DEFINABLE COMBINATORICS OF STRONG PARTITION CARDINALS

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Abstract. This is a draft containing a few sections with results referenced by [3] and [4].

1. Introduction

2. Partition Properties

This section will introduce the partition property, the partition measures, and some purely combinatorial consequences of partition properties. This section will work under ZF.

Definition 2.1. Let ON be the collection of ordinals. Let $\epsilon \in \text{ON}$ be an ordinal. A function $f : \epsilon \to \text{ON}$ has uniform cofinality ω if and only if there is a function $F : \epsilon \times \omega \to \text{ON}$ so that for all $n \in \omega$ and $\alpha < \epsilon$, $F(\alpha, n) < F(\alpha, n+1)$ and $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$. If $f : \epsilon \to \text{ON}$ and $\alpha \leq \epsilon$, then let $\sup(f \upharpoonright \alpha) = 0$ if $\alpha = 0$ and $\sup(f \upharpoonright \alpha) = \sup\{f(\beta) : \beta < \alpha\}$ if $\alpha > 0$. A function $f : \epsilon \to \text{ON}$ is discontinuous everywhere if and only if for all $\alpha < \epsilon$, $f(\alpha) > \sup(f \upharpoonright \alpha)$. A function $f : \epsilon \to \text{ON}$ is of the correct type if and only if f has uniform cofinality ω and discontinuous everywhere.

If X is a set of ordinals, then let $[X]^\epsilon$ be the collection of increasing functions from ϵ into X. Let $[X]^\epsilon_*$ denote the subset of $[X]^\epsilon$ consisting of those functions of the correct type.

Let κ be a cardinal. A subset $A \subseteq \kappa$ is unbounded in κ if and only if for all $\alpha < \kappa$, there is a $\beta \in A$ with $\alpha < \beta$. A set $A \subseteq \kappa$ is bounded in κ if and only if it is not unbounded in κ. A subset $A \subseteq \kappa$ is closed in κ if and only if for any $B \subseteq A$ so that B is bounded in κ, then $\sup(B) \in A$. A set $C \subseteq \kappa$ is a club subset of κ if and only if C is both closed and unbounded in κ.

Definition 2.2. Let κ be a cardinal, $\epsilon \leq \kappa$, and $\gamma < \kappa$. The ordinary partition relation, $\kappa \to (\kappa)^\epsilon_\gamma$, indicates that for every partition $P : [\kappa]^\epsilon \to \gamma$, there exists an $\alpha < \gamma$ and an $X \subseteq \kappa$ with $|X| = \kappa$ so that for all $f \in [X]^\epsilon$, $P(f) = \alpha$. Let $\kappa \to (\kappa)^\epsilon_{<\kappa}$ be the assertion that for all $\gamma < \kappa$, $\kappa \to (\kappa)^\epsilon_\gamma$ holds.

The correct type partition relation, $\kappa \to_* (\kappa)^\epsilon_\gamma$, indicates that for every $P : [\kappa]^\epsilon_* \to \gamma$, there exists a club $C \subseteq \kappa$ and an $\alpha < \gamma$ so that for all $f \in [C]^\epsilon_*$, $P(f) = \alpha$. In this situation, C is said to be a club homogeneous for P taking value α. Let $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$ be the assertion that for all $\gamma < \kappa$, $\kappa \to_* (\kappa)^\epsilon_\gamma$ holds.

A cardinal κ is called a strong partition cardinal if and only if $\kappa \to_* (\kappa)^\kappa_2$ holds.

Definition 2.3. If κ is a cardinal and $X \subseteq \kappa$ with $|X| = \kappa$. Let $\mathsf{enum}_X : \kappa \to X$ be the increasing enumeration of X. Let $\mathsf{next}_X : \kappa \to X$ be defined by $\mathsf{next}_X(\alpha)$ is the least element of X strictly greater than α. Let $\mathsf{next}^0_X : \kappa \to \kappa$ be defined by $\mathsf{next}^0_X(\alpha) = \alpha$. If $0 < \gamma < \kappa$, then let $\mathsf{next}^\gamma_X : \kappa \to X$ be defined by $\mathsf{next}^\gamma_X(\alpha)$ is the γ^{th} element of X strictly greater than α.

Definition 2.4. Let $\kappa \in \text{ON}$ and $\epsilon \leq \kappa$. Let $\mathsf{block} : {}^{\omega \cdot \epsilon}\text{ON} \to {}^\epsilon\text{ON}$ be defined by $\mathsf{block}(f)(\alpha) = \sup\{f(\omega \cdot \alpha + n) : n \in \omega\}$.

The correct type partition relation will be used in this article. The next result shows that the ordinary and correct type partition relations are closely related.

Fact 2.5. *Let κ be a cardinal and $\epsilon \leq \kappa$.*

(1) $\kappa \to_* (\kappa)^\epsilon_2$ implies $\kappa \to (\kappa)^\epsilon_2$.
(2) $\kappa \to (\kappa)^{\omega \cdot \epsilon}_2$ implies $\kappa \to_* (\kappa)^\epsilon_2$.

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Proof. (1) Suppose $P : [\kappa]^\epsilon \to 2$. By $\kappa \to_* (\kappa)^\epsilon_2$, there is a club $C \subseteq \kappa$ and an $i \in 2$ so that for all $f \in [C]^\kappa_*$, $P(f) = i$. Let $A \subseteq C$ be defined by $A = \{\mathsf{enum}_C(\omega \cdot \beta + \omega) : \beta < \kappa\}$. For each $\gamma \in A$, let β_γ be such that $\mathsf{enum}_C(\omega \cdot \beta_\gamma + \omega) = \gamma$. Suppose $f \in [A]^\epsilon$. Pick an $\alpha < \epsilon$ and let $\delta = \sup\{\beta_{f(\gamma)} + 1 : \gamma < \alpha\}$. Note that $\delta \leq \beta_{f(\alpha)}$. Then $\sup(f \upharpoonright \alpha) \leq \mathsf{enum}_C(\omega \cdot \delta) < \mathsf{enum}_C(\omega \cdot \delta + \omega) \leq \mathsf{enum}_C(\omega \cdot \beta_{f(\alpha)} + \omega) = f(\alpha)$. Thus for all $\alpha < \epsilon$, $\sup(f \upharpoonright \alpha) < f(\alpha)$ and hence f is discontinuous. Define $F : \epsilon \times \omega \to \kappa$ by $F(\alpha, n) = \mathsf{enum}_C(\omega \cdot \beta_{f(\alpha)} + n)$. Since C is a club, for each $\alpha < \epsilon$, $f(\alpha) = \mathsf{enum}_C(\omega \cdot \beta_{f(\alpha)} + \omega) = \sup\{\mathsf{enum}_C(\omega \cdot \beta_{f(\alpha)} + n) : n \in \omega\} = \sup\{F(\alpha, n) : n \in \omega\}$. Thus f has uniform cofinality ω. It has been shown that $[A]^\epsilon = [A]^\epsilon_*$. For any $f \in [A]^\epsilon$, $f \in [A]^\epsilon_* \subseteq [C]^\epsilon_*$ and thus $P(f) = i$. A is homogeneous for P in the ordinary sense.

(2) Suppose $P : [\kappa]^\epsilon_* \to 2$. Define $Q : [\kappa]^{\omega \cdot \epsilon} \to 2$ by $Q(g) = P(\mathsf{block}(g))$. By $\kappa \to (\kappa)^{\omega \cdot \epsilon}_2$, there is an $A \subseteq \kappa$ with $|A| = \kappa$ and an $i \in 2$ so that for all $g \in [\kappa]^{\omega \cdot \epsilon}$, $Q(g) = i$. Let $C = \{\alpha < \kappa : \sup(A \cap \alpha) = \alpha\}$ be the club of limit points of A. Suppose $f \in [C]^\epsilon_*$ and let $F : \epsilon \times \omega \to \kappa$ witness that f has uniform cofinality ω. Define $g : \omega \cdot \epsilon \to A$ as follows: For each $\alpha < \epsilon$, let $g(\omega \cdot \alpha) = \mathsf{next}_A(\max\{\sup(f \upharpoonright \alpha), F(\alpha, 0)\})$. If $g(\omega \cdot \epsilon + n)$ has been defined, then let $g(\omega \cdot \epsilon + n + 1) = \mathsf{next}_A(\max\{g(\omega \cdot \alpha + n), F(\alpha, n + 1)\})$. Note that $\mathsf{block}(g) = f$. Since $g \in [A]^{\omega \cdot \epsilon}$, $Q(g) = i$ and hence $P(\mathsf{block}(g)) = P(f) = i$. It has been shown that for all $f \in [C]^\epsilon_*$, $P(f) = i$ and thus C is homogeneous for P in the sense of the correct type partition relation. □

The correct type partition relation is preferable over the ordinary partition relation because it is directly related to partition measures.

Definition 2.6. Let κ be a regular cardinal and $\epsilon \leq \kappa$. Let μ^κ_ϵ be the filter on $[\kappa]^\epsilon_*$ defined by $X \in \mu^\kappa_\epsilon$ if and only if there exists a club $C \subseteq \kappa$ so that $[C]^\epsilon_* \subseteq X$. If μ^κ_ϵ is an ultrafilter, then it will be called the ϵ-partition measure on κ.

Fact 2.7. *If $\kappa \to_* (\kappa)^\epsilon_2$, then μ^κ_ϵ is an ultrafilter.*

Fact 2.8. *Let κ be cardinal and $\epsilon < \kappa$. If $\kappa \to_* (\kappa)^{\epsilon + \epsilon}_2$, then $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$.*

Proof. Let $\lambda < \kappa$ and $\Phi : [\kappa]^\epsilon_* \to \lambda$. If $h \in [\kappa]^{\epsilon+\epsilon}_*$, then let $h^0 \in [\kappa]^\epsilon_*$ and $h^1 \in [\kappa]^\epsilon_*$ be defined by $h^0(\alpha) = h(\alpha)$ and $h^1(\alpha) = h(\epsilon + \alpha)$. Define $P : [\kappa]^{\epsilon+\epsilon} \to 2$ by $P(h) = 0$ if and only if $\Phi(h^0) = \Phi(h^1)$. By $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$, there is a club $C \subseteq \kappa$ which is homogeneous for P. Suppose that C is homogeneous for P taking value 1. Let $Q : [C]^{\epsilon+\epsilon}_* \to 2$ be defined by $Q(h) = 0$ if and only if $Q(h^0) < Q(h^1)$. By $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$, there is a club $D \subseteq C$ which is homogeneous for Q. Let $E = \{\mathsf{enum}_D(\omega \cdot \alpha + \omega) : \alpha \in \kappa\}$ and note that $[E]^\epsilon = [E]^\epsilon_*$. For each $\delta < \kappa$, let $g_\delta \in [E]^\epsilon_*$ be defined by $g_\delta(\alpha) = \mathsf{enum}_E(\epsilon \cdot \delta + \alpha)$. For each $\delta_0 < \delta_1 \in \kappa$, let $h_{\delta_0,\delta_1} \in [E]^\epsilon_*$ be defined so that $h^0_{\delta_0,\delta_1} = g_{\delta_0}$ and $h^1_{\delta_0,\delta_1} = g_{\delta_1}$.

Suppose D is homogeneous for Q taking value 1. Then $Q(h_{n,n+1}) = 1$ implies that $\Phi(g_{n+1}) < \Phi(g_n)$. This contradicts the wellfoundedness of λ. Suppose D is homogeneous for Q taking value 0. Then for each $\delta_0 < \delta_1$, $Q(h_{\delta_0,\delta_1}) = 0$ and $P(h_{\delta_0,\delta_1}) = 1$ imply that $\Phi(g_{\delta_0}) < \Phi(g_{\delta_1})$. So the map $\Gamma : \kappa \to \lambda$ defined by $\Gamma(\delta) = \Phi(g_\delta)$ is an injection, which is impossible since $\lambda < \kappa$. Thus Q is a partition with no homogeneous club which violates $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$.

Thus C must be homogeneous for P taking value 0. Suppose $f_0, f_1 \in [C]^\epsilon_*$. Let $g \in [C]^\epsilon_*$ so that $\sup(f_0) < g(0)$ and $\sup(f_1) < g(0)$. Let $h_0, h_1 \in [C]^\epsilon_*$ be such that $h^0_0 = f_0$, $h^0_1 = f_1$, and $h^1_0 = h^1_1 = g$. $P(h_0) = P(h_1) = 0$ implies that $\Phi(f_0) = \Phi(g) = \Phi(f_1)$. Thus there is an $\eta < \lambda$ so that for all $f \in [C]^\epsilon_*$, $\Phi(f) = \eta$. □

Fact 2.9. *If $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$, then μ^κ_ϵ is a κ-complete ultrafilter.*

Proof. Suppose μ^κ_ϵ is not κ-complete. Then there is a $\lambda < \kappa$ and a sequence $\langle A_\alpha : \alpha < \lambda \rangle$ in μ^κ_ϵ so that $\bigcap_{\alpha<\lambda} A_\alpha = \emptyset$. Define $\Phi : [\kappa]^\epsilon_* \to \lambda$ by $\Phi(f)$ is the least $\alpha < \lambda$ so that $f \notin A_\alpha$. By Fact 2.9, there is a $\eta < \lambda$ and a club $C \subseteq \kappa$ so that $\Phi[[C]^\epsilon_*] = \{\eta\}$. Then $[C]^\epsilon_* \cap A_\eta = \emptyset$. This contradicts $A_\eta \in \mu^\kappa_\epsilon$. □

An ordinal γ is indecomposable if and only if for all $\alpha < \gamma$ and $\beta < \gamma$, $\alpha + \beta < \gamma$ and $\alpha \cdot \beta < \gamma$. (Here, an indecomposable ordinal is both additively and multiplicatively indecomposable.) Indecomposable ordinals are limit ordinals. If γ is indecomposable, then for each $\alpha < \gamma$, $\mathrm{ot}(\{\beta : \alpha < \beta < \gamma\}) = \gamma$. Also if $\alpha < \gamma$, then $\alpha + \gamma = \gamma$ and $\alpha \cdot \gamma = \gamma$.

The collection of indecomposable ordinals of a cardinal κ is a club subset of κ. The following lemma is useful for thinning out a club C_0 to a club subset $C_1 \subseteq C_0$ such that C_0 is sufficiently dense within C_1 for many constructions.

Lemma 2.10. *Let κ be a cardinal and $C_0 \subseteq \kappa$ is a club consisting entirely of indecomposable ordinals. Then $C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$ is a club subset of C_0 with the property that for all $\delta \in C_1$, $\beta < \delta$, and $\gamma < \delta$, $\mathsf{next}^{\beta}_{C_0}(\gamma) < \delta$.*

Proof. Since $\delta \in C_1 \subseteq C_0$ and C_0 consists entirely of indecomposable ordinals, δ is an indecomposable ordinal. Since $\delta = \mathsf{enum}_{C_0}(\delta)$, δ is a limit point of C_0. Since $\gamma < \delta$, there is an $\eta < \gamma$ so that $\gamma < \mathsf{enum}_{C_0}(\eta) < \mathsf{enum}_{C_0}(\gamma) = \delta$. Then $\mathsf{next}^{\beta}_{C_0}(\gamma) \leq \mathsf{enum}_{C_0}(\eta + \beta) < \mathsf{enum}_{C_0}(\delta) = \delta$ since $\eta + \beta < \delta$ as δ is indecomposable. □

Definition 2.11. Let $\epsilon \in \mathrm{ON}$ and $\delta_0, \delta_1 \leq \epsilon$ be such that $\delta_0 + \delta_1 = \epsilon$. If $f : \epsilon \to \mathrm{ON}$, then define $\mathsf{drop}(f, \delta_0) : \delta_1 \to \mathrm{ON}$ by $\mathsf{drop}(f, \delta_0)(\alpha) = f(\delta_0 + \alpha)$.

Fact 2.12. *Suppose $\epsilon \leq \kappa$, $\kappa \to_* (\kappa)^{1+\epsilon}_2$, and $\kappa \to_* (\kappa)^{\epsilon}_{<\kappa}$. If $\Phi : [\kappa]^{\epsilon}_* \to \kappa$ is a function so that for μ^{κ}_{ϵ}-almost all f, $\Phi(f) < f(0)$, then there is a $\zeta < \kappa$ so that for μ^{κ}_{ϵ}-almost all f, $\Phi(f) = \zeta$.*

Proof. Let $C_0 \subseteq \kappa$ be a club consisting entirely of indecomposable ordinals so that $\Phi(f) < f(0)$ for all $f \in [C_0]^{\epsilon}_*$. Define $P : [C_0]^{1+\epsilon}_* \to 2$ by $P(g) = 0$ if and only if $\Phi(\mathsf{drop}(g, 1)) < g(0)$. By $\kappa \to_* (\kappa)^{1+\epsilon}_2$, let $C_1 \subseteq C_0$ be a club homogeneous for P. Let $C_2 = \{\alpha \in C_1 : \mathsf{enum}_{C_1}(\alpha) = \alpha\}$. Pick an $f \in [C_2]^{\epsilon}_* \subseteq [C_0]^{\epsilon}_*$ and thus $\Phi(f) < f(0)$. Since $f(0) \in C_2$, Lemma 2.10 implies that $\mathsf{next}^{\omega}_{C_1}(\Phi(f)) < f(0)$. Let $g \in [C_0]^{1+\epsilon}_*$ be defined by $g(0) = \mathsf{next}^{\omega}_{C_1}(\Phi(f))$ and for all $\alpha < \epsilon$, $g(1+\alpha) = f(\alpha)$. Then $\Phi(\mathsf{drop}(g, 1)) = \Phi(f) < f(0) = g(0)$ and thus $P(g) = 0$. This shows that C_1 is homogeneous for P taking value 0. For all $f \in [C_2]^{\epsilon}_*$, Lemma 2.10 implies that $\mathsf{next}^{\omega}_{C_1}(0) < f(0)$. Let $g_f \in [C_1]^{1+\epsilon}_*$ be defined by $g_f(0) = \mathsf{next}^{\omega}_{C_1}(0)$ and for all $\alpha < \epsilon$, $g_f(1+\alpha) = f(\alpha)$. $P(g_f) = 0$ implies that $\Phi(g) < \mathsf{next}^{\omega}_{C_1}(0)$. Thus it has been shown that for all $f \in [C_2]^{\epsilon}_*$, $\Phi(f) < \mathsf{next}^{\omega}_{C_1}(0)$. By $\kappa \to_* (\kappa)^{\kappa}_{<\kappa}$, there is a club $C_3 \subseteq C_2$ and a $\zeta < \mathsf{next}^{\omega}_{C_1}(0)$ so that for all $f \in [C_3]^{\epsilon}_*$, $\Phi(f) = \zeta$. □

Fact 2.13. *(Solovay) Suppose κ is a cardinal and $\kappa \to_* (\kappa)^2_2$ holds. Then the ω-club filter on κ, μ^{κ}_1, is a κ-complete normal ultrafilter on κ.*

Proof. Fact 2.9 implies μ^{κ}_1 is κ-complete. Fact 2.8 implies $\kappa \to_* (\kappa)^1_{<\kappa}$. Let $\Phi : \kappa \to \kappa$ be a function which is μ^{κ}_1-almost everywhere regressive. Fact 2.12 implies there is a club $C_0 \subseteq \kappa$ and a $\zeta < \kappa$ so that for all $\beta \in [C_0]^1_*$, $\Phi(\beta) = \zeta$. Thus Φ is constant μ^{κ}_1-almost everywhere. □

Partition properties are useful for analyzing functions on partition spaces to establish properties of cardinalities for these sets. A set X is said to have regular cardinality if and only if there are no sets $Z \subseteq X$ with $|Z| < |X|$ and no family $\langle A_z : z \in Z\rangle$ of subsets of X with $|A_z| < |X|$ for each $z \in Z$ so that $X = \bigcup A_z$. Under AD, $|\mathbb{R}| = |\mathscr{P}(\omega)|$ is a nonwellorderable regular cardinality by the perfect set property. It is open if $|\mathscr{P}(\omega_1)|$ is a nonwellorderable regular cardinality under AD.

Zapletal asked whether a weaker wellordered regularity holds for $\mathscr{P}(\omega_1)$: If κ is an ordinal and $\langle X_\alpha : \alpha < \kappa\rangle$ is a sequence so that $\bigcup_{\alpha<\kappa} X_\alpha = \mathscr{P}(\omega_1)$, then is there an $\alpha < \kappa$ so that $|X_\alpha| = |\mathscr{P}(\omega_1)|$? When $\kappa = \omega_1$, this was solved in [2] as a consequence of the almost everywhere continuity property for ω_1. Since $|\mathscr{P}(\omega_1)| = |[\omega_1]^{\omega_1}| = |[\omega_1]^{\omega_1}_*|$, the presentation as $[\omega_1]^{\omega_1}_*$ will be favored for the sake of the partition property.

Many concrete subsets of $\mathscr{P}(\omega_1)$ are not regular and in fact fail the wellordered regularity of Zapletal's question.

Example 2.14. For $\epsilon < \omega_1$, there is a sequence $\langle X_\alpha : \alpha < \omega_1\rangle$ of subsets of $[\omega_1]^{\epsilon}$ so that $\bigcup_{\alpha<\omega_1} X_\alpha = [\omega_1]^{\epsilon}$ and for all $\alpha < \omega_1$, $|X_\alpha| \leq |\mathbb{R}|$. There is a sequence $\langle X_\alpha : \alpha < \omega_1\rangle$ of subsets of $[\omega_1]^{<\omega_1}$ so that $\bigcup_{\alpha<\omega_1} X_\alpha = [\omega_1]^{<\omega_1}$ and for all $\alpha < \omega_1$, $|X_\alpha| \leq |\mathbb{R}|$.

Proof. As an example, $[\omega_1]^{<\omega_1} = \bigcup_{\alpha \leq \beta < \omega_1} [\beta]^{\alpha}$ and one can check that $|[\beta]^{\alpha}| \leq |\mathbb{R}|$ when $\alpha \leq \beta < \omega_1$. □

If a set is a surjective image of $\mathbb{R}$, then it is a quotient of an equivalence relation on $\mathbb{R}$. The next result shows that any subset of $[\omega_1]^{<\omega_1}$ that contains a copy of $\mathbb{R} \sqcup \omega_1$ is an ω_1-length disjoint union of quotients of $\mathbb{R}$ by an equivalence relation on $\mathbb{R}$ where each quotient is in bijection with $\mathbb{R}$.

Fact 2.15. *([1]) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let $X \subseteq [\omega_1]^{<\omega_1}$. $|\mathbb{R} \sqcup \omega_1| \leq |X|$ if and only if there is a sequence $\langle E_\alpha : \alpha < \omega_1\rangle$ of equivalence relations on $\mathbb{R}$ so that for all $\alpha < \omega_1$, $|\mathbb{R}/E_\alpha| = |\mathbb{R}|$ and $|X| = |\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha|$.*

The proof of Fact 2.15 uses equivalence relations E_α that have at least one class which is uncountable. $\mathbb{R} \times \omega_1$ has a more natural presentation as $|\mathbb{R} \times \omega_1| = |\bigsqcup_{\alpha < \omega_1} \mathbb{R}/E_\alpha|$ where each E_α is the identity relation on $\mathbb{R}$ which has all equivalence classes countable and even size one. $|\mathbb{R} \times \omega_1|$ is the only cardinality obtainable this way by the following result.

Fact 2.16. *([1]) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let $\kappa < \mathrm{ON}$ and $\langle E_\alpha : \alpha < \kappa \rangle$ be a sequence of equivalence relations on $\mathbb{R}$ with all classes countable and $|\mathbb{R}/E_\alpha| = |\mathbb{R}|$. Then $|\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha| = |\mathbb{R} \times \kappa|$.*

Thus $\mathscr{P}(\omega_1)$ is the first natural cardinal after $\mathscr{P}(\omega)$ which could have this wellordered regularity. The next result establishes this for any strong partition cardinal.

Theorem 2.17. *Suppose δ satisfies $\delta \to_* (\delta)^\delta_2$. Let $\kappa \in \mathrm{ON}$. Then for every function $\Phi : [\delta]^\delta \to \kappa$, there is an $\alpha < \kappa$ so that $|\Phi^{-1}[\{\alpha\}]| = |[\delta]^\delta|$.*

Proof. Assume this result is not true. Let κ be the least ordinal so that there is a function $\Phi : [\delta]^\delta \to \kappa$ with the property that for each $\alpha < \kappa$, $|\Phi^{-1}[\{\alpha\}]| < |[\delta]^\delta|$.

Let $\mathcal{L}$ be $\delta \times 2$ with the lexicographic ordering. ($\mathcal{L}$ is isomorphic to δ.) If $F : \mathcal{L} \to \delta$, then let $F_0, F_1 : \delta \to \delta$ be defined by $F_i(\alpha) = F(\alpha, i)$. Define a partition $P_0 : [\delta]^{\mathcal{L}}_* \to 2$ by $P_0(F) = 0$ if and only if $\Phi(F_0) \leq \Phi(F_1)$. By $\delta \to_* (\delta)^\delta_2$, there is a club $C_0 \subseteq \delta$ which is homogeneous for P_0. C_0 must be homogeneous for P_0 taking value 0. To see this, suppose C_0 was homogeneous for P_0 taking value 1. Let $A = \{\mathsf{enum}_C(\omega \cdot \alpha + \omega) : \alpha \in \delta\}$. (Notice that every function $f \in [A]^\delta$ is of the correct type.) Let $g_n(\alpha) = \mathsf{enum}_A(\omega \cdot \alpha + n)$. Let $G^n : \mathcal{L} \to A$ be defined so that $G^n(\alpha, i) = g_{n+i}(\alpha)$ for both $i \in \{0, 1\}$. Note that for each $n \in \omega$, $G^n \in [A]^{\mathcal{L}}_* \subseteq [C_0]^{\mathcal{L}}_*$ and $G^n_i = g_{n+i}$ for $i \in \{0, 1\}$. Since $P_0(G^n) = 1$ for all $n \in \omega$, $\Phi(g_{n+1}) < \Phi(g_n)$. This violates the wellfoundedness of δ.

Now define $P_1 : [C_0]^{\mathcal{L}}_* \to 2$ by $P_1(F) = 0$ if and only if $\Phi(F_0) < \Phi(F_1)$. Again by $\delta \to_* (\delta)^\delta_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for P_1. Again let $A \subseteq C_1$ be defined by $A = \{\mathsf{enum}_{C_1}(\omega \cdot \alpha + \omega) : \alpha < \delta\}$. Let $B = \{\mathsf{enum}_A(\omega \cdot \alpha + i) : i \in \{0, 1\} \wedge \alpha < \delta\}$. Since $|[\delta]^\delta| = |\mathscr{P}(\delta)| = |{}^\delta 2|$, let $\Sigma : [\delta]^\delta \to {}^\delta 2$ be a bijection. Let $h : \delta \to A$ be defined by $h(\alpha) = \mathsf{enum}_A(\omega \cdot \alpha + 2)$. Define $\Psi : [\delta]^\delta \to [B]^\delta$ by $\Psi(f)(\alpha) = \mathsf{enum}_A(\omega \cdot \alpha + \Sigma(f)(\alpha))$. Note that Ψ is an injection. For each $f \in [\delta]^\delta$, let $G^f \in [C_1]^{\mathcal{L}}_*$ be defined by

$$G^f(\alpha, i) = \begin{cases} \Psi(f)(\alpha) & i = 0 \\ h(\alpha) & i = 1 \end{cases}$$

Note that $G^f_0 = \Psi(f)$ and $G^f_1 = h$.

Now suppose C_1 was homogeneous for P_1 taking value 1. One would have $P_1(G^f) = 1$ which implies that $\Phi(\Psi(f)) = \Phi(h)$ (since recall that $C_1 \subseteq C_0$ and C_0 is homogeneous for P_0 taking value 0). Let $\alpha = \Phi(h)$. Since $f \in [\delta]^\delta$ was arbitrary, one has that $\Psi[[\delta]^\delta] \subseteq \Phi^{-1}[\{\alpha\}]$. Since Ψ is an injection, one has that $|\Phi^{-1}[\{\alpha\}]| = |[\delta]^\delta|$. This contradicts the hypothesis on Φ. C_1 must be homogeneous for P_1 taking value 0.

Thus for any $f \in [\delta]^\delta$, $P_1(G^f) = 0$ and this implies that $\Phi(\Psi(f)) < \Phi(h)$. Let $\lambda = \Phi(h)$. Define $\Lambda : [\delta]^\delta \to \lambda$ by $\Lambda(f) = \Phi(\Psi(f))$. Since $\lambda < \kappa$ and κ is minimal with the above property, one has that there is an $\alpha < \lambda$ so that $|\Lambda^{-1}[\{\alpha\}]| = |[\delta]^\delta|$. However since Ψ is an injection and $\Psi[\Lambda^{-1}[\{\alpha\}]] \subseteq \Phi^{-1}[\{\alpha\}]$, one has that $|\Phi^{-1}[\{\alpha\}]| = |[\delta]^\delta|$. This contradicts the assumption on Φ.

Thus P_1 has no homogeneous club, which violates $\delta \to_* (\delta)^\delta_2$. □

Fact 2.18. *If κ has a κ-complete nonprincipal ultrafilter, then for all $\alpha \leq \beta < \kappa$, κ does not inject into ${}^\alpha\beta$, which is the collection of functions from α into β.*

Proof. Let μ be a κ-complete nonprincipal ultrafilter on κ. Let $\alpha \leq \beta < \kappa$. Suppose $\Phi : \kappa \to {}^\alpha\beta$ is an injection. For each $\gamma < \alpha$, by the κ-completeness of μ, there is a $u_\gamma < \beta$ and a set $A_\gamma \in \mu$ so that for all $\xi \in A_\gamma$, $\Phi(\xi)(\gamma) = u_\gamma$. Let $A = \bigcap_{\gamma<\alpha} A_\gamma$ and let $f \in {}^\alpha\beta$ be defined by $f(\gamma) = u_\gamma$. By the κ-completeness of μ, $A \in \mu$ and therefore contains at least two elements since μ is nonprincipal. Let $\xi_0, \xi_1 \in A$ be two distinct elements. Then $\Phi(\xi_0) = f = \Phi(\xi_1)$. This contradicts the injectiveness of Φ. □

The wellordered regularity property (Theorem 2.17) of $[\delta]^\delta$ when δ is a strong partition cardinal yields the following cardinality computation.

Fact 2.19. *Let δ be a cardinal satisfying $\delta \to_* (\delta)^\delta_2$. Then $|[\delta]^{<\delta}| < |[\delta]^\delta| = |\mathscr{P}(\delta)|$.*

Proof. The partition relation $\delta \to_* (\delta)^\delta_2$ implies that μ^δ_1 is a δ-complete nonprincipal measure by Fact 2.9. If $|[\delta]^{<\delta}| = |[\delta]^\delta|$, then let $\Psi : [\delta]^\delta \to [\delta]^{<\delta}$ be an injection. Fix a bijection $\pi : \delta \to \delta \times \delta$. Let $\pi_1, \pi_2 : \delta \times \delta \to \delta$ be the projections onto the first and second coordinate, respectively. Observe that $[\delta]^{<\delta} = \bigcup_{\alpha \leq \beta < \delta} [\beta]^\alpha$ by the regularity of δ. Define $\Phi : [\delta]^\delta \to \delta$ by $\Phi(f)$ is the least γ so that $\Psi(f) \in [\pi_2(\pi(\gamma))]^{\pi_1(\pi(\gamma))}$. By Theorem 2.17, there is an $\gamma < \delta$ so that $|[\pi_2(\pi(\gamma))]^{\pi_1(\pi(\gamma))}| = |\Phi^{-1}[\{\gamma\}]| = |[\delta]^\delta|$. Fact 2.18 implies this is not possible. ☐

Next, a few more club uniformization principles will be defined. Establishing some of these principles under AD at suitable cardinals will be the subject of later sections.

Fact 2.20. *μ^κ_ϵ is κ-complete ultrafilter if and only if $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$.*

Proof. ($\Leftarrow$) This is the argument from Fact 2.9.

($\Rightarrow$) Suppose $\lambda < \kappa$ and $\Phi : [\kappa]^\epsilon_* \to \lambda$. For $\alpha < \lambda$, let $A_\alpha = \Phi^{-1}[\{\alpha\}]$. Since $\bigcup_{\alpha<\lambda} A_\alpha = [\kappa]^\epsilon_*$, the κ-completeness of the ultrafilter μ^κ_ϵ implies that there some $\delta < \lambda$ so that $A_\delta \in \mu^\kappa_\epsilon$. Thus there is a club $C \subseteq \kappa$ with $[C]^\epsilon_* \subseteq A_\delta$. For all $f \in [C]^\epsilon_*$, $\Phi(f) = \delta$. ☐

It is not known if $\kappa \to_* (\kappa)^\kappa_2$ alone is sufficient to prove μ^κ_κ is κ-complete (or equivalently $\kappa \to_* (\kappa)^\kappa_{<\kappa}$). Some authors define κ to be a strong partition cardinal if $\kappa \to_* (\kappa)^\kappa_{<\kappa}$ holds. In this article, a strong partition cardinal will merely satisfy $\kappa \to_* (\kappa)^\kappa_2$. Under AD, Fact **??** uses pointclass arguments to establish an everywhere wellordered club uniformization which will imply in many cases μ^κ_κ is κ-complete.

If κ is a cardinal, then let club_κ denote the set of club subsets of κ. If $f \in [\kappa]^\kappa_*$, then let $\mathcal{C}_f$ be the closure of $f[\kappa]$, which is a club subset of κ. The following club uniformization principle is provable purely from the strong partition relation.

Fact 2.21. *(Almost everywhere fixed short length club uniformization) Suppose $\kappa \to_* (\kappa)^\kappa_2$ and $\epsilon < \kappa$. Let $R \subseteq [\kappa]^\epsilon_* \times \mathsf{club}_\kappa$ be $\subseteq$-downward closed in the club_κ-coordinate, which means that for all $\ell \in [\kappa]^\epsilon_*$, for all clubs $C \subseteq D$, if $R(\ell, D)$ holds, then $R(\ell, C)$ holds. There is a club $C \subseteq \kappa$ so that for all $\ell \in \mathrm{dom}(R) \cap [C]^\epsilon_*$, $R(\ell, C \setminus (\sup(\ell) + 1)$.*

Proof. Define a partition $P : [\kappa]^\kappa_* \to 2$ by $P(f) = 0$ if and only if $f \restriction \epsilon \in \mathrm{dom}(R)$ and $R(f \restriction \epsilon, \mathcal{C}_{\mathsf{drop}(f,\epsilon)})$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $D \subseteq \kappa$ which is homogeneous for P. Pick any $\ell \in \mathrm{dom}(R) \cap [D]^\epsilon_*$. There is a club $E \subseteq D$ so that $R(\ell, E)$. Pick any $h \in [E]^\kappa_*$ with $\sup(\ell) < h(0)$. Since R is $\subseteq$-downward closed, $\mathcal{C}_h \subseteq E$, and $R(\ell, E)$, one has $R(\ell, \mathcal{C}_h)$. Let $f \in [D]^\kappa_*$ be such that $f \restriction \epsilon = \ell$ and $\mathsf{drop}(f, \epsilon) = h$. Then $P(f) = 0$. Thus D is homogeneous for P taking value 0.

Let $h(\alpha) = \mathsf{enum}_D(\omega \cdot \alpha + \omega)$ and note that $h \in [D]^\kappa_*$. Let $C = \mathcal{C}_h$. Pick any $\ell \in [C]^\epsilon_*$. Let $\xi < \kappa$ be least so that $\sup(\ell) < h(\xi)$. Let $f = \ell\hat{\ }\mathsf{drop}(h, \xi)$. Now since $f \in [D]^\kappa_*$, $P(f) = 0$ and $\mathsf{drop}(f, \epsilon) = \mathsf{drop}(h, \xi)$ imply that $R(\ell, \mathcal{C}_{\mathsf{drop}(h,\xi)})$. Since $C \setminus (\sup(\ell) + 1) = \mathcal{C}_{\mathsf{drop}(h,\xi)}$, $R(\alpha, C \setminus (\sup(\ell) + 1))$ holds. ☐

Definition 2.22. Let κ be a cardinal. The everywhere wellordered club uniformization at κ is the assert that for every $R \subseteq \kappa \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate, there is a function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_\kappa$ so that for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, \Lambda(\alpha))$ holds.

The strong everywhere wellordered club uniformization at κ is the assertion that for every $R \subseteq \kappa \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate, there is a club $C \subseteq \kappa$ so that for $\alpha \in \mathrm{dom}(R)$, $R(\alpha, C \setminus (\alpha + 1))$.

Fact 2.23. *Let κ be a cardinal. The everywhere wellordered club uniformization at κ is equivalent to the strong everywhere wellordered club uniformization at κ.*

Proof. Assume the everywhere wellordered club uniformization holds for κ. Suppose $R \subseteq \kappa \times \mathsf{club}_\kappa$ is a relation which is $\subseteq$-downward closed in the club_κ-coordinate. Let $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_\kappa$ be a uniformization function with the property that for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, \Lambda(\alpha))$. For each $\alpha < \kappa$, let $C_\alpha = \Lambda(\alpha)$ if $\alpha \in \mathrm{dom}(R)$ and $C_\alpha = \kappa$ if $\alpha \notin \mathrm{dom}(R)$. Let $C = \triangle_{\alpha<\kappa} C_\alpha = \{\xi < \kappa : (\forall \alpha < \xi)(\xi \in C_\alpha)\}$ be the diagonal intersection of $\langle C_\alpha : \alpha < \kappa \rangle$ which is a club subset of κ. Note that for each $\alpha \in \mathrm{dom}(R)$, $C \setminus (\alpha + 1) \subseteq C_\alpha$ and $R(\alpha, C_\alpha)$. Since R is $\subseteq$-downward closed in the club_κ-coordinate, $R(\alpha, C \setminus (\alpha + 1))$ holds. ☐

Fact 2.24. *Suppose κ is a cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$. Then $\kappa \to_* (\kappa)^\kappa_{<\kappa}$ is equivalent to the everywhere wellordered club uniformization at κ.*

Proof. (⇐) Suppose $\lambda < \kappa$ and $\Phi : [\kappa]^\kappa_* \to \lambda$. Assume that there is no $\alpha < \lambda$ with a club $C \subseteq \kappa$ so that for all $f \in [C]^\kappa_*$, $\Phi(f) = \alpha$. This implies for all $\alpha < \lambda$, $\Phi^{-1}[\{\alpha\}] \notin \mu^\kappa_\kappa$. Since $\kappa \to_* (\kappa)^\kappa_2$ implies μ^κ_κ is an ultrafilter, for all $\alpha < \lambda$, $[\kappa]^\kappa_* \setminus \Phi^{-1}[\{\alpha\}] \in \mu^\kappa_\kappa$. Define $R \subseteq \kappa \times \mathsf{club}_\kappa$ by $R(\alpha, C)$ if and only if $\alpha < \lambda$ and $[C]^\kappa_* \subseteq [\kappa]^\kappa_* \setminus \Phi^{-1}[\{\alpha\}]$. Observe $\mathrm{dom}(R) = \lambda$. By the hypothesis, there is a $\Lambda : \lambda \to \mathsf{club}_\kappa$ such that for all $\alpha < \lambda$, $R(\alpha, \Lambda(\alpha))$. Since the intersection of less than κ many club subsets of κ is a club, $C = \bigcap_{\alpha<\lambda} \Lambda(\alpha)$ is a club subset. $C \subseteq \bigcap_{\alpha<\lambda} [\kappa]^\kappa_* \setminus \Phi^{-1}[\{\alpha\}] = [\kappa]^\kappa_* \setminus \bigcup_{\alpha<\lambda} \Phi^{-1}[\{\alpha\}] = [\kappa]^\kappa_* \setminus [\kappa]^\kappa_* = \emptyset$ which is a contradiction.

(⇒) Suppose $R \subseteq \kappa \times \mathsf{club}_\kappa$ is a relation which is $\subseteq$-downward closed in the club_κ-coordinate. Define $P_0 : [\kappa]^\kappa_* \to 2$ by $P_0(f) = 0$ if and only if for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, \mathcal{C}_{\mathsf{drop}(f,\alpha)})$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_0 \subseteq \kappa$ homogeneous for P_0. Suppose C_0 is homogeneous for P_0 taking value 1. Define $P_1 : [C_0]^\kappa_* \to 2$ by $P_1(f) = 0$ if and only if there exists an $\alpha < f(0)$ so that $\alpha \in \mathrm{dom}(R)$ and $\neg R(\alpha, \mathcal{C}_f)$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_1 \subseteq C_0$ homogeneous for P_1. Take any $f \in [C_1]^\kappa$. Since $P_0(f) = 1$, there is an $\alpha \in \mathrm{dom}(R)$ with $\neg R(\alpha, \mathcal{C}_{\mathsf{drop}(f,\alpha)})$. Then $P_1(\mathsf{drop}(f,\alpha)) = 0$ and since $\mathsf{drop}(f,\alpha) \in [C_1]^\kappa_*$, C_1 must be homogeneous for P_1 taking value 0. Define $\Psi : [C_1]^\kappa_* \to \kappa$ by $\Psi(f)$ is the least $\alpha < f(0)$ so that $\alpha \in \mathrm{dom}(R)$ and $\neg R(\alpha, \mathcal{C}_f)$. Ψ has the property that for all $f \in [C_1]^\kappa_*$, $\Psi(f) < f(0)$. By Fact 2.12, there is a club $C_2 \subseteq C_1$ and a $\zeta < \kappa$ so that for all $f \in [C_2]^\kappa_*$, $\Psi(f) = \zeta$. This implies that $\zeta \in \mathrm{dom}(R)$. There is some club $D \subseteq \kappa$ with $R(\zeta, D)$. Pick an $h \in [D \cap C_2]^\kappa_*$. Then $R(\zeta, \mathcal{C}_h)$ holds since $\mathcal{C}_h \subseteq D \cap C_2 \subseteq D$ and R is $\subseteq$-downward closed in the club_κ-coordinate. This contradicts $\Psi(h) = \zeta$. Thus C_0 must have been homogeneous for P_0 taking value 0. Take any $f \in [C_0]^\kappa_*$. Define $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_\kappa$ by $\Lambda(\alpha) = \mathcal{C}_{\mathsf{drop}(f,\alpha)}$. $P_0(f) = 0$ implies that for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, \Lambda(\alpha))$. ☐

The following summarizes some equivalences of everywhere wellordered club uniformization.

Fact 2.25. *Suppose κ is a cardinal and assume $\kappa \to_* (\kappa)^\kappa_2$. Then the following are equivalent.*

- $\kappa \to_* (\kappa)^\kappa_{<\kappa}$.
- *μ^κ_κ is a κ-complete ultrafilter.*
- *Everywhere wellordered club uniformization at κ*
- *Strong everywhere wellordered club uniformization at κ.*

Definition 2.26. Let κ be a cardinal.

- (Almost everywhere short length club uniformization at κ) For every relation $R \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate, there is a club $C \subseteq \kappa$ and a function $\Lambda : \mathrm{dom}(R) \cap [C]^{<\kappa}_* \to \mathsf{club}_\kappa$ so that for all $\ell \in \mathrm{dom}(R) \cap [C]^{<\kappa}_*$, $R(\ell, \Lambda(\ell))$.
- (Strong almost everywhere short length club uniformization at κ) For every relation $R \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate, there is a club $C \subseteq \kappa$ so that for all $\ell \in \mathrm{dom}(R) \cap [C]^{<\kappa}_*$, $R(\ell, C \setminus (\sup(\ell) + 1))$.

Fact 2.27. *Suppose κ is a cardinal and $\kappa \to_* (\kappa)^\kappa_2$. Then almost everywhere short length club uniformization at κ is equivalent to the strong almost everywhere short length club uniformization at κ.*

Proof. Assume the almost everywhere short length club uniformization at κ. Let $R \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ be a relation which is $\subseteq$-downward closed in the club_κ-coordinate. By the hypothesis, let $C_0 \subseteq \kappa$ be a club and $\Lambda : \mathrm{dom}(R) \cap [C_0]^{<\kappa}_*$ have the property that for all $\ell \in \mathrm{dom}(R) \cap [C_0]^{<\kappa}_*$, $R(\ell, \Lambda(\ell))$.

Define a partition $P : [C_0]^\kappa_* \to 2$ by $P(f) = 0$ if and only if for all $\alpha < \kappa$, if $f \restriction \alpha \in \mathrm{dom}(R)$, then $R(f \restriction \alpha, \mathcal{C}_{\mathsf{drop}(f,\alpha)})$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for P. Now suppose that C_1 is homogeneous for P taking value 1. This means for all $f \in [C_1]^\kappa_*$, there exists an $\alpha < \kappa$ so that $f \restriction \alpha \in \mathrm{dom}(R)$ and $\neg R(f \restriction \alpha, \mathcal{C}_{\mathsf{drop}(f,\alpha)})$. Define $\Phi : [C_1]^\kappa_* \to \kappa$ by $\Phi(f)$ is the least α with the above property.

A function $h \in [C_1]^\kappa_*$ will be defined by recursion. If $h \restriction 0 = \emptyset \notin \mathrm{dom}(R)$, then let $F_0 = C_1$. If $h \restriction 0 = \emptyset \in \mathrm{dom}(R)$, then let $F_0 = C_1 \cap \Lambda(\emptyset)$. In either case, $h \restriction 0$ has the property that for all $g \in [F_0]^\kappa_*$, $\Phi(h \restriction 0\hat{\ }g) > 0$ since if $h \restriction 0 \in \mathrm{dom}(R)$, then $R(h \restriction 0, \mathcal{C}_g)$ holds because $R(h \restriction 0, \Lambda(\emptyset))$, $\mathcal{C}_g \subseteq \Lambda(0)$, and R is $\subseteq$-downward closed. Let $h(0) = \mathsf{next}^\omega_{F_0}(0)$. Suppose for $\alpha < \kappa$, $h \restriction \alpha$ and $\langle F_\beta : \beta < \alpha \rangle$ have been defined with the property that for all $\beta < \alpha$, if $g \in [F_\beta]^\kappa_*$, then $\Phi(h \restriction \beta\hat{\ }g) > \beta$. If $h \restriction \alpha \notin \mathrm{dom}(R)$, then let $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$. If $h \restriction \alpha \in \mathrm{dom}(R)$, then let $F_\alpha = \bigcap_{\beta<\alpha} F_\beta \cap \Lambda(h \restriction \alpha)$. In either case, for all $g \in [F_\alpha]^\kappa_*$, $\Phi(h \restriction \alpha\hat{\ }g) > \alpha$. To see this: If $\beta < \alpha$, note that $\mathsf{drop}(h \restriction \alpha, \beta)\hat{\ }g \subseteq F_\beta$ which implies that $\Phi(h \restriction \alpha\hat{\ }g) = \Phi((h \restriction \beta)\hat{\ }\mathsf{drop}(h \restriction \alpha, \beta)\hat{\ }g) > \beta$ by the induction hypothesis. Thus $\Phi(h \restriction \alpha\hat{\ }g) \geq \alpha$.

If $h \upharpoonright \alpha \in \text{dom}(R)$, then $R(h \upharpoonright \alpha, \mathcal{C}_g)$ holds because $R(h \upharpoonright \alpha, \Lambda(h \upharpoonright \alpha))$ holds, $\mathcal{C}_g \subseteq \Lambda(h \upharpoonright \alpha)$, and R is $\subseteq$-downward closed. This implies $\Phi(h \upharpoonright \alpha\hat{}g) > \alpha$. If $f \upharpoonright \alpha \notin \text{dom}(R)$, then by definition of Φ, $\Phi(h \upharpoonright \alpha\hat{}g) > \alpha$. Let $h(\alpha) = \mathsf{next}^{\omega}_{F_\alpha}(\sup(h \upharpoonright \alpha))$. This completes the definition of h and $\langle F_\alpha : \alpha < \kappa \rangle$. For any α, $\Phi(h) = \Phi(h \upharpoonright \alpha\hat{}\mathsf{drop}(h, \alpha)) > \alpha$ since $\mathsf{drop}(h, \alpha) \in [F_\alpha]^\kappa_*$. As $\alpha < \kappa$ is arbitrary, $\Phi(h) \geq \kappa$ which contradicts the fact that $\Phi : [C_1]^\kappa_* \to \kappa$.

This implies that C_1 must be homogeneous for P taking value 0. Let $h \in [C_1]^\kappa_*$ be defined by $h(\alpha) = \mathsf{enum}_{C_1}(\omega \cdot \alpha + \omega)$. Let $C_2 = \mathcal{C}_h$. Suppose $\ell \in \text{dom}(R) \cap [C_2]^{<\kappa}_*$. Let α be least so that $h(\alpha) > \sup(\ell)$. Let $f = \ell\hat{}\mathsf{drop}(h, \alpha)$. Since $f \in [C_1]^\kappa_*$, $P(f) = 0$. Since $\ell \in \text{dom}(R)$ and $f \upharpoonright |\ell| = \ell$, $R(f \upharpoonright |\ell|, \mathcal{C}_{\mathsf{drop}(f,|\ell|)})$ holds and thus $R(\ell, \mathcal{C}_{\mathsf{drop}(h,\alpha)})$ holds. However $\mathcal{C}_{\mathsf{drop}(h,\alpha)} = C_2 \setminus (\sup(\ell) + 1)$. Thus $R(\ell, C_2 \setminus (\sup(\ell) + 1))$ holds. C_2 is the desired club. ☐

Fact 2.28. *Let κ be a cardinal. Almost everywhere short length club uniformization at κ implies the everywhere wellordered club uniformization at κ*

Proof. Suppose $R \subseteq \kappa \times \mathsf{club}_\kappa$ is $\subseteq$-downward closed in the club_κ-coordinate. Define $S \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ by $S(\ell, D)$ if and only if $|\ell| \in \text{dom}(R)$ and $R(|\ell|, D)$. By the almost everywhere short length club uniformization and Fact 2.27, there is a club C so that for all $\ell \in \text{dom}(S) \cap [C]^{<\kappa}_*$, $S(\ell, C \setminus (\sup(\ell) + 1))$. For each $\alpha < \omega_1$, let $\ell_\alpha \in [C]^\alpha_*$ be defined by recursion as follows. Let $\ell_\alpha(0) = \mathsf{next}^\omega_C(0)$. If $\beta < \alpha$ and $\ell_\alpha \upharpoonright \beta$ has been defined, then let $\ell_\alpha(\beta) = \mathsf{next}^\omega_C(\sup(\ell_\alpha \upharpoonright \beta))$. Let $\Lambda : \text{dom}(R) \to \mathsf{club}_\kappa$ be defined by $\Lambda(\alpha) = C \setminus (\sup(\ell_\alpha) + 1)$. Suppose $\alpha \in \text{dom}(R)$. Since $|\ell_\alpha| = \alpha$, $\ell_\alpha \in \text{dom}(S)$ and thus $S(\ell_\alpha, C \setminus (\sup(\ell_\alpha) + 1))$. By definition of S, $R(\alpha, \Lambda(\alpha))$. ☐

These uniformization results can be used to prove a mixed everywhere wellordered and almost everywhere short length club uniformization.

Fact 2.29. *Suppose the almost everywhere short length club uniformization holds at κ. Let $R \subseteq \kappa \times [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ be $\subseteq$-downward closed in the club_κ-coordinate. Then there is a club $C \subseteq \kappa$ so that for all $\alpha < \kappa$ and $\ell \in [C]^{<\kappa}_*$, if $(\alpha, \ell) \in \text{dom}(R)$, then $R(\alpha, \ell, C \setminus (\max\{\sup(\ell), \alpha\} + 1))$.*

Proof. For each $\ell \in [\kappa]^{<\kappa}_*$, define $S_\ell \subseteq \kappa \times \mathsf{club}_\kappa$ by $S_\ell(\alpha, D)$ if and only if $R(\alpha, \ell, D)$. S_ℓ is $\subseteq$-downward closed in the club_κ-coordinate. By Fact 2.28 and Fact 2.25, there is a club $E \subseteq \kappa$ so that for all $\alpha \in \text{dom}(S_\ell)$, $S_\ell(\alpha, E \setminus (\alpha + 1))$. Define $T \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ by $T(\ell, E)$ if and only if for all $\alpha < \kappa$, if $\alpha \in \text{dom}(S_\ell)$, then $S_\ell(E \setminus (\alpha + 1))$. By the previous discussion, $\text{dom}(T) = [\kappa]^{<\kappa}_*$. By Fact 2.27, there is a club $C \subseteq \kappa$ so that for all $\ell \in [C]^{<\kappa}_*$, $T(\ell, C \setminus (\sup(\ell) + 1))$. Now suppose $\alpha < \kappa$ and $\ell \in [C]^{<\kappa}_*$ with $(\alpha, \ell) \in \text{dom}(R)$. One has $T(\ell, C \setminus (\sup(\ell) + 1))$. Since $\alpha \in \text{dom}(S_\ell)$, $S_\ell(\alpha, (C \setminus (\sup(\ell) + 1)) \setminus (\alpha + 1))$. By definition of S_ℓ and since $(C \setminus (\sup(\ell) + 1)) \setminus (\alpha + 1) = C \setminus (\max\{\sup(\ell), \alpha\}) + 1)$, one has $R(\alpha, \ell, C \setminus (\max\{\sup(\ell), \alpha\} + 1))$. ☐

3. Almost Everywhere Continuity Properties

Definition 3.1. Let $\Phi : [\kappa]^\kappa_* \to \kappa$ and $C \subseteq \kappa$ be a club. Say that $\sigma \in [C]^{<\kappa}_*$ is a continuity point for Φ relative to C if and only if for all $g_0, g_1 \in [C]^\kappa_*$ such that $g_0 \upharpoonright |\sigma| = \sigma = g_1 \upharpoonright |\sigma|$, $\Phi(g_0) = \Phi(g_1)$. Say that σ is a minimal continuity point for Φ relative to C if and only if no proper initial segment of σ is a continuity point for Φ relative to C.

Theorem 3.2. *Let κ be a cardinal so that $\kappa \to_* (\kappa)^\kappa_2$ and the almost everywhere short length club uniformization at κ holds. Let $\Phi : [\kappa]^\kappa_* \to \kappa$. Then there is a club $C \subseteq \kappa$ with the following properties.*

(a) *$\Phi \upharpoonright [C]^\kappa_*$ is continuous: For every $f \in [C]^\kappa_*$, there exists an $\alpha < \kappa$ so that for all $g \in [C]^\kappa_*$, if $f \upharpoonright \alpha = g \upharpoonright \alpha$, then $\Phi(g) = \Phi(f)$.*
(b) *For any $f \in [C]^\kappa_*$, let β_f be the unique β so that $\sup(f \upharpoonright \beta) \leq \Phi(f) < f(\beta)$. Then $f \upharpoonright \beta_f$ is a minimal continuity point for Φ relative to C.*
(c) *For any $\sigma \in [C]^{<\kappa}_*$, if there is a $g \in [C]^\kappa_*$ so that $\sup(\sigma) < g(0)$ and $\Phi(\sigma\hat{}g) < g(0)$, then σ is a continuity point of Φ relative to C.*

Proof. Under the hypothesis, Fact 2.27 implies strong almost everywhere short length club uniformization at κ. For each $\sigma \in [\kappa]^{<\kappa}_*$, let $\Phi_\sigma : [\kappa \setminus (\sup(\sigma) + 1)]^\kappa_* \to \kappa$ be defined by $\Phi_\sigma(g) = \Phi(\sigma\hat{}g)$. Let K be the set of $\sigma \in [\kappa]^{<\kappa}_*$ so that for all club $D \subseteq \kappa$, there exists a $g \in [D]^\kappa_*$ with $\sup(\sigma) < g(0)$ and $\Phi_\sigma(g) < g(0)$.

Claim 1: For each $\sigma \in K$, there is unique $c_\sigma \in \kappa$ so that there exists a club D with the property that for all $g \in [D]^\kappa_*$, $\Phi_\sigma(g) = c_\sigma < g(0)$.

Proof. Let $Q_\sigma : [\kappa \setminus (\sup(\sigma)+1)]^\kappa_* \to 2$ by $Q_\sigma(g) = 0$ if and only if $\Phi_\sigma(g) < g(0)$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club D_0 which is homogeneous for Q_σ. Since $\sigma \in K$, there is a $g \in [D_0]^\kappa_*$ so that $\sup(\sigma) < g(0)$ and $\Phi_\sigma(g) < g(0)$. Thus D_0 is homogeneous for Q_σ taking value 0. For all $g \in [D_0]^\kappa_*$, $Q_\sigma(g) = 0$ implies that $\Phi_\sigma(g) < g(0)$. By Fact 2.12, there is a club $D_1 \subseteq D_0$ and $c_\sigma \in \kappa$ so that for all $g \in [D_1]^\kappa_*$, $\Phi_\sigma(g) = c_\sigma$. (Note that c_σ depends only on σ and does not depend on D_1.) ☐

Let $R_0 \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ be defined by $R(\sigma, D)$ if and only if the conjunction of the following holds.

(1) If $\sigma \in K$, then for all $g \in [D]^\kappa_*$, $\sup(\sigma) < g(0)$ and $\Phi_\sigma(g) = c_\sigma < g(0)$.
(2) If $\sigma \notin K$, then for all $g \in [D]^\kappa_*$, $\sup(\sigma) < g(0)$ and $\Phi_\sigma(g) \geq g(0)$.

Note that R is $\subseteq$-downward closed in the club_κ-coordinate. If $\sigma \in K$, then Claim 1 implies $\sigma \in \mathrm{dom}(R_0)$. If $\sigma \notin K$, then by definition $\sigma \in \mathrm{dom}(R_0)$. Thus $\mathrm{dom}(R_0) = [\kappa]^{<\kappa}_*$. By Theorem **??**, there is a club $C_0 \subseteq \kappa$ so that for all $\sigma \in [\kappa]^{<\kappa}_*$, $R_0(\sigma, C_0 \setminus (\sup(\sigma)+1))$.

Claim 2: If $\sigma \in K \cap [C_0]^{<\kappa}_*$, then σ is a continuity point for Φ relative to C_0. Moreover, for all $f \in [C_0]^\kappa_*$ with $\sigma = f \restriction |\sigma|$, $\Phi(f) = c_\sigma < f(|\sigma|)$.

Proof. Suppose $\sigma \in K \cap [C_0]^{<\kappa}_*$. Let $f, g \in [C_0]^\kappa_*$ be such that $f \restriction |\sigma| = \sigma = g \restriction |\sigma|$. Since $R_0(\sigma, C_0 \setminus (\sup(\sigma)+1))$, $\mathsf{drop}(f, |\sigma|) \in C_0 \setminus (\sup(\sigma)+1)$, and $\mathsf{drop}(g, |\sigma|) \in C_0 \setminus (\sup(\sigma)+1)$,

$$\Phi(f) = \Phi_{f\restriction|\sigma|}(\mathsf{drop}(f \restriction |\sigma|)) = \Phi_\sigma(\mathsf{drop}(f, |\sigma|)) = c_\sigma = \Phi_\sigma(\mathsf{drop}(g, |\sigma|)) = \Phi_{g\restriction|\sigma|}(\mathsf{drop}(g \restriction |\sigma|)) = \Phi(g).$$

The properties of R_0 also imply $\Phi(f) = c_\sigma < \mathsf{drop}(f, |\sigma|)(0) = f(|\sigma|)$. ☐

Let K^* be the set of $\sigma \in K$ such that for all proper initial segments $\tau \subset \sigma$, $\tau \notin K$.

Claim 3: For any club $C \subseteq C_0$, if $\sigma \in K^* \cap [C]^{<\kappa}_*$, then σ is a minimal continuity point relative to C.

Proof. Fix $C \subseteq C_0$. Suppose $\sigma \in K^* \cap [C]^{<\kappa}_*$. Since $\sigma \in K$, Claim 2 implies that σ is a continuity point for Φ relative to C_0 and hence also relative to C which is subset of C_0. Let $\tau \subset \sigma$ be a proper initial segment. Then $\tau \notin K$. Let $g_0 \in [C \setminus (\sup(\tau)+1)]^\kappa_*$. Pick $g_1 \in [C \setminus (\sup(\tau)+1)]^\kappa_*$ so that $g_1(0) > \Phi_\tau(g_0)$. Since $C \subseteq C_0$, $R_0(\tau, C_0 \setminus (\sup(\tau)+1))$ holds, and R_0 is $\subseteq$-downward closed in the club_κ-coordinate, one has $R_0(\tau, C \setminus (\sup(\tau+1)))$. Since $\tau \notin K$, the definition of R_0 implies

$$\Phi(\tau\hat{\ }g_0) = \Phi_\tau(g_0) < g_1(0) \leq \Phi_\tau(g_1) = \Phi(\tau\hat{\ }g_1).$$

Thus $\Phi(\tau\hat{\ }g_0) \neq \Phi(\tau\hat{\ }g_1)$ and hence τ is not a continuity point for Φ relative to C. ☐

Claim 4: There is a club $C_1 \subseteq C_0$ so that for all $f \in [C_1]^\kappa_*$, there exists some $\alpha < \kappa$ with $f \restriction \alpha \in K$.

Proof. Let $P_0 : [C_0]^\kappa_* \to 2$ be defined by $P_0(f) = 0$ if and only if there exists an $\alpha < \kappa$ so that $f \restriction \alpha \in K$. By the partition relation $\kappa \to_* (\kappa)^\kappa_2$, let $C_1 \subseteq C_0$ be homogeneous for P_0. Suppose C_1 is homogeneous for P_0 taking value 1. Fix an $f \in [C_1]^\kappa_*$. Let $\alpha < \kappa$. Since $P_0(f) = 1$, $f \restriction \alpha \notin K$. Because $R_0(f \restriction \alpha, C_0 \setminus (\sup(f \restriction \alpha)+1))$, $C_1 \subseteq C_0$, and R_0 is $\subseteq$-downward closed in the club_κ-coordinate, $R_0(f \restriction \alpha, C_1 \setminus (\sup(f \restriction \alpha)+1))$. Since $f \restriction \alpha \notin K$ and $\mathsf{drop}(f, \alpha) \in C_1 \setminus (\sup(f \restriction \alpha)+1)$, the definition of R_0 implies that $\Phi(f) = \Phi_{f\restriction\alpha}(\mathsf{drop}(f, \alpha)) \geq \mathsf{drop}(f, \alpha)(0) = f(\alpha)$. It has been shown that for all $\alpha < \kappa$, $\Phi(f) \geq \alpha$. Thus $\Phi(f) \geq \kappa$ which is impossible since Φ takes values in κ. C_0 is homogeneous for P_0 taking value 0 which establishes the claim. ☐

Using Claim 4, for each $f \in [C_1]^\kappa_*$, let α_f be the least $\alpha < \kappa$ so that $f \restriction \alpha \in K$. Thus $f \restriction \alpha_f \in K^*$. Recall that for all $f \in [\kappa]^\kappa_*$, β_f is the unique β so that $\sup(f \restriction \beta) \leq \Phi(f) < f(\beta)$.

Claim 5: For all $f \in [C_1]^\kappa_*$, $\beta_f \leq \alpha_f$.

Proof. Let $f \in [C_1]^\kappa_*$. $f \restriction \alpha_f \in K$. By Claim 2, one has $\Phi(f) < f(\alpha_f)$. Thus the unique β so that $\sup(f \restriction \beta) \leq \Phi(f) < f(\beta)$ is less than or equal to α_f. Hence $\beta_f \leq \alpha_f$. ☐

Claim 6: There is a club $C_2 \subseteq C_1$ so that for all $f \in [C_2]^\kappa_*$, $\alpha_f = \beta_f$.

Proof. Let $P_1 : [C_1]^\kappa_* \to 2$ by $P_1(f) = 0$ if and only if $\alpha_f = \beta_f$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_2 \subseteq C_1$ which is homogeneous for P_1. Suppose C_2 is homogeneous for P_1 taking value 1.

Subclaim 6.1: For any $\sigma \in K^* \cap [C_2]^{<\kappa}_*$, there is an ordinal $\gamma < |\sigma|$ so that $c_\sigma < \sigma(\gamma)$.

Proof. Let $f \in [C_2]^\kappa_*$ so that $\sigma \subseteq f$. Since $\sigma \in K^*$, $\alpha_f = |\sigma|$. Since $P_1(f) = 1$, one has that $\beta_f < \alpha_f = |\sigma|$. Thus using Claim 2,

$$\sup(f \restriction \beta_f) \leq \Phi(f) = \Phi_\sigma(\mathsf{drop}(f, \alpha_f)) = c_\sigma < f(\beta_f) \leq \sup(f \restriction \alpha_f).$$

So β_f is an ordinal $\gamma < |\sigma|$ so that $c_\sigma < \sigma(\gamma)$. ☐

Let $f \in [C_2]^\kappa_*$. Suppose $f \restriction 0 = \emptyset \in K$. Then $f \restriction 0 \in K^*$ and thus $\alpha_f = 0$. $P_1(f) = 1$ implies that $\beta_f < \alpha_f = 0$ which is impossible. It has been shown that $f \restriction 0 \notin K$.

Suppose $\epsilon < \kappa$ and for all $\delta < \epsilon$, it has been shown that $f \restriction \delta \notin K$. Suppose $f \restriction \epsilon \in K$. Thus $f \restriction \epsilon \in K^*$ and hence by Claim 2, $\Phi(f) = c_{f \restriction \epsilon}$. By Subclaim 6.1, there is a $\gamma < \epsilon$ so that $\Phi(f) = c_{f \restriction \epsilon} < f(\gamma)$. Since $\gamma < \epsilon$, $f \restriction \gamma \notin K$. Because $C_2 \subseteq C_1$, one has $R_0(f \restriction \gamma, C_2 \setminus (\sup(f \restriction \gamma) + 1))$. Since $\mathsf{drop}(f, \gamma) \in [C_2 \setminus (\sup(f \restriction \gamma) + 1)]^\kappa_*$, the definition of R_0 implies that $\Phi(f) = \Phi_{f \restriction \gamma}(\mathsf{drop}(f, \gamma)) \geq \mathsf{drop}(f, \gamma)(0) = f(\gamma)$. Thus we have shown $\Phi(f) < f(\gamma)$ and $\Phi(f) \geq f(\gamma)$. Contradiction. Thus C_2 must be homogeneous for P_1 taking value 0. ☐

C_2 is the desired club satisfying property (a), (b), and (c):

Property (b) following from Claim 3 and Claim 6. Property (a) follows from property (b).

Suppose $\sigma \in [C_2]^{<\kappa}_*$ and there is a $g \in [C_2]^\kappa_*$ with $\sup(\sigma) < g(0)$ and $\Phi(\sigma \hat{\ } g) < g(0)$. Let $f = \sigma \hat{\ } g$. Then $\beta_f \leq |\sigma|$. By Claim 6, $\alpha_f = \beta_f \leq |\sigma|$. Thus $\sigma \restriction \alpha_f$ is a minimal continuity point for Φ relative to C_2 by Claim 3 and thus σ is also a continuity point for Φ relative to C_2. This establishes property (c). ☐

Fact 3.3. *Assume* ZF. *Suppose κ is a cardinal such that for all $\Phi : [\kappa]^\kappa_* \to \kappa$, there is a club $C \subseteq \kappa$ so that for all $f \in [C]^\kappa_*$, $f \restriction \beta_f$ is a minimal continuity point for Φ relative to C where β_f is the unique β so that $\sup(f \restriction \beta) \leq \Phi(f) < f(\beta)$. Then $\kappa \to_* (\kappa)^\kappa_{<\kappa}$ holds.*

Proof. Let $\lambda < \kappa$ and $\Phi : [\kappa]^\kappa_* \to \lambda$. Let C be a club satisfying the expressed continuity property with $\lambda < \min(C)$. For all $f \in [C]^\kappa_*$, $\Phi(f) < \lambda < f(0)$ and thus $\beta_f = 0$. For all $f \in [C]^\kappa_*$, $\emptyset$ is a minimal continuity point for Φ relative to C. For all $f, g \in [C]^\kappa_*$, $\Phi(f) = \Phi(g)$. Pick any $h \in [C]^\kappa_*$ and let $\delta = \Phi(h)$. Then for all $f \in [C]^\kappa_*$, $\Phi(f) = \delta$. ☐

[2] Theorem 5.3 shows that every function $\Phi : [\omega_1]^{\omega_1}_* \to {}^{\omega_1}\omega_1$ is continuous μ^κ_κ-almost everywhere in a natural sense. Using Fact 2.29 and the ideas of [2] Theorem 5.3, the following analogous almost everywhere continuity result can be shown. The proof is omitted since this result will not be used in the paper.

Theorem 3.4. *Assume $\kappa \to_* (\kappa)^\kappa_2$ and the almost everywhere short length club uniformization holds at κ. Let $\Phi : [\kappa]^\kappa_* \to {}^\kappa\kappa$. Then there is a club $C \subseteq \kappa$ so that for all $f \in [C]^\kappa_*$ and $\beta < \kappa$, there exists an $\alpha < \kappa$ so that for all $g \in [C]^\kappa_*$, if $f \restriction \alpha = g \restriction \alpha$, then $\Phi(f) \restriction \beta = \Phi(g) \restriction \beta$.*

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