leq i < N_{n+1}} \left(\frac{1}{i+1} + 2^{-n-2}\right) < 3^{n+2} + 1 + \sum_{N_n \leq i < N_{n+1}} 2^{-i-2} < 3^{n+2} + \frac{3}{2}$$

using equation (1) for the second inequality.

Note also

$$\varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(z)) = \varsigma(m_{N_n}, m_{N_n+1}, \Phi(x), \Phi(y)) + \sum_{N_n < i < N_{n+1}} \varsigma(m_i, m_{i+1}, \Phi(x), \Phi(z))$$

$$< \frac{1}{N_n+1} + 2^{-N_n-2} + \frac{1}{2} \sum_{N_n < i < N_{n+1}} \frac{1}{i+1} + \sum_{N_n < i < N_{n+1}} 2^{-i-2}$$

$$= \frac{1}{2} \sum_{N_n \leq i < N_{n+1}} \frac{1}{i+1} + \frac{1}{2} \left(\frac{1}{N_n + 1} \right) + \sum_{N_n \leq i < N_{n+1}} 2^{-i-2} < \frac{1}{2}(3^{n+2}) + \frac{3}{2}$$

using equation (1). In summary,

$$\varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(z)) < \frac{1}{2}(3^{n+2}) + \frac{3}{2} \tag{4}$$

Similarly, $\varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(y), \Phi(z)) < \frac{1}{2}(3^{n+2}) + \frac{3}{2}$. The case for $v(n) = 1$ and $v(n) = 2$ are similar.

It remains to show that $P(\Phi(x), \Phi(y), \Phi(z)) = v$. In the following, let $(L_n : n \in \omega)$ be the sequence defined as above using $(\Phi(x), \Phi(y), \Phi(z))$. The following statements will be proved by induction on n:
(I) $m_{N_n} < L_{n+1} \leq m_{N_{n+1}}$.
(II) $P(\Phi(x), \Phi(y), \Phi(z))(n) = v(n)$.
(III) The following holds:

$$\max\{\varsigma(L_{n+1}, m_{N_{n+1}}, \Phi(x), \Phi(y)), \varsigma(L_{n+1}, m_{N_{n+1}}, \Phi(x), \Phi(z)), \varsigma(L_{n+1}, m_{N_{n+1}}, \Phi(y), \Phi(z))\} < \frac{1}{2}(3^{n+2}) + \frac{3}{2}.$$

Suppose properties (I), (II), and (III) holds for all $k < n$. Suppose $v(n) = 0$. (The other cases are similar.)
$L_n \leq m_{N_n}$ by definition if $n = 0$ and by the induction hypothesis otherwise. Therefore,

$$\varsigma(L_n, m_{N_{n+1}}, \Phi(x), \Phi(y)) \geq \varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(y)) > 3^{n+2}$$

using equation (2). This shows $L_{n+1} \leq S_{L_n,n}(\Phi(x), \Phi(y)) \leq m_{N_{n+1}}$. Using the induction hypothesis or the definition when $n = 0$,

$$\max\{\varsigma(L_n, m_{N_n}, \Phi(x), \Phi(y)), \varsigma(L_n, m_{N_n}, \Phi(x), \Phi(z)), \varsigma(L_n, m_{N_n}, \Phi(y), \Phi(z))\} < \frac{1}{2}(3^{n+1}) + \frac{3}{2} < 3^{n+2}.$$

Hence $L_{n+1} \leq m_{N_n}$ is impossible. This proves (I).

Observe that

$$\begin{aligned}\varsigma(L_n, m_{N_{n+1}}, \Phi(x), \Phi(z)) &= \varsigma(L_n, m_{N_n}, \Phi(x), \Phi(z)) + \varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(z)) \\ &< \frac{1}{2}(3^{n+1}) + \frac{3}{2} + \frac{1}{2}(3^{n+2}) + \frac{3}{2} \leq 3^{n+2}\end{aligned}$$

using the induction hypothesis and equation (4). This shows $S_{L_n,n}(\Phi(x), \Phi(z)) > m_{N_{n+1}}$. Similarly, $S_{L_n,n}(\Phi(y), \Phi(z)) > m_{N_{n+1}}$. $S_{L_n,n}(\Phi(x), \Phi(y)) \leq m_{N_{n+1}}$ has already been shown above. Thus

$$P(\Phi(x), \Phi(y), \Phi(z))(n) = 0 = v(n).$$

This shows (II).

Note that

$$\varsigma(L_{n+1}, m_{N_{n+1}}, \Phi(x), \Phi(z)) \leq \varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(z)) < \frac{1}{2}(3^{n+2}) + \frac{3}{2}$$

using equation (4). Similarly,

$$\varsigma(L_{n+1}, m_{N_{n+1}}, \Phi(y), \Phi(z)) < \frac{1}{2}(3^{n+2}) + \frac{3}{2}.$$

Finally,

$$\begin{aligned}\varsigma(L_n, m_{N_{n+1}}, \Phi(x), \Phi(y)) &= \varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(y)) - \big(\varsigma(L_n, L_{n+1}, \Phi(x), \Phi(y)) - \varsigma(L_n, m_{N_n}, \Phi(x), \Phi(y))\big) \\ &< 3^{n+2} + \frac{3}{2} - 3^{n+2} + \frac{1}{2}(3^{n+1}) + \frac{3}{2} < \frac{1}{2}(3^{n+2}) + \frac{3}{2}\end{aligned}$$

using equation (3), the definition of the sequence $(L_n : n \in \omega)$, and the induction hypothesis. This proves (III). □

Theorem 16.4. *There is a continuous function $Q : [{}^\omega 2]^3_{E_2} \to {}^\omega 2$ so that for any E_2-tree p, $Q[[[p]]^3_{E_2}] = {}^\omega 2$.*

There is a Δ^1_1 function $K : {}^3({}^\omega 2) \to {}^\omega 2$ so that on any $\boldsymbol{\Sigma}^1_1$ set A with $E_2 \leq_{\boldsymbol{\Delta}^1_1} E_2 \upharpoonright A$, $K[[A]^3_{E_2}] = {}^\omega 2$ (and in particular the image meets each E_2-equivalence class).

Proof. Q can be obtained by composing the function from Fact 16.3 with a homeomorphism from ${}^\omega 3 \to {}^\omega 2$.

K can be obtained by mapping elements of ${}^3({}^\omega 2) \setminus [{}^\omega 2]^3_{E_2}$ to $\tilde{0}$ and mapping elements in $[{}^\omega 2]^3_{E_2}$ according to Q. Note that by Fact 14.14, every such set A contains an E_2-tree. □

Lemma 16.5. *Let X and Y be topological spaces. Let $\mathscr{B} \subseteq \mathscr{P}(X)$ be a nonempty family of subsets of X. Let $A \subseteq Y$ be Borel. Suppose $f : X \to Y$ is a Borel function with the property that for all $B \in \mathscr{B}$ and open $U \subseteq X$ with $U \cap B \neq \emptyset$, there exists an $x \in B$ with $f(x) \notin A$ and an $x' \in U \cap B$ with $f(x') \in A$. Then there is a Borel function $g : X \to Y$ such that $g \restriction B$ is not continuous for all $B \in \mathscr{B}$.*

Proof. The assumption above implies that A is Borel but not equal to either $\emptyset$ or Y. The topology of Y is not $\{\emptyset, Y\}$. There exists some $y_1, y_2 \in Y$ and open set $V \subseteq Y$ with $y_1 \in V$ and $y_2 \notin V$. Define

$$g(x) = \begin{cases} y_1 & f(x) \notin A \\ y_2 & f(x) \in A \end{cases}.$$

Suppose there was some $B \in \mathscr{B}$ so that $g \restriction B$ is a continuous function. By the assumptions, there is a $x \in B$ so that $f(x) \notin A$. So $g(x) = y_1$. By continuity, $g^{-1}[V] \cap B$ is a nonempty open set containing $x \in B$. There is some $U \subseteq X$ open so that $g^{-1}[V] \cap B = U \cap B$. By the assumptions, there some $x' \in U \cap B$ so that $f(x') \in A$. Hence $g(x') = y_2 \notin V$. Contradiction. □

Fact 16.6. *There is a $\mathbf{\Delta}_1^1$ function $P' : [{}^\omega 2]^3_{E_2} \to {}^\omega 3$ so that on any E_2-tree p, $P' \restriction [[p]]^3_{E_2}$ is not continuous.*

Proof. This result is proved by applying Lemma 16.5 to $X = [{}^\omega 2]^3_{E_2}$, $Y = {}^\omega 3$, $\mathscr{B} = \{[[p]]^3_{E_2} : p \text{ is an } E_2\text{-tree}\}$, f is the function P from Fact 16.3, and $A = \{z \in {}^\omega 3 : (\exists k)(\forall n > k)(z(n) = 0)\}$. It remains to show that these objects satisfy the required properties of Lemma 16.5.

Fix an E_2-tree p. Let $(m_k : k \in \omega)$, $g : {}^{<\omega}2 \to {}^{<\omega}2$, and Φ be the objects associated with p from the definition of an E_2-tree. Fact 16.3 implies $P[[[p]]^3_{E_2}] = {}^\omega 3$. Hence there is some $(x, y, z) \in [[p]]^3_{E_2}$ so that $P(x, y, z) \notin A$. Let $U \subseteq [{}^\omega 2]^3_{E_2}$ be open so that $U \cap [[p]]^3_{E_2} \neq \emptyset$. There are some $s, t, u \in {}^{<\omega}2$ so that $\emptyset \neq N_{s,t,u} \cap [[p]]^3_{E_2} \subseteq U \cap [[p]]^3_{E_2}$. Let $x' = s\hat{\ }\tilde{0}$, $y' = t\hat{\ }\tilde{1}$, and $z' = u\hat{\ }\widetilde{01}$. Using the computation from the proof of Fact 16.3, if k is chosen so that $L_k \geq m_{|s|}$, then for all $n > k$, $P(\Phi(x'), \Phi(y'), \Phi(z'))(n) = 0$. Hence $(\Phi(x'), \Phi(y'), \Phi(z')) \in U \cap [[p]]^3_{E_2}$ and $P(\Phi(x'), \Phi(y'), \Phi(z')) \in A$. □

Theorem 16.7. *There is a $\mathbf{\Delta}_1^1$ function $K : {}^3({}^\omega 2) \to {}^\omega 2$ so that for any $\mathbf{\Sigma}_1^1$ set A with $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \restriction A$, $K \restriction A$ is not continuous.*

Proof. Use the usual arguments to adjust the domain and range of the function from Fact 16.6. Then apply Fact 14.14. □

Corollary 16.8. *E_2 does not have the 3-Mycielski proprety.*

Proof. Let $C \subseteq {}^3({}^\omega 2)$ be any comeager set so that $K \restriction C$ is continuous. Then C witnesses the failure of the 3-Mycielski property for E_2. □

17. The Structure of E_3

This section will give the characterization of E_3-big $\mathbf{\Sigma}_1^1$ sets coming from its dichotomy result. See the references mentioned below for the details.

Definition 17.1. E_3 is the equivalence relation on ${}^\omega({}^\omega 2)$ defined by $x \; E_3 \; y$ if and only if $(\forall n)(x(n) \; E_0 \; y(n))$.

Definition 17.2. Let $\langle \cdot, \cdot \rangle : {}^2\omega \to \omega$ be some recursive pairing function.

Let $\pi_1, \pi_2 : {}^2\omega \to \omega$ be projections onto the first and second coordinate, respectively.

Let $A \subseteq \omega$. Define $\text{dom}(A) = \{(i, j) : \langle i, j \rangle \in A\}$.

If $A \subseteq \omega$ is finite, let $L(A) = \sup \pi_1[\text{dom}(A)]$.

If $s \in {}^n 2$, let $\text{grid}(s) : \text{dom}(n) \to 2$ be defined by $\text{grid}(s)(i, j) = s(\langle i, j \rangle)$.

Definition 17.3. $\mathbb{Z}_2$ is the group $(2, +^{\mathbb{Z}_2}, 0)$ where $+^{\mathbb{Z}_2}$ is modulo 2 addition and 0 denotes the identity element.

Let ${}^\omega\mathbb{Z}_2 = ({}^\omega 2, +^{{}^\omega\mathbb{Z}_2}, \tilde{0})$ where $+^{{}^\omega\mathbb{Z}_2}$ is the coordinate-wise addition of $+^{\mathbb{Z}_2}$ and $\tilde{0}$ is the constant 0 function.

Let ${}^\omega({}^\omega\mathbb{Z}_2) = ({}^\omega({}^\omega 2), +^{{}^\omega({}^\omega\mathbb{Z}_2)}, \bar{0})$ where $+^{{}^\omega({}^\omega\mathbb{Z}_2)}$ is the coordinate-wise addition of $+^{{}^\omega\mathbb{Z}_2}$ and $\bar{0} \in {}^\omega({}^\omega 2)$ is defined by $\bar{0}(k)(j) = 0$ for all $k, j \in \omega$.

Let $Z \subseteq {}^\omega 2$ be defined by $Z = \{x \in {}^\omega 2 : (\exists k)(\forall j > k)(x(j) = 0)\}$.

$\bigoplus_{n \in \omega} \mathbb{Z}_2 = (Z, +^{{}^\omega\mathbb{Z}_2}, \tilde{0})$ is the ω-direct product of $\mathbb{Z}_2$.

${}^{\omega}(\bigoplus_{n\in\omega}\mathbb{Z}_2) = ({}^{\omega}Z, +^{{}^{\omega}({}^{\omega}\mathbb{Z}_2)}, \bar{0})$ is the ω-product of $\bigoplus_{n\in\omega}\mathbb{Z}_2$.

If $g \in {}^{\omega}(\bigoplus_{n\in\omega}\mathbb{Z}_2)$, define $\text{supp}(g) = \{n \in \omega : g(n) \neq \tilde{0}\}$.

${}^{\omega}(\bigoplus_{n\in\omega}\mathbb{Z}_2)$ acts on ${}^{\omega}({}^{\omega}2)$ by left addition when ${}^{\omega}({}^{\omega}2)$ is considered as ${}^{\omega}({}^{\omega}\mathbb{Z}_2)$. That is, $g \cdot x = g +^{{}^{\omega}({}^{\omega}\mathbb{Z}_2)} x$.

Fact 17.4. *$x\ E_3\ y$ if and only if there is a $g \in {}^{\omega}(\bigoplus_{n\in\omega}\mathbb{Z}_2)$ so that $x = g \cdot y$.*

Definition 17.5. A grid system is a sequence $(g_{s,t} : s,t \in {}^{<\omega}2 \wedge |s| = |t|\})$ in ${}^{\omega}(\bigoplus_{n\in\omega}\mathbb{Z}_2)$ with the following properties:

(I) If $s, t, u \in {}^{n}2$ for some $n \in \omega$, then $g_{s,u} = g_{t,u} +^{{}^{\omega}(\bigoplus_{n\in\omega}\mathbb{Z}_2)} g_{s,t}$.

(II) For all $m \leq n$, $s, t \in {}^{n}2$, $u, v \in {}^{m}2$ and $l \in \pi_1[\text{dom}(n)]$ with $u \subseteq s$, and $v \subseteq t$, if

$$\text{grid}(s) \restriction \text{dom}(n \setminus m) \cap ((l+1) \times \omega) = \text{grid}(t) \restriction \text{dom}(n \setminus m) \cap ((l+1) \times \omega)$$

then for all $i \leq l$, $g_{s,t}(i) = g_{u,v}(i)$.

Fact 17.6. *Let $B \subseteq {}^{\omega}({}^{\omega}2)$ be $\mathbf{\Sigma}^1_1$ so that $E_3 \restriction B \equiv_{\mathbf{\Delta}^1_1} E_3$. Then there is a continuous injective map $\Phi : {}^{\omega}({}^{\omega}2) \to {}^{\omega}({}^{\omega}2)$, a grid system $(g_{s,t} : s,t \in {}^{<\omega}2 \wedge |s| = |t|)$, and sequences $(k_i : i \in \omega)$ and $(p_{m,i} : m, i \in \omega)$ in ω with the following properties:*

(i) $\Phi[{}^{\omega}({}^{\omega}2)] \subseteq B$.

(ii) If $s, t \in {}^{n}2$, then $\text{supp}(g_{s,t}) \subseteq (k_{L(n)} + 1)$.

(iii) For each $m \in \omega$, $(k_i : i \in \omega)$ and $(p_{m,i} : i \in \omega)$ are strictly increasing sequences.

(iv) For all $x, y \in {}^{\omega}({}^{\omega}2)$ and $m, j \in \omega$, if $x(m)(j) = 0$ and $y(m)(j) = 1$, then $\Phi(x)(k_m)(p_{m,j}) = 0$ and $\Phi(y)(k_m)(p_{m,j}) = 1$.

(v) Let $x, y \in {}^{\omega}({}^{\omega}2)$ and $l \in \omega$. Suppose

$$\Big(\forall (i,j) \in ((l+1) \times \omega) \setminus dom(n)\Big)\Big(x(i)(j) = y(i)(j)\Big).$$

Let $s, t \in {}^{n}2$ be such that for all $(i,j) \in \text{dom}(n)$, $\text{grid}(s)(i,j) = x(i)(j)$ and $\text{grid}(t)(i,j) = y(i)(j)$. Then $(g_{s,t} \cdot \Phi(x))(l) = \Phi(y)(l)$.

Proof. This is implicit in [8]. See the presentation in [12] Chapter 14, especially Section 14.5 and 14.6. ☐

Note that Φ as above is an E_3 reduction.

18. E_3 Does Not Have the 2-Mycielski Property

Definition 18.1. For each $s \in {}^{<\omega}2$, let $N_{\text{grid}(s)} = \{x \in {}^{\omega}({}^{\omega}2) : (\forall (i,j) \in \text{dom}(|s|))(x(i)(j) = s(\langle i,j \rangle))\}$.

Each $N_{\text{grid}(s)}$ is an open neighborhood of ${}^{\omega}({}^{\omega}2)$ and also the collection $\{N_{\text{grid}(s)} : s \in {}^{<\omega}2\}$ forms a basis for the topology of ${}^{\omega}({}^{\omega}2)$.

When $\sigma : m \to {}^{<\omega}2$, then N_σ will refer to the usual basic open neighborhood of ${}^{\omega}({}^{\omega}2)$. Both types of open sets will be used in the proof of the following result.

Theorem 18.2. *Let $D = \{(x,y) \in {}^{2}({}^{\omega}({}^{\omega}2)) : x(0) \neq y(0)\}$. D is dense open.*

For all $\mathbf{\Sigma}^1_1$ sets $B \subseteq {}^{\omega}({}^{\omega}2)$ with $E_3 \restriction B \equiv_{\mathbf{\Delta}^1_1} E_3$, $[B]^2_{E_3} \not\subseteq D$.

E_3 does not have the 2-Mycielski property.

Proof. Suppose $(x,y) \in D$. There is some n so that $x(0)(n) \neq y(0)(n)$. Let $\sigma, \tau : 1 \to {}^{<\omega}2$ be defined by $\sigma(0) = x(0) \restriction (n+1)$ and $\tau(0) = y(0) \restriction (n+1)$. Then $(x,y) \in N_{\sigma,\tau} \subseteq D$. D is open.

Suppose $\sigma, \tau : m \to {}^{<\omega}2$ have the property that for all $k < m$, $|\sigma(k)| = |\tau(k)|$. Define $\sigma', \tau' : m \to {}^{<\omega}2$ by

$$\sigma'(k) = \begin{cases} \sigma(k) & k \neq 0 \\ \sigma(k)\hat{}0 & k = 0 \end{cases} \quad \text{and} \quad \tau'(k) = \begin{cases} \tau(k) & k \neq 0 \\ \tau(k)\hat{}1 & k = 0 \end{cases}$$

$N_{\sigma',\tau'} \subseteq N_{\sigma,\tau}$ and $N_{\sigma',\tau'} \subseteq D$. D is dense open.

Fix Φ and the other objects specified by Fact 17.6. Note that for any $s \in {}^{<\omega}2$, $g_{s,s} = \bar{0}$. In particular, $g_{\emptyset,\emptyset} = \bar{0}$.

Let $\rho_n : 1 \to {}^{<\omega}2$ be defined by $\rho_n(0) = \Phi(\bar{0})(0) \restriction n$. If $s \in {}^{<\omega}2$, then define $x_s \in {}^{\omega}({}^{\omega}2)$ by

$$x_s(i)(j) = \begin{cases} s(\langle i,j \rangle) & (i,j) \in \text{dom}(|s|) \\ 0 & \text{otherwise} \end{cases}.$$

Let $s_0 = \emptyset$.

Suppose $s_n \in {}^{<\omega}2$ has been defined so that $x_{s_n}(0) = \tilde{0}$ and $\Phi(x_{s_n})(0) = \Phi(\bar{0})(0)$. By continuity, find some $u \in {}^{<\omega}2$ with $s_n \subseteq u$ and $x_{s_n} \in N_{\text{grid}(u)}$ so that $N_{\text{grid}(u)} \subseteq \Phi^{-1}[N_{\rho_{n+1}}]$. Now find the least $k > |u|$ so that $k = \langle 1, q\rangle$ for some $q \in \omega$. Let $s_{n+1} \supseteq u$ be of length $k+1$ defined by

$$s_{n+1}(j) = \begin{cases} u(j) & j < |u| \\ 1 & j = k \\ 0 & \text{otherwise} \end{cases}.$$

Note that since $x_{s_{n+1}}(0) = \tilde{0} = \bar{0}(0)$, $(g_{\emptyset,\emptyset} \cdot \Phi(x_{s_{n+1}}))(0) = \Phi(\bar{0})(0)$ by condition (v) of Definition 17.6. This implies that $\Phi(x_{s_{n+1}})(0) = \Phi(\bar{0})(0)$.

Now define $x \in {}^{\omega}({}^{\omega}2)$ by

$$x(i)(j) = \left(\bigcup_{n\in\omega} \text{grid}(s_n)\right)(i,j).$$

Since $N_{\text{grid}(s_n)} \subseteq \Phi^{-1}[N_{\rho_n}]$ for all $n \in \omega$, $\Phi(x)(0) = \Phi(\bar{0})(0)$. Hence $(\Phi(x), \Phi(\bar{0})) \notin D$. However, there are infinitely many $q \in \omega$ so that $x(1)(q) = 1$. Since Φ is an E_3 reduction, $\neg(\Phi(x)\ E_3\ \Phi(\bar{0}))$.

It has been shown that for any map Φ as in Fact 17.6, $[\Phi[{}^{\omega}({}^{\omega}2)]]^2_{E_3} \not\subseteq D$. Since any $\mathbf{\Sigma}^1_1$ set $B \subseteq {}^{\omega}({}^{\omega}2)$ with the property that $E_3 \upharpoonright B \equiv_{\mathbf{\Delta}^1_1} E_3$ has some such map Φ so that $\Phi[{}^{\omega}({}^{\omega}2)] \subseteq B$, no such B can have the property that $[B]^2_{E_3} \subseteq D$. E_3 does not have the 2-Mycielski property. ☐

19. Completeness of Ultrafilters on Quotients

Without the axiom of choice, the notion of completeness of ultrafilters needs to be defined with care.

Definition 19.1. Let X be a set. Let U be an ultrafilter on X. Let I be a set. U is I-complete if and only if for any set J which inject into I but is not in bijection with I, and any injective function $f : J \to U$, $\bigcap_{j\in J} f(j) \in U$.

U is I^+-complete if and only if for all J which inject into I and all injective functions $f : J \to U$, $\bigcap_{j\in J} f(j) \in U$.

$\aleph_1$-complete is often called countably complete. A well-known result is that there are no countably complete ultrafilters on ${}^{\omega}2$. There are countably complete ultrafilters on quotients of Polish spaces by equivalence relations.

Fact 19.2. *Let $\mathcal{C} \subseteq \mathcal{P}({}^{\omega}2/E_0)$ be defined by $A \in \mathcal{C}$ if and only if $\bigcup A$ belongs to the comeager filter on ${}^{\omega}2$. $\mathcal{C}$ is a countably complete ultrafilter on ${}^{\omega}2/E_0$.*

Proof. $\mathcal{C}$ is an ultrafilter follows from the generic ergodicity of E_0. Countable completeness is clear; in fact under AD, every ultrafilter is countably complete. ☐

A natural question is whether this ultrafilter or any ultrafilter on ${}^{\omega}2/E_0$ could be more than just countably complete. $\mathbb{R}$ injects into ${}^{\omega}2/E_0$. Is $\mathcal{C}$ $\mathbb{R}^+$-complete? Note the function f in Definition 19.1 is required to be injective. Otherwise this notion becomes clearly trivial using the function $f : \mathbb{R} \to \mathcal{C}$ defined by $x \mapsto ({}^{\omega}2/E_0)\setminus\{[x]_{E_0}\}$. The next fact will show using a modification of the above function that there are no nonprincipal $\mathbb{R}^+$-complete ultrafilters on quotients of Polish spaces.

Fact 19.3. $(\mathsf{ZF} + \mathsf{AD})$ *Suppose E is an equivalence relation on a Polish space X so that $= \leq E$ (where $\leq$ denotes the existence of a reduction). Then no nonprincipal ultrafilter on X/E is $\mathbb{R}^+$-complete.*

Proof. Let U be a nonprincipal $(\mathbb{R})^+$-complete ultrafilter on X/E. Let $\Psi : {}^{\omega}2 \to X$ be a reduction witnessing $= \leq E$. Let $\Phi : {}^{\omega}2 \to X/E$ be defined by $\Phi(x) = [\Psi(x)]_E$. Φ is an injective function.

Let $\tilde{L} = (X/E) \setminus \Phi[{}^{\omega}2] = \bigcap_{x\in{}^{\omega}2}(X/E) \setminus \{\Phi(x)\}$. $\tilde{L} \in U$ since U is both nonprincipal and $\mathbb{R}^+$-complete.

Let $L = \bigcup \tilde{L}$. L must be uncountable. Hence L is in bijection with ${}^{\omega}2$. Define $f : L \to (X/E)$ by

$$f(x) = (X/E) \setminus \{[x]_E, \Phi(x)\}$$

To show f is injective, it suffices to show that the map on L defined by $x \mapsto \{[x]_E, \Phi(x)\}$ is injective: Suppose $x \neq y$ and $\{[x]_E, [\Psi(x)]_E\} = \{[y]_E, [\Psi(y)]_E\}$. Since Ψ is a reduction, $\neg(\Psi(x)\ E\ \Psi(y))$. Therefore, one must have that $x\ E\ \Psi(y)$. This is impossible since $[x]_E \in (X/E) \setminus \Phi[^\omega 2]$. This shows f is injective.

Since for all $x \in L$, $[x]_E \notin f(x)$, $\tilde{L} \cap \bigcap_{x \in L} f(x) = \emptyset$. Since $\tilde{L} \in U$, $\bigcap_{x \in L} f(x) \notin U$. U is not $\mathbb{R}^+$-complete. Contradiction. □

Fact 19.4. ($\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$ *or* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$) *Let X be a Polish space and E be an equivalence relation on $^\omega 2$. If $^\omega 2/E$ is not well-ordered, then there is no $\mathbb{R}^+$-complete nonprincipal ultrafilter on X/E.*

Proof. Under $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$, results of Woodin and Martin show that every set of reals is κ-Suslin for some $\kappa < \Theta$. So the complement of E is κ-Suslin for some $\kappa < \Theta$. In $\mathsf{ZF} + \mathsf{AD}$, [5] showed that if the complement of E is κ-Suslin, then either the identity reduces into E or $^\omega 2/E$ is in bijection with a cardinal less than or equal to κ (and hence can be well-ordered).

Under $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$, [2] Theorem 1.4 (along with [2] Corollary 3.2) states that for any set X, either X is wellordered or $\mathbb{R}$ injects into X.

In either case, the result now follows from 19.3. □

20. Conclusion

This section includes some questions.

Question 20.1. Under $\mathsf{ZF} + \neg\mathrm{AC}_\omega^\mathbb{R}$, can there be ω-Jónsson functions for $^\omega 2$?

In particular, is there an ω-Jónsson function for $^\omega 2$ in the Cohen-Halpern-Lévy model H (see Question 4.2)?

Question 20.2. It was shown that E_2 does not have the 2-Mycielski property. An interesting question would be: what is the relation between the n-Mycielski property, n-Jónsson property, and the surjectivity properties in dimension n for E_2?

In particular, does the 2-dimensional version of the results in Section 16 hold?

Does $^\omega 2/E_2$ have the 2-Jónsson property, 3-Jónsson property, or full Jónsson property?

Question 20.3. For E_1 and E_3, this paper only considers the Mycielski property. One can ask about some of the other properties of E_1 or E_3 which had been studied for E_0 and E_2. For example:

Does $^\omega(^\omega 2)/E_1$ or $^\omega(^\omega 2)/E_3$ have the Jónsson property?

Question 20.4. Assuming determinacy, if R/E_0 injects into a set X, can X have the Jónsson property? More specifically, if E is a $\boldsymbol{\Delta}^1_1$ equivalence relation on $\mathbb{R}$ so that $E_0 \leq_{\boldsymbol{\Delta}^1_1} E$, can $\mathbb{R}/E$ have the Jónsson property?

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