EQUIVALENCE RELATIONS WHICH ARE BOREL SOMEWHERE

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Abstract. The following will be shown: Let I be a σ-ideal on a Polish space X so that the associated forcing of I^+ $\mathbf{\Delta}^1_1$ sets ordered by $\subseteq$ is a proper forcing. Let E be a $\mathbf{\Sigma}^1_1$ or a $\mathbf{\Pi}^1_1$ equivalence relation on X with all equivalence classes $\mathbf{\Delta}^1_1$. If for all $z \in H_{(2^{\aleph_0})^+}$, $z^\sharp$ exists, then there exists an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq X$ such that $E \restriction C$ is a $\mathbf{\Delta}^1_1$ equivalence relation.

1. Introduction

The basic question addressed here in its most naive form is:

Question: If E is an equivalence relation on a Polish space X, is there a large and nice set $C \subseteq X$ such that $E \restriction C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

Here, $E \restriction C = E \cap (C \times C)$.

Two immediate concerns about the question arise from the phrase "large and nice":

The basic idea of the question is that given an equivalence relation E, can one find a subset C such that $E \restriction C$ is a simpler equivalence relation, in particular $\mathbf{\Delta}^1_1$. One does not want to hide any complexity of $E \restriction C$ inside the set C. Therefore C should be "nice" in the sense that it is a $\mathbf{\Delta}^1_1$ subset of X.

Every equivalence relation restricted to a countable set is a $\mathbf{\Delta}^1_1$ equivalence relation. Conditions must be imposed on C to make the question meaningful. σ-ideals on the Polish space X would include all countable subsets of X. So if one demands that C be a non-small set according to a σ-ideal I on X, then the most egregious trivialities vanish. Subsets C of X with $C \notin I$ are called I^+ sets. In the question, a reasonable largeness requirement on C should be that it is I^+ and $\mathbf{\Delta}^1_1$.

Without I having some useful properties, there seems to be no particular reason to expect any interesting answer. Some conditions should be imposed on I: Given a σ-ideal I on a Polish space X, there is a natural forcing $\mathbb{P}_I$ associated with I that has been used extensively in descriptive set theory and cardinal characteristics of the continuum. $\mathbb{P}_I$ consists of all I^+ $\mathbf{\Delta}^1_1$ subsets of X ordered by $\subseteq$. Motivated by works in cardinal characteristics, one could require I to have the property that $\mathbb{P}_I$ is a proper forcing.

In cardinal characteristics, properness is used for preservation of certain properties under countable support iterations. This will not be how properness is used in this paper. Rather, properness will be used to produce I^+ $\mathbf{\Delta}^1_1$ subsets for which forcing and absoluteness can be used to derive meaningful information. The main tool that makes this approach possible is the following result:

Fact 2.4. (Zapletal, [34] Proposition 2.2.2.) *Let I be a σ-ideal on a Polish space X. The following are equivalent:*

(i) $\mathbb{P}_I$ is a proper forcing.

(ii) For any sufficiently large cardinal Θ, for every $B \in \mathbb{P}_I$, and for every countable $M \prec H_\Theta$ with $\mathbb{P}_I \in M$ and $B \in M$, the set $C := \{x \in B : x$ is $\mathbb{P}_I$-generic over $M\}$ is I^+ $\mathbf{\Delta}^1_1$.

With this result, the question is now asked with respect to a σ-ideal such that $\mathbb{P}_I$ is proper. Beyond properness of $\mathbb{P}_I$, the ideal does not necessarily have any definability restrictions. The desired set C of the question should be I^+ and $\mathbf{\Delta}^1_1$ but not necessarily E-invariant.

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A natural place to begin exploring this question is with the simplest class of definable equivalence relations just beyond $\mathbf{\Delta}_1^1$ equivalence relations: If I is a σ-ideal such that $\mathbb{P}_I$ is proper and E is an $\mathbf{\Sigma}_1^1$ equivalence relation, is there an I^+ $\mathbf{\Delta}_1^1$ set C such that $E \restriction C$ is $\mathbf{\Delta}_1^1$?

Unfortunately, the answer is no.

Fact 1.1. ([19]) *There exists an $\mathbf{\Sigma}_1^1$ equivalence relation E and a σ-ideal I with $\mathbb{P}_I$ proper such that for all I^+ $\mathbf{\Delta}_1^1$ set C, $E \restriction C$ is not $\mathbf{\Delta}_1^1$.*

Proof. (See [19], Example 4.25.) Let K be a $\mathbf{\Sigma}_1^1$ but not $\mathbf{\Delta}_1^1$ ideal on ω. Define E_K on ${}^\omega({}^\omega 2)$ by $x\ E_K\ y$ if and only if $\{n \in \omega : x(n) \neq y(n)\} \in K$. E_K is a $\mathbf{\Sigma}_1^1$ but not $\mathbf{\Delta}_1^1$ equivalence relation. (Note that it has non-$\mathbf{\Delta}_1^1$ classes.)

Let $\mathbb{S}^\omega$ denote countable product of Sacks forcing, i.e. $p \in \mathbb{S}^\omega$ if and only if p is a function on ω so that for each n, $p(n)$ is a perfect tree. If $p \in \mathbb{S}^\omega$, let $[p] = \{x \in {}^\omega({}^\omega 2) : (\forall n)(x(n) \in [p(n)])\}$. Let I be the σ-ideal generated by $\mathbf{\Delta}_1^1$ sets that do not contain $[p]$ for any $p \in \mathbb{S}^\omega$. [19] Fact 9.25 (iii) shows that any I^+ $\mathbf{\Delta}_1^1$ set contains $[p]$ for some $p \in \mathbb{S}^\omega$. This can be used to show that if B is I^+ $\mathbf{\Delta}_1^1$, then $E_K \leq_{\mathbf{\Delta}_1^1} E_K \restriction B$. Hence for every I^+ $\mathbf{\Delta}_1^1$ B, $E_K \restriction B$ is not a $\mathbf{\Delta}_1^1$ equivalence relation. $\square$

This suggests that in order to possibly obtain a positive answer, the equivalence relation considered should more closely resemble $\mathbf{\Delta}_1^1$ equivalence relations. An obvious feature of $\mathbf{\Delta}_1^1$ equivalence relations is that all their equivalence classes are $\mathbf{\Delta}_1^1$. Kanovei, Sabok, and Zapletal then asked the following question of $\mathbf{\Sigma}_1^1$ equivalence relations which share this feature:

Question 1.2. ([19] Question 4.28) If I is a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper and E is a $\mathbf{\Sigma}_1^1$ equivalence relation with all classes $\mathbf{\Delta}_1^1$, then is there an I^+ $\mathbf{\Delta}_1^1$ set C such that $E \restriction C$ is $\mathbf{\Delta}_1^1$?

Similarly, the question can be asked for the dual class of equivalence relations on the same projective level:

Question 1.3. If I is a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper and E is a $\mathbf{\Pi}_1^1$ equivalence relation with all classes $\mathbf{\Delta}_1^1$, then is there an I^+ $\mathbf{\Delta}_1^1$ set C such that $E \restriction C$ is $\mathbf{\Delta}_1^1$?

Again having all $\mathbf{\Delta}_1^1$ classes is necessary for a positive answer by considering the example from the proof of Fact 1.1 using a $\mathbf{\Pi}_1^1$ ideal on ω.

With these restrictions, the initial naive question becomes a rather robust question. Throughout the paper, the term "main question" will refer to questions of the former type for various classes of definable equivalence relations on Polish spaces. For concreteness, the reader should perhaps keep in mind the following explicit instance of the main question: If E is a $\mathbf{\Sigma}_1^1$ equivalence relation with all classes $\mathbf{\Delta}_1^1$, is there a nonmeager or positive measure $\mathbf{\Delta}_1^1$ set C such that $E \restriction C$ is a $\mathbf{\Delta}_1^1$ equivalence relation?

Section 2 will provide the basic concepts from idealized forcing including the main tool about proper idealized forcings used throughout the paper. Some useful notations for expressing the main question are also introduced. The main question in a slightly stronger form is formalized.

Section 3 will provide known results and examples to show that the main question has a positive answer for the most natural $\mathbf{\Sigma}_1^1$ equivalence relations with all $\mathbf{\Delta}_1^1$ classes. In particular, [19] showed that $\mathbf{\Sigma}_1^1$ equivalence relations with all classes countable and equivalence relations $\mathbf{\Delta}_1^1$ reducible to orbit equivalence relations of Polish group actions have a positive answer to the main question.

Section 4 will show that a positive answer to the main question for $\mathbf{\Sigma}_1^1$ equivalence relations with all $\mathbf{\Delta}_1^1$ classes follows from some large cardinal assumptions. In particular, it can be proved from iterability principles (such as the existence of a measurable cardinal):

Theorem 4.22. *Let I be a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper. Let E be a $\mathbf{\Sigma}_1^1$ equivalence relation on X with all classes $\mathbf{\Delta}_1^1$. If for all $z \in {}^\omega\omega$, $z^\sharp$ exists and $(\chi_E^I)^\sharp$ exists, then there is an I^+ $\mathbf{\Delta}_1^1$ set C such that $E \restriction C$ is $\mathbf{\Delta}_1^1$.*

Here χ_E^I is a set depending on I and E. This set χ_I^E is in $H_{(2^{\aleph_0})^+}$ so it is a fairly small set. More explicitly, χ_E^I is a triple $\langle \mathbb{P}_I, \mu_E^I, \sigma_E^I \rangle$, where $\mu_E^I, \sigma_E^I \in V^{\mathbb{P}_I}$ are names that witness two existential formulas. In fact, these two names can be chosen a bit more constructively using the fullness or maximality property of forcing. In particular, there is a positive answer to the main question for $\mathbf{\Sigma}^1_1$ equivalence relations with all $\mathbf{\Delta}^1_1$ classes if there exists a Ramsey cardinal.

After showing the positive answer follows from certain large cardinal principles, a natural question would be whether the negative answer to the main question holds in L or forcing extensions of L, only assuming mild large cardinal assumptions if necessary. The next sections give partial results for a positive answer using different and weaker consistency assumptions for a restricted class of equivalence relations or ideals. Although these results are inherently interesting, these sections should be understood as an attempt to find situations that can not be used to produce a counterexample to a positive answer to the main question. These results seem to enforce the intuition that a universe with very weak large cardinals may be the ideal place to search for such a counterexample.

In Section 5, it will be shown that $I_{\text{countable}}$, the ideal of countable sets, and I_{E_0}, the σ-ideal generated by $\mathbf{\Delta}^1_1$ sets on which E_0 is smooth, will always give a positive answer to the main question for $\mathbf{\Sigma}^1_1$ equivalence relation with all $\mathbf{\Delta}^1_1$ classes. The associated forcings for these two σ-ideals are Sacks forcing and Prikry-Silver forcing, respectively. The meager ideal, I_{meager}, and the Lebesgue null ideal, I_{null}, have as their associated forcing, the Cohen forcing and random real forcing, respectively. Under $\mathsf{MA} + \neg\mathsf{CH}$, the main question has a positive answer for the meager and null ideal:

Theorem 5.15. ($\mathsf{ZFC} + \mathsf{MA} + \neg\mathsf{CH}$) *Let I be either I_{null} or I_{meager}. Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation with all classes $\mathbf{\Delta}^1_1$. Then there exists an I^+ $\mathbf{\Delta}^1_1$ set C such that $E \restriction C$ is $\mathbf{\Delta}^1_1$.*

Section 6 will consider thin $\mathbf{\Sigma}^1_1$ equivalence relations, i.e., equivalence relations with no perfect set of inequivalent elements. Burgess showed that such equivalence relations have at most $\aleph_1$ many equivalence classes. This suggests that the main question for thin $\mathbf{\Sigma}^1_1$ equivalence relations with all $\mathbf{\Delta}^1_1$ classes should be approached combinatorially using covering numbers and the properness of $\mathbb{P}_I$. For example assuming PFA, there is a positive answer for all σ-ideals I with $\mathbb{P}_I$ proper and E a thin $\mathbf{\Sigma}^1_1$ equivalence relation with all classes $\mathbf{\Delta}^1_1$. However, the combinatorial approach is not the right one. Using definability ideas, the main question for thin $\mathbf{\Sigma}^1_1$ equivalence relations (even without all $\mathbf{\Delta}^1_1$ classes) has a strong positive answer:

Theorem 6.8. (ZFC) *If I is a σ-ideal such that $\mathbb{P}_I$ is proper and E is a thin $\mathbf{\Sigma}^1_1$ equivalence relation, then there exists a I^+ $\mathbf{\Delta}^1_1$ set C such that C is contained in a single E-class.*

Section 7 will show that a positive answer for $\mathbf{\Pi}^1_1$ equivalence relations with all $\mathbf{\Delta}^1_1$ classes follows from sharps in much the same way as in the $\mathbf{\Sigma}^1_1$ case:

Theorem 7.12. *Let I be a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper. Let E be a $\mathbf{\Pi}^1_1$ equivalence relation on X with all classes $\mathbf{\Delta}^1_1$. If for all $z \in {}^\omega\omega$, $z^\sharp$ exists and $(\chi^I_E)^\sharp$ exists, then there is an I^+ $\mathbf{\Delta}^1_1$ set C such that $E \restriction C$ is $\mathbf{\Delta}^1_1$.*

The set χ^I_E is defined similarly to the $\mathbf{\Sigma}^1_1$ case.

Section 8 will consider $\mathbf{\Pi}^1_1$ equivalence relations with all classes countable. As mentioned above, ZFC can provide a positive answer to the main question for $\mathbf{\Sigma}^1_1$ equivalence relation with all countable classes. In the $\mathbf{\Pi}^1_1$ case, there is insufficient absoluteness to carry out the same proof. However, from the consistency of a remarkable cardinal, one can obtain the consistency of a positive answer to the main question for $\mathbf{\Pi}^1_1$ equivalence relation with all countable classes:

Theorem 8.10. *Let κ be a remarkable cardinal in L. Let $G \subseteq Coll(\omega, <\kappa)$ be $Coll(\omega, <\kappa)$ generic over L. In $L[G]$, if I is a σ-ideal with $\mathbb{P}_I$ proper and E is a $\mathbf{\Pi}^1_1$ equivalence relation with all classes countable, then there exists some I^+ $\mathbf{\Delta}^1_1$ set C such that $E \restriction C$ is $\mathbf{\Delta}^1_1$.*

There is also a similar result using a weakly compact cardinal but $\mathbb{P}_I$ must be a $\aleph_1$-c.c. forcing.

In fact, much more holds in the model from the above theorem. Using some ideas of Neeman and Norwood, there is actually a positive answer to the main question for all equivalence relations in $L(\mathbb{R})$ with all classes $\mathbf{\Delta}^1_1$ in the above model. (See [6].)

Section 9 will show that in L, the main question for $\mathbf{\Delta}_2^1$ equivalence relations with all classes $\mathbf{\Delta}_1^1$ (in fact countable) is false.

Theorem 9.9. *In L, there is a $\mathbf{\Delta}_2^1$ equivalence relation with all classes countable such that for all σ-ideals I and all I^+ $\mathbf{\Delta}_1^1$ sets C, $E \upharpoonright C$ is not $\mathbf{\Delta}_1^1$.*

[8] has shown that a positive answer to the main question for projective equivalence relations with all classes $\mathbf{\Delta}_1^1$ holds under strong large cardinal assumptions.

Finally, the last section will summarize the work of the paper from the point of view of showing the consistency of a negative answer to the main question. Related questions will be introduced. Some dubious speculations about how a negative answer could be obtained will be discussed.

Drucker, in [11], has independently obtained some results that are very similar to those which appear in this paper: He has shown that a positive answer to the main question follows from a measurable cardinal using similar ideas to those appearing in Section 4. He has obtained results for σ-ideals whose forcings are provably $\aleph_1$-c.c. which are similar to Section 5. Drucker also proved the results of Section 9 of this paper using a very similar equivalence relation. In [11], Drucker also considers more general forms of canonization than that which appears in this paper.

Since this paper, more progress has been made concerning positive answers to the main question for larger classes of equivalence relations. [8] used arguments involving homogeneous trees and games to establish a positive answer to the main question for projective equivalence relations (and beyond) with all classes $\mathbf{\Delta}_1^1$ under strong large cardinal and determinacy assumptions. [6] shows that an upper on the consistency strength of a positive answer to the main question for $L(\mathbb{R})$ equivalence relations with all classes $\mathbf{\Delta}_1^1$ is the existence of a remarkable cardinal. [27] has shown that a positive answer holds for more general forms of the main question for more abstract pointclasses under AD^+.

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2. Basic Concepts

This section reviews the basics of idealized forcing and formalizes the main question of interest.

Definition 2.1. Let I be a σ-ideal on a Polish space X. Let $\mathbb{P}_I$ be the collection of all I^+ $\mathbf{\Delta}_1^1$ subsets of X. Let $\leq_{\mathbb{P}_I} = \subseteq$. Let $1_{\mathbb{P}_I} = X$. $(\mathbb{P}_I, \leq_{\mathbb{P}_I}, 1_{\mathbb{P}_I})$ is the forcing associated with the ideal I.

Fact 2.2. *Let I be a σ-ideal on a Polish space X. There is a $\mathbb{P}_I$-name $\dot{x}_{\text{gen}}$ such that for all $\mathbb{P}_I$-generic filters G over V and all B which are $\mathbf{\Delta}_1^1$ coded in V, $B \in G$ if and only if $\dot{x}_{\text{gen}}[G] \in B$.*

Proof. See [34], Proposition 2.1.2. ☐

Definition 2.3. Let I be a σ-ideal on a Polish space X. Let $M \prec H_\Theta$ be a countable elementary substructure for some cardinal Θ. $x \in X$ is $\mathbb{P}_I$-generic over M if and only if the set $\{A \in \mathbb{P}_I \cap M : x \in A\}$ is a $\mathbb{P}_I$-generic filter over M.

Fact 2.4. *Let I be a σ-ideal on a Polish space X. The following are equivalent:*

(i) $\mathbb{P}_I$ is a proper forcing.

(ii) For any sufficiently large cardinal Θ, for every $B \in \mathbb{P}_I$, and for every countable $M \prec H_\Theta$ with $\mathbb{P}_I \in M$ and $B \in M$, the set $C := \{x \in B : x \text{ is } \mathbb{P}_I\text{-generic over } M\}$ is I^+ $\mathbf{\Delta}_1^1$.

Proof. See [34], Proposition 2.2.2. Since this is the most important tool in this paper, a proof will be sketched:

(i) $\Rightarrow$ (ii) Let $B \in \mathbb{P}_I \cap M$ be arbitrary. It is straightforward to show that C is $\mathbf{\Delta}_1^1$. Suppose $C \in I$. Then by Fact 2.2, $B \Vdash_{\mathbb{P}_I}^V \dot{x}_{\text{gen}} \notin C$. This implies that there is some $D \subseteq \mathbb{P}_I$ which is dense, $D \in M$, and $B \Vdash_{\mathbb{P}_I}^V \check{M} \cap \check{D} \cap \dot{G} = \emptyset$. Therefore, there can be no $(M, \mathbb{P}_I)$-generic condition below B. $\mathbb{P}_I$ is not proper.

(ii) $\Rightarrow$ (i) Let $B \in \mathbb{P}_I \cap B$ be arbitrary. Suppose $C \notin I$. Then $C \Vdash_{\mathbb{P}_I}^V \dot{x}_{\text{gen}} \in C$. So for all $D \subseteq \mathbb{P}_I$ with D dense and $D \in M$, $C \Vdash_{\mathbb{P}_I}^V \check{M} \cap \check{D} \cap \dot{G} \neq \emptyset$. C is an $(M, \mathbb{P}_I)$-generic condition below B. $\mathbb{P}_I$ is proper. ☐

The following is some convenient notation:

Definition 2.5. ([19] Definition 1.15) Let Λ and Γ be classes of equivalence relations defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces. Let I be a σ-ideal on a Polish space X. Define $\Lambda \to_I \Gamma$ to mean: for all B which are I^+ $\mathbf{\Delta}^1_1$ subsets of X and every equivalence relation E defined on X such that $E \upharpoonright B \in \Lambda$, there exists an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ such that $E \upharpoonright C \in \Gamma$.

The following are some of the classes of equivalence relations that will appear later.

Definition 2.6. For any Polish space X, ev denotes the full equivalence relation on X consisting of a single class.

For any Polish space X, id is the equality equivalence relation.

$\mathbf{\Delta}^1_1$ denotes the class of all $\mathbf{\Delta}^1_1$ equivalence relations defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces. (In context, it should be clear when $\mathbf{\Delta}^1_1$ refers to the class of equivalence relations or just the $\mathbf{\Delta}^1_1$ definable subsets.)

${\mathbf{\Sigma}^1_1}^{\mathbf{\Delta}^1_1}$ is the class of all $\mathbf{\Sigma}^1_1$ equivalence relations defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces with all classes $\mathbf{\Delta}^1_1$.

${\mathbf{\Pi}^1_1}^{\mathbf{\Delta}^1_1}$ is the class of all $\mathbf{\Pi}^1_1$ equivalence relations defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces with all classes $\mathbf{\Delta}^1_1$.

${\mathbf{\Delta}^1_2}^{\mathbf{\Delta}^1_1}$ is the class of all $\mathbf{\Delta}^1_2$ equivalence relations defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces with all classes $\mathbf{\Delta}^1_1$.

A thin equivalence relation is an equivalence relation with no perfect set of inequivalent elements.

${\mathbf{\Sigma}^1_1}^{\text{thin}}$ is the class of all thin $\mathbf{\Sigma}^1_1$ equivalence relations defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces.

${\mathbf{\Sigma}^1_1}^{\text{thin } \mathbf{\Delta}^1_1}$ is the class of all thin $\mathbf{\Sigma}^1_1$ equivalence relations defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces and have all classes $\mathbf{\Delta}^1_1$.

${\mathbf{\Pi}^1_1}^{\aleph_0}$ denotes the class of all $\mathbf{\Pi}^1_1$ equivalence relations with all classes countable and defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces.

A thin set is a set without a perfect subset.

Let ${\mathbf{\Pi}^1_1}^{\text{thin}}$ denote the class of all $\mathbf{\Pi}^1_1$ equivalence relations with all classes thin and defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces.

Kanovei, Sabok, and Zapletal asked the following question:

Question 2.7. ([19] Question 4.28) If I is a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper, then does ${\mathbf{\Sigma}^1_1}^{\mathbf{\Delta}^1_1} \to_I \mathbf{\Delta}^1_1$ hold?

This paper will address this question and its various related forms for other classes of definable equivalence relations.

3. Examples

This section gives known results concerning the main question and some examples.

Proposition 3.1. *Let Γ_1 denote the class of equivalence relations $\mathbf{\Delta}^1_1$ reducible to orbit equivalence relations of Polish group actions. Then $\Gamma_1 \to_I \mathbf{\Delta}^1_1$ for any σ-ideal I on X such that $\mathbb{P}_I$ is proper.*

Proof. See [19], Theorem 4.26. □

The main question also holds for $\mathbf{\Sigma}^1_1$ equivalence relations with all classes countable. The following is an example of a Σ^1_1 but not $\mathbf{\Delta}^1_1$ equivalence relation with classes of size at most two: Let $A \subseteq {}^\omega 2$ be any Σ^1_1 but not $\mathbf{\Delta}^1_1$ set. Define the equivalence relation E on ${}^\omega 2 \times 2$ by

$$(x, i)\ E\ (y, j) \Leftrightarrow (x = y \wedge i = j) \vee (x = y \wedge x \in A).$$

E is Σ^1_1. For all $z \in {}^\omega 2$, $z \in A$ if and only if $(z, 0)\ E\ (z, 1)$. E can not be $\mathbf{\Delta}^1_1$.

Proposition 3.2. *Let Γ_2 denote the class of $\mathbf{\Sigma}^1_1$ equivalence relations with all classes countable. Then $\Gamma_2 \to_I \mathbf{\Delta}^1_1$ for any σ-ideal I on X such that $\mathbb{P}_I$ is proper.*

Proof. See [19], Theorem 4.27. The proof is provided below to emphasize a particular observation.

Fix a $B \subseteq X$ which is I^+ $\mathbf{\Delta}^1_1$. As E is $\mathbf{\Sigma}^1_1$, there exists some $z \in {}^\omega 2$ such that E is $\Sigma^1_1(z)$. For each $x \in X$, $[x]_E$ is $\Sigma^1_1(x,z)$. Since every $\Sigma^1_1(x,z)$ set with a non-$\Delta^1_1(x,z)$ element has a perfect subset (see [24] Theorem 6.3), the statement "all E-classes are countable" is equivalent to

$$(\forall x)(\forall y)(y \; E \; x \Rightarrow y \in \Delta^1_1(x,z))$$

As the relation "$y \in \Delta^1_1(x,z)$" in variables x and y is $\Pi^1_1(z)$, the above is $\Pi^1_1(z)$. By Mostowski absoluteness, $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I}$ "all E-classes are countable". There is some $\mathbb{P}_I$-name τ such that $B \Vdash_{\mathbb{P}_I} \tau \in {}^\omega X \wedge \tau$ enumerates $[\dot{x}_{\text{gen}}]_E$. By [34] Proposition 2.3.1, there exists some $B' \subseteq B$ with $B' \in \mathbb{P}_I$ and a $\mathbf{\Delta}^1_1$ function f such that $B' \Vdash_{\mathbb{P}_I} f(\dot{x}_{\text{gen}}) = \tau$. Choose $M \prec H_\Theta$ with Θ sufficiently large and M contains $\mathbb{P}_I$, B', τ, and the code for f. By Fact 2.4, let $C \subseteq B'$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic over M elements in B'.

The claim is that for $x, y \in C$, $x \; E \; y$ if and only if $(\exists n)(f(n) = y)$. This is because, for all $x \in C$, $M[x] \models f(x)$ enumerates $[x]_E$. Let N be the Mostowski collapse of $M[x]$. One can always assume the transitive closure of elements of X is a subset of M (for instance, one could have identified X with ${}^\omega\omega$). Therefore the Mostowski collapse map does not move elements of X. Hence $N \models f(x)$ enumerates $[x]_E$. The statement "$f(x)$ enumerates $[x]_E$" is the conjunction of a $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ formula coded in N. By Mostowski absoluteness, $f(x)$ enumerates $[x]_E$ in V. This proves the claim. Thus $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation. $\square$

By the work above, for each x which is $\mathbb{P}_I$-generic over M, $M[x] \models [x]_E$ is countable. So in $M[x]$, there exists some real u_x such that u_x codes an enumeration of $[x]_E$. In $M[x]$, $[x]_E$ is $\Delta^1_1(u_x)$. In the above proof, one showed that u_x remains an enumeration of $[x]_E$ even in V. So $[x]_E$ is $\Delta^1_1(u_x)$ even in V. This observation is the quintessential idea of the proof of the positive answer for the main question assuming large cardinal properties. Note that in the above proof, there was a $\mathbf{\Delta}^1_1$ function f which uniformly provided the enumeration of $[x]_E$ for each $x \in C$. This feature is not necessary.

Below, positive answers to the main question will be demonstrated for some specific equivalence relations.

Definition 3.3. For each $x \in {}^\omega 2$, ω_1^x is the least x-admissible ordinal above ω. Define the equivalence relation F_{ω_1} on ${}^\omega 2$ by $x \; F_{\omega_1} \; y$ if and only if $\omega_1^x = \omega_1^y$.

F_{ω_1} is an Σ^1_1 equivalence relation with all classes $\mathbf{\Delta}^1_1$.

Example 3.4. Let I be a σ-ideal on ${}^\omega 2$ with $\mathbb{P}_I$ proper. Then $\{F_{\omega_1}\} \rightarrow_I \{\text{ev}\}$, i.e. there is an I^+ class.

Proof. Let B be an arbitrary I^+ $\mathbf{\Delta}^1_1$ set. Choose $M \prec H_\Theta$ where Θ is a sufficiently large cardinal and $\mathbb{P}_I, B \in M$. By Fact 2.4, let $C \subseteq B$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic over M reals in B. For each $x \in C$, $\omega_1^x \in M[x] \cap \text{ON}$. Since the ground model and the forcing extension have the same ordinals (by properness), $\omega_1^x \in M \cap \text{ON}$. For each $\alpha \in \text{ON}$, let $F^\alpha_{\omega_1} = \{x \in {}^\omega 2 : \omega_1^x = \alpha\}$. Each $F^\alpha_{\omega_1} = \emptyset$ or is an F_{ω_1}-class. $C = \bigcup_{\alpha \in M \cap \text{ON}} F^\alpha_{\omega_1} \cap C$. Since F_{ω_1}-classes are $\mathbf{\Delta}^1_1$, $F^\alpha_{\omega_1} \cap C$ is $\mathbf{\Delta}^1_1$ for all α. As M is countable, $M \cap \text{ON}$ is countable. There exists some $\alpha \in M \cap \text{ON}$ such that $F^\alpha_{\omega_1} \cap C$ is I^+ since I is a σ-ideal. For this α, $F_{\omega_1} \upharpoonright F^\alpha_{\omega_1} \cap C = \text{ev} \upharpoonright F^\alpha_{\omega_1} \cap C$. $\square$

Since $M \prec H_\Theta$, for each $x \in C$, there exists a countable admissible ordinal $\alpha > \omega_1^x$ with $\alpha \in M$. By Sacks theorem applied in M, let $y \in M$ be such that $\omega_1^y = \alpha$. Then $[x]_{F_{\omega_1}}$ is $\Delta^1_1(y)$. Again the phenomenon described above occurs: there exist some $y \in M[x]$ (in fact $y \in M$) such that $M[x] \models [x]_{F_{\omega_1}}$ is $\Delta^1_1(y)$ and $V \models [x]_{F_{\omega_1}}$ is $\Delta^1_1(y)$.

Actually, F_{ω_1} is classifiable by countable structures. Proposition 3.1 would have already shown $\{F_{\omega_1}\} \rightarrow_I \mathbf{\Delta}^1_1$. See [7] for more information about F_{ω_1}.

Definition 3.5. Define the equivalence relation E_{ω_1} on ${}^\omega 2$ by

$$x \; E_{\omega_1} \; y \Leftrightarrow (x \notin \text{WO} \wedge y \notin \text{WO}) \vee (\text{ot}(x) = \text{ot}(y))$$

where WO is the set of reals coding well-orderings and, for $x \in \text{WO}$, $\text{ot}(x)$ is the order type of the linear order coded by x.

E_{ω_1} is a Σ^1_1 equivalence relation with all classes $\mathbf{\Delta}^1_1$ except for one Σ^1_1 class consisting of the reals that do not code wellfounded linear orderings.

Example 3.6. Let I be a σ-ideal on ${}^\omega 2$ with $\mathbb{P}_I$ proper. Then $\{E_{\omega_1}\} \rightarrow_I \{\text{ev}\}$.

Proof. Let $B \subseteq {}^{\omega}2$ be I^+ $\mathbf{\Delta}_1^1$.

(Case I) There exists some $B' \leq_{\mathbb{P}_I} B$ such that $B' \Vdash_{\mathbb{P}_I} \dot{x}_{\text{gen}} \notin \text{WO}$: Let $M \prec H_\Theta$ be a countable elementary structure with Θ a sufficiently large cardinal and $\mathbb{P}_I, B' \in M$. By Fact 2.4, let $C \subseteq B'$ be the I^+ $\mathbf{\Delta}_1^1$ set of all $\mathbb{P}_I$-generic over M reals in B'. Let $x \in C$. By Fact 2.2, let $G_x \subseteq \mathbb{P}_I$ be the generic filter associated with x. $B' \in G_x$ since $x \in B'$. $B' \Vdash_{\mathbb{P}_I} \dot{x}_{\text{gen}} \notin \text{WO}$ implies that $M[x] \models x \notin \text{WO}$. Let N be the Mostowski collapse of $M[x]$. Since the Mostowski collapse map does not move reals, $x \in N$. Also $N \models x \notin \text{WO}$. Since WO is Π_1^1, $V \models x \notin \text{WO}$. Hence $E \restriction C$ consists of a single class. So $E \restriction C = \text{ev} \restriction C$.

(Case II) $B \Vdash_{\mathbb{P}_I} \dot{x}_{\text{gen}} \in \text{WO}$: Then choose $M \prec H_\Theta$, a countable elementary substructure, and Θ a sufficiently large cardinal. By Fact 2.4, let $C \subseteq B$ be the I^+ $\mathbf{\Delta}_1^1$ set of $\mathbb{P}_I$-generic over M reals in B. As in Case I, $B \Vdash_{\mathbb{P}_I} \dot{x}_{\text{gen}} \in \text{WO}$ implies that $V \models x \in \text{WO}$. So when $x \in C$, $\text{ot}(x) \in M[x] \cap \text{ON}$. For each ordinal $\alpha < \omega_1$, $E_{\omega_1}^\alpha := \{x \in \text{WO} : \text{ot}(x) = \alpha\}$ is a $\mathbf{\Delta}_1^1$ set. Since $M[x]$ and M have the same ordinals and M is countable, $M[x]$ has only countably many ordinals. $C = \bigcup_{\alpha \in M \cap \text{ON}} E_{\omega_1}^\alpha \cap C$. $E_{\omega_1}^\alpha \cap C$ is $\mathbf{\Delta}_1^1$ for each α. Since I is a σ-ideal, there is some $\alpha \in M \cap \text{ON}$ such that $E_{\omega_1}^\alpha \cap C$ is I^+. So for this α, $E_{\omega_1} \restriction E_{\omega_1}^\alpha \cap C = \text{ev} \restriction E_{\omega_1}^\alpha \cap C$. ☐

Note that E_{ω_1} does not have all classes $\mathbf{\Delta}_1^1$. However, it is a thin Σ_1^1 equivalence relation. It will be shown later that the main question can be answered positively for thin $\mathbf{\Sigma}_1^1$ equivalence relation regardless of whether the classes are all $\mathbf{\Delta}_1^1$.

Next, there is one further enlightening example which does not fall under the scope of Proposition 3.1 or Proposition 3.2.

Fact 3.7. *There exist a Π_1^1 set $D \subseteq {}^{\omega}\omega$, a Π_1^1 set $P \subseteq ({}^{\omega}\omega)^3$, and a Σ_1^1 set $S \subseteq ({}^{\omega}\omega)^3$ such that:*

(1) If $z \in D$, then for all $x, y \in {}^{\omega}\omega$, $P(z,x,y) \Leftrightarrow S(z,x,y)$.

(2) For all $z \in D$, the relation $x \; E_z \; y$ if and only $P(z,x,y)$ is an equivalence relation, which is $\mathbf{\Delta}_1^1$ by (1).

(3) If E is a $\mathbf{\Delta}_1^1$ equivalence relation, then there is a z such that $x \; E \; y \Leftrightarrow P(z,x,y)$.

Proof. See [9], Definition 14. ☐

Definition 3.8. ([9] Definition 29) Define the equivalence relation $E_{\mathbf{\Delta}_1^1}$ on $({}^{\omega}\omega)^2$ by

$$(z_1, x_1) \; E_{\mathbf{\Delta}_1^1} \; (z_2, x_2) \Leftrightarrow (z_1 = z_2) \wedge (\neg D(z_1) \vee S(z_1, x_1, x_2))$$

$E_{\mathbf{\Delta}_1^1}$ is a Σ_1^1 equivalence relation. For each z, define E_z by $x \; E_z \; y$ if and only if $(z,x) \; E_{\mathbf{\Delta}_1^1} \; (z,y)$. For all z, E_z is a $\mathbf{\Delta}_1^1$ equivalence relation. If E is a $\mathbf{\Delta}_1^1$ equivalence relation, then there exists a $z \in D$ such that $E = E_z$. $E_{\mathbf{\Delta}_1^1}$ has all classes $\mathbf{\Delta}_1^1$.

Fact 3.9. *If E is a $\mathbf{\Delta}_1^1$ equivalence relation, then $E \leq_{\mathbf{\Delta}_1^1} E_{\mathbf{\Delta}_1^1}$.*

Proof. (See [9]) Let $z \in D$ such that $x \; E \; y \Leftrightarrow S(z,x,y)$ for all $x, y \in {}^{\omega}\omega$. Then $f : {}^{\omega}\omega \to ({}^{\omega}\omega)^2$ defined by $f(x) = (z,x)$ is the desired reduction. ☐

Proposition 3.10. *$E_{\mathbf{\Delta}_1^1}$ is a Σ_1^1 equivalence relation with all classes $\mathbf{\Delta}_1^1$, has uncountable classes, and is not reducible to the orbit equivalence relation of a Polish group action.*

Proof. All except the last statement have been mentioned above. Let E_1 be the equivalence relation on ${}^{\omega}({}^{\omega}2)$ defined by $x \; E_1 \; y$ and and only if $(\exists m)(\forall n \geq m)(x(n) = y(n))$. E_1 is a Δ_1^1 equivalence relation so by Fact 3.9, $E_1 \leq_{\mathbf{\Delta}_1^1} E_{\mathbf{\Delta}_1^1}$. E_1 is not $\mathbf{\Delta}_1^1$ reducible to any orbit equivalence relation of a Polish group action, by [22] Theorem 4.2. ☐

$E_{\mathbf{\Delta}_1^1}$ is a Σ_1^1 equivalence relation which does not fall under Proposition 3.1 or Proposition 3.2. Next, it will be shown that the main question formulated for $E_{\mathbf{\Delta}_1^1}$ has a positive answer.

Theorem 3.11. *Let I be a σ-ideal on $({}^{\omega}\omega)^2$ such that $\mathbb{P}_I$ is a proper forcing, then $\{E_{\mathbf{\Delta}_1^1}\} \to_I \mathbf{\Delta}_1^1$.*

Proof. Since D is Π_1^1, let T be a recursive tree on $\omega \times \omega$ such that $x \in D$ if and only if T^x is well-founded. Define

$$D_\alpha := \{x : \text{rk}(T^x) < \alpha\}$$

Each D_α is $\mathbf{\Delta}^1_1$. Define

$$(z_1,x_1)\ E^\alpha_{\mathbf{\Delta}^1_1}(z_2,x_2) \Leftrightarrow z_1 = z_2 \wedge (\neg D_\alpha(z_1) \vee P(z_1,x_1,x_2)) \Leftrightarrow z_1 = z_2 \wedge (\neg D_\alpha(z_1) \vee (S(z_1,x_1,x_2)))$$

Each $E^\alpha_{\mathbf{\Delta}^1_1}$ is $\mathbf{\Delta}^1_1$ and $E_{\mathbf{\Delta}^1_1} = \bigcap_{\alpha<\omega_1} E^\alpha_{\mathbf{\Delta}^1_1}$.

Let $B \subseteq ({}^\omega\omega)^2$ be a $\mathbf{\Delta}^1_1$ I^+ set. Let $\pi_1 : ({}^\omega\omega)^2 \to {}^\omega\omega$ be the projection onto the first coordinate.

(Case I) $B \not\Vdash_{\mathbb{P}_I} \pi_1(\dot{x}_{\text{gen}}) \in D$: Then there exists some $B' \leq_{\mathbb{P}_I} B$ such that $B' \Vdash_{\mathbb{P}_I} \pi_1(\dot{x}_{\text{gen}}) \notin D$. Now let $M \prec H_\Theta$ be a countable elementary substructure with $B', \mathbb{P}_I \in M$ and Θ some sufficiently large cardinal. By Fact 2.4, let $C \subseteq B'$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic over M elements in B'. By elementarity, $B' \Vdash^M_{\mathbb{P}_I} \pi_1(\dot{x}_{\text{gen}}) \notin D$. So for all $x \in C$, $M[x] \models \pi_1(x) \notin D$. For each $x \in C$, let N_x denote the Mostowski collapse of $M[x]$. Note that the Mostowski collapse map does not move reals. Hence $N_x \models \pi_1(x) \notin D$. By Mostowski absoluteness, $\pi_1(x) \notin D$. So for all $(z_1,x_1),(z_2,x_2) \in C$, $(z_1,x_2)\ E_{\mathbf{\Delta}^1_1}(z_2,x_2) \Leftrightarrow z_1 = z_2$. So $E_{\mathbf{\Delta}^1_1} \restriction C$ is $\mathbf{\Delta}^1_1$.

(Case II) Otherwise $B \Vdash_{\mathbb{P}_I} \pi_1(\dot{x}_{\text{gen}}) \in D$: Let $M \prec H_\Theta$ be a countable elementary substructure with $B, \mathbb{P}_I \in M$ and Θ some sufficiently large cardinal. By Fact 2.4, let $C \subseteq B$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic over M elements in B. By elementarity, $B \Vdash^M_{\mathbb{P}_I} \pi_1(\dot{x}_{\text{gen}}) \in D$. For all $x \in C$, $M[x] \models \pi_1(x) \in D$. $M[x] \models (\exists\alpha < \omega_1)(\text{rk}(T^{\pi_1(x)}) < \alpha)$. Let $\beta = M \cap \omega_1$. Since $\mathbb{P}_I$ preserves $\aleph_1$, $\omega_1^{M[x]} = \omega_1^M$ for all $x \in C$. For each $x \in C$, let N_x be the Mostowski collapse of $M[x]$. Note that the Mostowski collapse map does not move any reals. Then for all $x \in C$, $N_x \models (\exists \alpha < \omega_1^{N_x} = \beta)(\text{rk}(T^{\pi_1(x)}) < \alpha)$. For each $x \in C$, there is some $\alpha < (\omega_1)^{N_x}$ such that $N_x \models \text{rk}(T^{\pi_1(x)}) < \alpha$. After expressing this statement using a real in N_x that codes the countable (in N_x) ordinal α, Mostowski absoluteness implies that $\text{rk}(T^{\pi_1(x)}) < \alpha$. It has been shown that for all $x \in C$, $\text{rk}(T^{\pi_1(x)}) < \beta$. For all $x \in C$, $\pi_1(x) \in D_\beta$. $E_{\mathbf{\Delta}^1_1} \restriction C = E^\beta_{\mathbf{\Delta}^1_1} \restriction C$. The latter is $\mathbf{\Delta}^1_1$. □

The above proof motivates the ideas used in the next section.

4. Positive Answer for $\mathbf{\Sigma}^1_1$ $^{\mathbf{\Delta}^1_1}$

Using some of the ideas from the earlier examples, it will be shown that a positive answer to the main question follows from large cardinals. Avoiding any explicit mention of iteration principles, a crude result for the positive answer is first given assuming some generic absoluteness and the existence of tree representations that behave very nicely with generic extensions. This result will illustrate all the main ideas before going into the more optimal but far more technical proof using interable structures.

For simplicity, assume that E is a $\mathbf{\Sigma}^1_1$ equivalence relation on ${}^\omega\omega$.

First, a classical result about $\mathbf{\Sigma}^1_1$ equivalence relations.

Fact 4.1. *Let E be a $\Sigma^1_1(z)$ equivalence relation on ${}^\omega\omega$. Then there exists $\mathbf{\Delta}^1_1$ relations E_α, for $\alpha < \omega_1$, with the property that if $\alpha < \beta$, then $E_\alpha \supseteq E_\beta$, $E = \bigcap_{\alpha<\omega_1} E_\alpha$, and there exists a club set $C \subseteq \omega_1$ such that for all $\alpha \in C$, E_α is an equivalence relation.*

Proof. See [3]. Since E is $\Sigma^1_1(z)$, let T be a z-recursive tree on $\omega \times \omega \times \omega$ such that $(x,y) \in E$ if and only if $T^{(x,y)}$ is illfounded. For each $\alpha < \omega_1$, define E_α by $(x,y) \in E_\alpha \Leftrightarrow \text{rk}(T^{(x,y)}) > \alpha$. Observe that E_α is $\Delta^1_1(z,c)$ for any c which codes the ordinal α.

The verification of the rest of the theorem is an application of the boundedness theorem and can be found in any reference on the descriptive set theory of equivalence relations. (See Lemma 7.2 for a similar result in the $\mathbf{\Pi}^1_1$ case.) □

For the rest of this section, fix a z-recursive tree T as in the proof above. $\{E_\alpha : \alpha < \omega_1\}$ will refer to the sequence of $\mathbf{\Delta}^1_1$ equivalence relations obtained from T.

Lemma 4.2. *Let E be a $\Sigma^1_1(z)$ equivalence relation. Let $x, y \in {}^\omega\omega$ be such that $[x]_E$ is $\Pi^1_1(y)$. Let δ be an ordinal such that $\omega_1^{x\oplus y\oplus z} < \delta$ and E_δ is an equivalence relation. Then $[x]_E = [x]_{E_\delta}$.*

Proof. Define $E' \subseteq ({}^\omega\omega)^2$ by

$$a\ E'\ b \Leftrightarrow (a \in [x]_E \wedge b \in [x]_E) \vee (a \notin [x]_E \wedge b \notin [x]_E)$$

E' is $\Pi^1_1(x \oplus y \oplus z)$. $({}^\omega\omega)^2 - E'$ is then $\Sigma^1_1(x \oplus y \oplus z)$. $({}^\omega\omega)^2 - E' \subseteq ({}^\omega\omega)^2 - E$. By the effective boundedness theorem, there exists an $\alpha < \omega_1^{x \oplus y \oplus z} < \delta$ such that for all $(x, y) \in ({}^\omega\omega)^2 - E'$, $\text{rk}(T^{(x,y)}) \leq \alpha$. Hence $E' \supseteq E_\alpha$.

Since $E \subseteq E_\delta$, $[x]_E \subseteq [x]_{E_\delta}$. Since $E_\delta \subseteq E_\alpha \subseteq E'$, $[x]_{E_\delta} = \{u : (u, x) \in E_\delta\} \subseteq \{u : (u, x) \in E'\} = [x]_{E'} = [x]_E$. Therefore, $[x]_E = [x]_{E_\delta}$. □

Lemma 4.2 gives an upper bound on the ordinal level of the sequence $\{E_\alpha : \alpha < \omega_1\}$ where a $\mathbf{\Pi}^1_1$ E-class stabilizes. According to this lemma, a crucial piece of information in finding this bound is a real y which can be used as a parameter in some Π^1_1 definition of the $\mathbf{\Pi}^1_1$ E-class. Rather than knowing a particular Π^1_1 code, it suffices to know where some particular code lives:

Lemma 4.3. *Let E be a $\Sigma^1_1(z)$ equivalence relation. Let $\mathbb{P}$ be a forcing in M which adds a generic real. Choose Θ to be a regular cardinal greater than $|\mathbb{P}|^+$. Let $M \prec H_\Theta$ be a countable elementary substructure with $z, \mathbb{P} \in M$ and contains $|\mathbb{P}|^+$. Suppose for all g which are $\mathbb{P}$-generic over M, there exists a $y \in M[g]$ such that $V \models [g]_E$ is $\Pi^1_1(y)$. Then there exists a countable ordinal α such that for all $\mathbb{P}$-generic over M reals g, $[g]_E = [g]_{E_\alpha}$.*

Proof. Let M' be the Mostowski collapse of M. Let α be the image under the Mostowski collapse map of $|\mathbb{P}|^+$. α is an uncountable successor cardinal in $M'[g]$ for all g which are $\mathbb{P}$-generic over M. Let γ be the image of $|\mathbb{P}|$ under this Mostowski collapse. Of course, $\alpha = \gamma^+$.

Now fix such a g. Let h be $\text{Coll}(\omega, \gamma)$-generic over $M[g]$. Note that $\aleph_1^{M[g][h]} = \alpha$. α is a $(g \oplus y \oplus z)$-admissible ordinal greater than $\omega_1^{g \oplus y \oplus z}$ since α is a cardinal of $M'[g][h]$. Note that since $M'[g][h]$ is countable, $\alpha < \omega_1^V$.

E is a $\Sigma^1_1(z)$ equivalence relation in M'. Therefore, the statement "E is an equivalence relation" is $\Pi^1_2(z)$. By Schoenfield absoluteness, this statement is absolute to any forcing extension of M'. So E remains an equivalence relation in $M'[g][h]$. Therefore, Fact 4.1 holds in $M'[g][h]$. There exists a club set of $\beta < (\aleph_1)^{M'[g][h]} = \alpha$ so that E_β is an equivalence relation in $M'[g][h]$. As β is countable in $M'[g][h]$, there is some real $c \in M'[g][h]$ which codes β. E_β is $\Delta^1_1(z, c)$. Therefore, the statement "E_β is an equivalence relation" is $\mathbf{\Pi}^1_1$ in $M'[g][h]$. By Mostowski absoluteness between $M'[g][h]$ and V, E_β is an equivalence relation in V. $E_\alpha = \bigcap_{\beta < \alpha} E_\beta$ is an equivalence relation in V since it is an intersection of equivalence relations in V.

Now Lemma 4.2 can be applied to show that $[g]_E = [g]_{E_\alpha}$. This α is as required and the proof is complete.

A close inspection of the argument shows that if there was a common ordinal α so that $\alpha = \aleph_1^{M'[g]}$ for all g's which are $\mathbb{P}$-generic over M, then one could do the argument above using α and $M[g]$ without introducing any Lévy collapses. For instance, this would hold if $\mathbb{P}$ was $\aleph_1$-preserving.

The argument above actually shows that any $\alpha \in M'$ which is a successor cardinal in M' greater than the image of $|\mathbb{P}|$ under the Mostowski collapse map would also work. Therefore, if the initial model M was chosen as above with the additional property that M has no largest cardinal, then $\alpha = M' \cap \text{ON}$ would also work. □

Remark 4.4. Also there are more careful versions of Lemma 4.1 in which all the E_α's are equivalence relations which could be used to avoid this issue entirely. However, the simpler form of Lemma 4.1 was used so that it could be more easily applied to the less familiar $\mathbf{\Pi}^1_1$ setting in Lemma 7.2.

Now returning to the setting of the main question: Suppose E is a $\Sigma^1_1(z)$ equivalence relation with all classes $\mathbf{\Delta}^1_1$. Let I be a σ-ideal with $\mathbb{P}_I$-proper. According to Lemma 4.3, if one could find some $M \prec H_\Theta$ such that whenever x is $\mathbb{P}_I$-generic over M, a Π^1_1 code for $[x]_E$ resides inside $M[x]$, then letting C be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic reals over M, there would exist some $\alpha < \omega_1$ such that $E \restriction C = E_\alpha \restriction C$. Hence $E \restriction C$ is $\mathbf{\Delta}^1_1$.

A plausible candidate for the Π^1_1 code of $[x]_E$ which is an element of $M[x]$ would be some y such that $M[x] \models [x]_E$ is $\Pi^1_1(y)$. However, $M[x]$ may not think $[x]_E$ is $\mathbf{\Delta}^1_1$. The statement "all E-class are $\mathbf{\Delta}^1_1$" is $\Pi^1_4(z)$. If V satisfies $\mathbf{\Pi}^1_4$-generic absoluteness, one can choose $M \prec H_\Theta$ such that some particular $\Pi^1_4(z)$ statement becomes absolute between M and all its generic extensions. So in such a structure M, $M[x]$ will think $[x]_E$ is $\mathbf{\Delta}^1_1$.

Now in $M[x]$, there is some y such $M[x] \models [x]_E$ is $\Pi^1_1(y)$. In general, it is not clear if $[x]_E$ is $\Pi^1_1(y)$ in V. The formula "$[x]_E$ is $\Pi^1_1(y)$" is $\Pi^1_2(z)$. One can not use Schoenfield absoluteness between $M[x]$ (or rather its transitive collapse) and V since it is not the case that $\omega_1^V \subseteq M[x]$ because $M[x]$ is countable in V. So what is needed is some $M \prec H_\Theta$ such that for all $\mathbb{P}_I$-generic over M real x, a certain $\Pi^1_2(z)$ formula is absolute between $M[x]$ and V. The concept of universal Baireness can be used to remedy this issue.

Definition 4.5. ([12]) $A \subseteq {}^\omega\omega$. A is universally Baire if and only if there exists $\alpha, \beta \in \mathrm{ON}$ and trees U on $\omega \times \alpha$ and W on $\omega \times \beta$ such that
(1) $A = p[U]$. ${}^\omega\omega - A = p[W]$.
(2) For all $\mathbb{P}$, $1_\mathbb{P} \Vdash_\mathbb{P} p[\check{U}] \cup p[\check{W}] = {}^\omega\omega$,
where p of a tree denotes the projection of the tree.

Fact 4.6. *Suppose A is a $\mathbf{\Sigma}^1_2$ set defined by a $\mathbf{\Sigma}^1_2$ formula $\varphi(x)$. Let U and W be trees witnessing that A is universally Baire. Then $1_\mathbb{P} \Vdash_\mathbb{P} (\forall x)(\varphi(x) \Leftrightarrow x \in p[\check{U}])$.*

Proof. See [12], page 221-222. □

Definition 4.7. Let E be a $\Sigma^1_1(z)$ equivalence relation. Define the set D by

$$(x, T) \in D \Leftrightarrow (T \text{ is a tree on } \omega \times \omega) \wedge (\forall y)(y \; E \; x \Leftrightarrow T^y \in \mathrm{WF})$$

D is $\Pi^1_2(z)$.

Finally, the first result showing that a positive answer follows from some strong set theoretic assumptions:

Proposition 4.8. *Assume all $\mathbf{\Pi}^1_2$ sets are universally Baire and $\mathbf{\Pi}^1_4$-generic absoluteness holds. Let I be a σ-ideal such that $\mathbb{P}_I$ is proper. Then ${\mathbf{\Sigma}^1_1}^{\mathbf{\Delta}^1_1} \rightarrow_I \mathbf{\Delta}^1_1$.*

Proof. Let E be a $\Sigma^1_1(z)$ equivalence relation. Since all $\mathbf{\Pi}^1_2$ sets are universally Baire, let U and W be trees on $\omega \times \omega \times \alpha$ and $\omega \times \omega \times \beta$, respectively, where $\alpha, \beta \in \mathrm{ON}$, giving the universally Baire representations for the $\Pi^1_2(z)$ set D from Definition 4.7.

Suppose $B \in \mathbb{P}_I$. Using the reflection theorem, choose Θ large enough so that B, $\mathbb{P}_I$, z, U, and W are contained in H_Θ and H_Θ satisfies $\mathbf{\Pi}^1_4$-generic absoluteness for the statement "$(\forall x)(\exists T)((x, T) \in D)$". Let $M \prec H_\Theta$ be a countable elementary substructure containing B, $\mathbb{P}_I$, z, U, and W.

By Fact 2.4, let C be the I^+ $\mathbf{\Delta}^1_1$ subset of $\mathbb{P}_I$-generic over M reals in B. Let $g \in C$. Since E has all classes $\mathbf{\Delta}^1_1$, M satisfies $(\forall x)(\exists T)((x, T) \in D)$. Because M has generic absoluteness for this formula, $M[g] \models (\forall x)(\exists T)((x, T) \in D)$. There exists some $T \in M[g]$ such that $M[g] \models (g, T) \in D$.

By Fact 4.6, $M[g] \models (g, T) \in p[U]$. There exists $\Phi : \omega \to \alpha$ with $\Phi \in M[g]$ such that $(g, T, \Phi) \in [U]$. For each $n \in \omega$, $(g \upharpoonright n, T \upharpoonright n, \Phi \upharpoonright n) \in M$. For each n, $M[g] \models (g \upharpoonright n, T \upharpoonright n, \Phi \upharpoonright n) \in U$. By absoluteness, for each n, $M \models (g \upharpoonright n, T \upharpoonright n, \Phi \upharpoonright n) \in U$. Since $M \prec H_\Theta$, for all n, $(g \upharpoonright n, T \upharpoonright n, \Phi \upharpoonright n) \in U$ in the true universe V. Therefore, in V, $(g, T, \Phi) \in [U]$. $(g, T) \in p[U]$. $(g, T) \in D$. Note that $(g, T) \in D$ implies that $[g]_E$ is $\Pi^1_1(T)$.

It has been shown that for the chosen M, whenever g is $\mathbb{P}_I$-generic over M, there exists some $z \in M[g]$ such that $[g]_E$ is $\Pi^1_1(z)$. By Lemma 4.3, there is a countable ordinal α such that for all $\mathbb{P}_I$-generic over M reals g, $[g]_E = [g]_{E_\alpha}$. Hence $E \upharpoonright C = E_\alpha \upharpoonright C$. $E \upharpoonright C$ is $\mathbf{\Delta}^1_1$. □

Remark 4.9. By [2], Theorem 8, $\mathbf{\Pi}^1_4$ generic absoluteness is equiconsistent with every set having a sharp and the existence of a reflecting cardinal. The proof of [2], Theorem 8, shows that any structure satisfying Σ^1_4 generic absoluteness is closed under sharps. By [12], Theorem 3.4, all $\mathbf{\Pi}^1_2$ sets are universally Baire is equivalent to the existence of sharps for all sets. Hence, the hypothesis of Proposition 4.8 is equiconsistent with all sets having sharps and the existence of a reflecting cardinal.

Observe that $\mathbf{\Pi}^1_4$ generic absoluteness can be avoided for those $\mathbf{\Sigma}^1_1$ equivalence relation such that the statement "all E-classes are $\mathbf{\Delta}^1_1$" hold in any model of ZFC containing the defining parameters for E.

Proposition 4.10. *The consistency of ZFC, sharps of all sets exists, and there exists a reflecting cardinal implies the consistency of ${\mathbf{\Sigma}^1_1}^{\mathbf{\Delta}^1_1} \rightarrow_I \mathbf{\Delta}^1_1$ for all σ-ideal I on a Polish space such that $\mathbb{P}_I$ is proper.*

Proof. See Remark 4.9. □

Next, a positive answer to the main question will be obtained from assumptions with weaker consistency strength. The result above illustrates all the main ideas but used stronger than necessary assumptions: $\mathbf{\Pi}^1_4$ generic absolutness and all $\mathbf{\Pi}^1_2$ sets are universally Baire. $\mathbf{\Pi}^1_4$ generic absoluteness was used to preserve the statement "all E-classes are $\mathbf{\Delta}^1_1$". Below, it will be shown how sharps can be used to give a $\mathbf{\Pi}^1_3$ statement which is equivalent. Sharps will also be used to make the statement "all E-classes are $\mathbf{\Delta}^1_1$" true in the desired generic extensions, which is more subtle than just applying Martin-Solovay absoluteness. As observed above, sharps play an important role in $\mathbf{\Pi}^1_2$ sets being universally Baire. In the following, a much more careful analysis will be given to determine exactly which sharps are needed.

For the more optimal proof, iterable structures will be the main tools. Familar examples of iterable structures are V itself when V has a measurable cardinal, certain elementary substructures of V_Θ when V contains a measurable cardinal, and mice that come from the existence of sharps. In the first two, the measure exists in the structure, but in the latter, the measure is external. Some references for this material are [28], [1], and any text on inner model theory.

Let X be a set. A simple formulation of the statement "$X^\sharp$ exists" is that there is an elementary embedding $j : L[X] \to L[X]$ which fixes the elements of the transitive closure of X. Another classical formulation is that there is a closed unbounded class of indiscernible (called the Silver's indiscernibles) for $L[X]$. When $x \in {}^\omega 2$, the object $x^\sharp$ can be considered as a real coding statements about indiscernibles (in a language with countably many new constant symbols to be interpreted as a countably infinite subset of indiscernibles) true in $L[x]$. Another very useful characterization of $X^\sharp$ is given by mice:

Definition 4.11. (See [28], Definitions 10.18, 10.30, and 10.37.) Let $\mathscr{L} = \{\dot\in, \dot E, \dot U\}$ where $\dot\in$ is a binary relation symbol, and both $\dot E$ and $\dot U$ are unary predicates. Let X be a set. An X-mouse is a $\mathscr{L}$-structure, $\mathcal{M} = \langle J_\alpha[X], \in, X, U\rangle$, where $J_\alpha[X]$ is the α^{th} level of Jensen's fine structural hierarchy of $L[X]$, $\dot E^{\mathcal{M}} = X$, and $\dot U^{\mathcal{M}} = U$, with the following additional properties:

(a) $\mathcal{M}$ is an amenable structure, i.e., for all $z \in J_\alpha[X]$, $z \cap X \in J_\alpha[X]$ and $z \cap U \in J_\alpha[X]$.

(b) In the language $\{\dot\in\}$, $(J_\alpha[X], \in) \models$ "$\mathsf{ZFC} - \mathsf{P}$ and there is a largest cardinal".

(c) If κ is the largest cardinal of $(J_\alpha[X], \in)$, then $\mathcal{M} \models U$ is a κ-complete normal non-trivial ultrafilter on κ.

(d) $\mathcal{M}$ is iterable, i.e., every structure appearing in any putative linear iteration of $\mathcal{M}$ (by U) is wellfounded.

The statement $X^\sharp$ exists is also equivalent to the existence of an X-mouse. $X^\sharp$ will sometimes also denote the smallest X-mouse $\mathcal{M}$ in the sense that if $\mathcal{N}$ is an X-mouse, then there is an α such that the α^{th} iteration $\mathcal{M}_\alpha$ is $\mathcal{N}$.

Under the condition that sharps of all reals exist, the statement "all E-classes are $\mathbf{\Delta}^1_1$" will be shown to be $\mathbf{\Pi}^1_3$. This is a significant improvement since $\mathbf{\Pi}^1_3$ generic absoluteness is much easier to obtain.

Proposition 4.12. *Let E be a $\Sigma^1_1(z)$ equivalence relation. There is a $\Pi^1_3(z)$ formula $\varpi(v)$ in free variable v such that:*

Let $x \in {}^\omega 2$. If $(x \oplus z)^\sharp$ exists, then the statement "$[x]_E$ is $\mathbf{\Delta}^1_1$" is equivalent to $\varpi(x)$.

Assume for all $r \in {}^\omega\omega$, $r^\sharp$ exists. The statement "all E-classes are $\mathbf{\Delta}^1_1$" is equivalent to $(\forall x)\varpi(x)$. In particular, this statement is $\Pi^1_3(z)$.

Proof. For simplicity, assume E is a Σ^1_1 equivalence relation on ${}^\omega\omega$. Let T be a recursive tree on $\omega \times \omega \times \omega$ such that

$$(x, y) \in E \Leftrightarrow T^{x,y} \text{ is illfounded}$$

Claim : Assume $x^\sharp$ exists, then

$$\text{“}V \models [x]_E \text{ is } \mathbf{\Delta}^1_1\text{”} \Leftrightarrow \text{“}1_{\mathrm{Coll}(\omega,<c_1)} \Vdash_{\mathrm{Coll}(\omega,<c_1)} (\exists c)(c \in \mathrm{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c)))\text{”} \in x^\sharp$$

Here c_1 comes from $\{c_n : n \in \omega\}$, which is a collection of constant symbols used to denote indiscernibles. In the above, $x^\sharp$ is considered as a theory consisting of the statements about indiscernibles true in $L[x]$.

Proof of Claim: Assume $[x]_E$ is $\mathbf{\Delta}^1_1$. Then

$$(\exists \xi < \omega_1)(\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \xi)$$

Since $x^\sharp$ exists, ω_1 is inaccessible in $L[x]$ and $|\mathscr{P}^{L[x]}(\mathrm{Coll}(\omega, \xi))| = \aleph_0$. In V, there is a $g \subseteq \mathrm{Coll}(\omega, \xi)$ which is $\mathrm{Coll}(\omega, \xi)$ generic over $L[x]$. Since $g \in V$, there is a $c \in L[x][g] \subseteq V$ such that $c \in \mathrm{WO}$ and $\mathrm{ot}(c) = \xi$.

$$V \models (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c))$$

Since this statement above is Π^1_2, Schoenfield absoluteness gives

$$L[x][g] \models (\forall y)(\neg(y\ E\ x) \Rightarrow \text{rk}(T^{x,y}) < \text{ot}(c))$$

Using the weak homogeneity of $\text{Coll}(\omega, \xi)$,

$$L[x] \models 1_{\text{Coll}(\omega,\xi)} \Vdash_{\text{Coll}(\omega,\xi)} (\exists c)(c \in \text{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \text{rk}(T^{x,y}) < \text{ot}(c)))$$

The statement forced above is Σ^1_3. By upward absoluteness of $\boldsymbol{\Sigma}^1_3$ statements

$$L[x] \models 1_{\text{Coll}(\omega,<\omega_1)} \Vdash_{\text{Coll}(\omega,<\omega_1)} (\exists c)(c \in \text{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \text{rk}(T^{x,y}) < \text{ot}(c)))$$

$$\text{“}1_{\text{Coll}(\omega,<c_1)} \Vdash_{\text{Coll}(\omega,<c_1)} (\exists c)(c \in \text{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \text{rk}(T^{x,y}) < \text{ot}(c)))\text{”} \in x^\sharp$$

($\Leftarrow$) Assume

$$\text{“}1_{\text{Coll}(\omega,<c_1)} \Vdash_{\text{Coll}(\omega,<c_1)} (\exists c)(c \in \text{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \text{rk}(T^{x,y}) < \text{ot}(c)))\text{”} \in x^\sharp$$

Let $\xi < \omega_1$ be a Silver indiscernible for $L[x]$. Then

$$L[x] \models 1_{\text{Coll}(\omega,<\xi)} \Vdash_{\text{Coll}(\omega,<\xi)} (\exists c)(c \in \text{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \text{rk}(T^{x,y}) < \text{ot}(c)))$$

Again since $\xi < \omega_1$ and ω_1 is inaccessible in $L[x]$, $\mathscr{P}^{L[x]}(\text{Coll}(\omega, < \xi))$ is countable in V. In V, there exists $g \subseteq \text{Coll}(\omega, < \xi)$ which is $\text{Coll}(\omega, < \xi)$-generic over V.

$$L[x][g] \models (\exists c)(c \in \text{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \text{rk}(T^{x,y}) < \text{ot}(c)))$$

Since $g \in V$, $L[x][g] \subseteq V$ and there exists a $c \in L[x][g]$ such that

$$L[x][g] \models (\forall y)(\neg(y\ E\ x) \Rightarrow \text{rk}(T^{x,y}) < \text{ot}(c))$$

This statement is $\boldsymbol{\Pi}^1_2$. Since $L[x][g] \subseteq V$, Schoenfield absoluteness can be applied to give

$$V \models (\forall y)(\neg(y\ E\ x) \Rightarrow \text{rk}(T^{x,y}) < \text{ot}(c))$$

Therefore,

$$V \models [x]_E \text{ is } \boldsymbol{\Delta}^1_1$$

This concludes the proof of the claim.

The statement in variables v and w expressing “$w = v^\sharp$” is Π^1_2. Therefore

$$[x]_E \text{ is } \boldsymbol{\Delta}^1_1$$

if and only if

$$(\forall y)((y = x^\sharp) \Rightarrow \text{“}1_{\text{Coll}(\omega,<c_1)} \Vdash_{\text{Coll}(\omega,<c_1)} (\exists c)(c \in \text{WO} \wedge (\forall y)(\neg(y\ Ex) \Rightarrow \text{rk}(T^{x,y}) < \text{ot}(c)))\text{”} \in y)$$

The latter is $\Pi^1_3(x)$.

Similarly

$$(\forall x)([x]_E \text{ is } \boldsymbol{\Delta}^1_1)$$

if and only if

$$(\forall x)(\forall y)((y = x^\sharp) \Rightarrow \text{“}1_{\text{Coll}(\omega,<c_1)} \Vdash_{\text{Coll}(\omega,<c_1)} (\exists c)(c \in \text{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \text{rk}(T^{x,y}) < \text{ot}(c)))\text{”} \in y)$$

The latter is Π^1_3.

Let $\varpi(v)$ be the statement:

$$(\forall y)(y = v^\sharp \Rightarrow \text{“}1_{\text{Coll}(\omega,<c_1)} \Vdash_{\text{Coll}(\omega,<c_1)} (\exists c)(c \in \text{WO} \wedge (\forall y)(\neg(y\ E\ v) \Rightarrow \text{rk}(T^{v,y}) < \text{ot}(c)))\text{”} \in y)$$

By the above results, this works. □

So assuming for all $x \in {}^\omega 2$, $x^\sharp$ exists, the statement “all E-classes are $\boldsymbol{\Delta}^1_1$” is $\boldsymbol{\Pi}^1_3$. Below, some conditions for $\boldsymbol{\Pi}^1_3$ generic absoluteness will be explored. However, there is still a subtle point to be noted. Assume all sharps of reals exist and generic $\boldsymbol{\Pi}^1_3$ absoluteness holds for a forcing $\mathbb{P}$. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Then in $V[G]$, the statement $(\forall x)(\varpi(x))$ remains true by $\boldsymbol{\Pi}^1_3$ absoluteness. But if $V[G]$ does not satisfy all sharps of reals exist, then it may not be true that $(\forall v)(\varpi(v))$ is equivalent to the statement “all E-classes are $\boldsymbol{\Delta}^1_1$”. For the main question in the case of $\mathbb{P}_I$, only the fact that $[\dot{x}_{\text{gen}}]_E$ is $\boldsymbol{\Delta}^1_1$ will be of any concern. In $V[G]$, one has $(\forall x)(\varpi(x))$. In particular, if g is the generic real added by G, then $\varpi(g)$ holds. If $(g \oplus z)^\sharp$ exists, then Proposition 4.12 implies “$[g]_E$ is $\boldsymbol{\Delta}^1_1$” is equivalent to $\varpi(g)$. Hence $[g]_E$ is $\boldsymbol{\Delta}^1_1$ in $V[G]$. The following is a situation (applicable later) for which $(g \oplus z)^\sharp$ exists.

Fact 4.13. *Let A be a set. Suppose $A^\sharp$ exists and $j : L[A] \to L[A]$ is a nontrivial elementary embedding fixing the elements in the transitive closure of A. Suppose $\mathbb{P} \in L[A]$ is a forcing in $(V_{crit(j)})^{L[A]}$. Suppose $G \subseteq \mathbb{P}$ is generic over V (or just $L[A]$), then $\langle A, G\rangle^\sharp$ exists in $V[G]$.*

Proof. Since $\mathbb{P} \in (V_{\text{crit}(j)})^{L[A]}$, $j''\ \text{tc}(\{\mathbb{P}\}) = \text{tc}(\{\mathbb{P}\})$. Define the lift $\tilde{j} : L[\langle A, G\rangle] \to L[\langle A, G\rangle]$ by

$$\tilde{j}(\tau[G]) = j(\tau)[G]$$

By the usual arguments, $\tilde{j}$ is a nontrivial elementary embedding definable in $V[G]$. Hence $\langle A, G\rangle^\sharp$ exists. □

Next, a few more basic properties of iterable structures:

Fact 4.14. *Let $\mathscr{L} = \{\dot{\in}, \dot{U}\}$. Suppose $\mathcal{N} = (N, \in, V)$ is an iterable structure. Suppose $\mathcal{M} = (M, \in, U)$ is an $\mathscr{L}$-structure such that there exists an $\mathscr{L}$-elementary embedding $j : \mathcal{M} \to \mathcal{N}$. Then $\mathcal{M}$ is iterable.*

Proof. See [1], Lemma 18 for a proof. □

Fact 4.15. *Let $M \prec H_\Theta$ be a countable elementary substructure where Θ is some sufficienty large cardinal. Let $\mathcal{U}$ be an iterable structure and $\mathcal{U} \in M$. Let $\mathcal{U}^M = \mathcal{U} \cap M$. Then $\mathcal{U}^M$ is iterable.*

Proof. Let φ be some $\mathscr{L} = \{\dot{\in}, \dot{U}\}$ sentence. For any $x \in \mathcal{U} \cap M$, $\mathcal{U}^M \models \varphi(x)$ if and only if $M \models \mathcal{U} \models \varphi(x)$. Since $M \prec H_\Theta$, if and only if $V \models \mathcal{U} \models \varphi(x)$. Hence $\mathcal{U}^M \prec \mathcal{U}$ as an $\mathscr{L}$-structure. By Fact 4.14, $\mathcal{U}^M$ is iterable. □

As mentioned above, it is not possible in general to claim that $\mathbf{\Pi}^1_2$ statements are absolute between a countable model M and the universe V since Schoenfield absoluteness can not be applied when it is not the case that $\omega_1^V \subseteq M$. However, ω_1-iterable structures can be used to solve this problem by applying Schoenfield absoluteness in the ω_1 iteration.

Fact 4.16. *Let X be a set. Suppose $\mathcal{M} = (J_\alpha[X], \in, X, U)$ is an X-mouse. Then $J_\alpha[X]$ is Π^1_2-correct, that is, if φ is a $\mathbf{\Pi}^1_2$ sentence with parameters in $J_\alpha[X]$, then $J_\alpha[X] \models \varphi$ if and only if $V \models \varphi$.*

Let κ be the largest cardinal of $(J_\alpha[X], \in)$. Suppose $\mathbb{P} \in J_\alpha[X]$ is a forcing such that $(J_\alpha[X], \in) \models \mathbb{P} \in V_\kappa$. Then $J_\alpha[X]$ is $\mathbb{P}$-generically Π^1_2-correct, that is, for all $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over $J_\alpha[X]$ and $G \in V$, and any $\mathbf{\Pi}^1_2$-formula coded in $J_\alpha[X][G]$, $J_\alpha[X][G] \models \varphi$ if and only if $V \models \varphi$.

Proof. Let $\mathcal{M}_0 = \mathcal{M}$. Let $j_{0,\omega_1} : \mathcal{M}_0 \to \mathcal{M}_{\omega_1}$ denote the ω_1-iteration. $\mathcal{M}_{\omega_1}$ is well-founded, so let $\mathcal{M}_{\omega_1} = (J_\beta[X], \in, X, U_{\omega_1})$. Note that $\beta \geq \omega_1$. By [28], Lemma 10.21 (d), j_{0,ω_1} is a full (Σ_ω) elementary embedding in the language $\{\dot{\in}, \dot{E}\}$. So if φ is a $\mathbf{\Pi}^1_2$ statement with parameter in $J_\alpha[X]$ then $J_\alpha[X] \models \varphi$ if and only if $J_\beta[X] \models \varphi$. Since $\omega_1^V \subseteq J_\beta[X]$, $J_\beta[X] \models \varphi$ if and only if $V \models \varphi$.

For the second statement: Since $(J_\alpha[X], \in) \models \mathbb{P} \in V_\kappa$, j_{0,ω_1} does not move any elements in the transitive closure of $\mathbb{P}$. Also no new subsets of $\mathbb{P}$ appear in $J_\beta[X]$. Thus if G is $\mathbb{P}$-generic over $J_\alpha[X]$, then G is $\mathbb{P}$-generic over $J_\beta[X]$. Lift the elementary embedding $j_{0,\omega_1} : J_\alpha[X] \to J_\beta[X]$ to $\tilde{j}_{0,\omega_1} : J_\alpha[X][G] \to J_\beta[X][G]$ in the usual way: if $\tau \in (J_\alpha[X])^{\mathbb{P}}$, then

$$\tilde{j}_{0,\omega_1}(\tau[G]) = j_{0,\omega_1}(\tau)[G]$$

$\tilde{j}_{0,\omega_1}$ is a well-defined elementary embedding. Let φ be a Π^1_2 formula coded in $J_\alpha[X][G]$. Using this elementary embedding, $J_\alpha[X][G] \models \varphi$ if and only if $J_\beta[X][G] \models \varphi$. Since $\omega_1^V \subseteq J_\beta[X][G]$ and using Schoenfield absoluteness, $J_\beta[X][G] \models \varphi$ if and only if $V \models \varphi$. □

Fact 4.17. *Let $\mathbb{P}$ be a forcing. Suppose $\varphi(v)$ is a formula with one free variable. By fullness or the maximality principle (see* [23] *Theorem IV.7.1), there exists a name $\tau^{\mathbb{P}}_\varphi$ such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\exists v)(\varphi(v))$ if and only if $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \varphi(\tau^{\mathbb{P}}_\varphi)$. If $\tau^{\mathbb{P}}_\varphi$ is a name for a real, one may assume that it is a nice name for a real.*

Note that $\tau^{\mathbb{P}}_\varphi$ is not unique. $\tau^{\mathbb{P}}_\varphi$ will just refer to any $\mathbb{P}$-name that satisfies the above property.

Fact 4.18. *Consider the $\mathbf{\Sigma}^1_3$ sentence $(\exists v)(\varphi(v))$ where $\varphi(v)$ is Π^1_2. If $\langle \mathbb{P}, \tau^{\mathbb{P}}_\varphi\rangle^\sharp$ exists, then $(\exists v)(\varphi(v))$ is absolute between the ground model and $\mathbb{P}$-extensions.*

Proof. This is originally proved using the Martin-Solovay tree, which were implicit in [25]. The proof from [5] Theorem 3 is sketched below to make explicit what sharps are necessary.

Suppose $1_{\mathbb{P}} \Vdash^V_{\mathbb{P}} (\exists v)\varphi(v)$. Then $1_{\mathbb{P}} \Vdash^V_{\mathbb{P}} \varphi(\tau^{\mathbb{P}}_\varphi)$.

Note that $\mathbb{P} \in \langle \mathbb{P}, \tau_\varphi^{\mathbb{P}} \rangle^\sharp$ (where $\langle \mathbb{P}, \tau_\varphi^{\mathbb{P}} \rangle^\sharp$ is considered as a mouse as in Definition 4.11) and $\langle \mathbb{P}, \tau_\varphi^{\mathbb{P}} \rangle^\sharp \models \mathbb{P} \in V_\kappa$, where κ is the largest cardinal of $\langle \mathbb{P}, \tau_\varphi^{\mathbb{P}} \rangle^\sharp$.

Using some standard way of coding, let T be a tree of attempts to build a tuple $(\mathcal{M}, \mathbb{Q}, H, y, j)$ with the following properties:

(1) $\mathcal{M}$ is a countable structure satisfying (a), (b), and (c) from Definition 4.11.
(2) $\mathbb{Q} \in \mathcal{M}$ is a forcing.
(3) H is $\mathbb{Q}$-generic over $\mathcal{M}$.
(4) y is a real in $\mathcal{M}[H]$ and $\mathcal{M}[H] \models \varphi(y)$.
(5) $j : \mathcal{M} \to \langle \mathbb{P}, \tau_\varphi^{\mathbb{P}} \rangle^\sharp$ is an elementary embedding in the language $\{\dot\in, \dot E, \dot U\}$ with $j(\mathbb{Q}) = \mathbb{P}$.

Let G be an arbitrary $\mathbb{P}$-generic over V. Since $V[G] \models (\exists v)\varphi(v)$, $V[G] \models \varphi(\tau_\varphi^{\mathbb{P}}[G])$. By the downward absoluteness of Π_2^1 statements (which follows from Mostowski absoluteness), $\langle \mathbb{P}, \tau_\varphi^{\mathbb{P}} \rangle^\sharp[G] \models \varphi(\tau_\varphi^{\mathbb{P}}[G])$. By Downward-Lowenheim-Skolem, let $\mathcal{N}$ be a countable $\{\dot\in, \dot E, \dot U\}$ elementary substructure of $\langle \mathbb{P}, \tau_\varphi^{\mathbb{P}} \rangle^\sharp$ containing $\mathbb{P}$ and $\tau_\varphi^{\mathbb{P}}$. Let $\mathcal{M}$ be the Mostowski collapse of $\mathcal{N}$, and $j : \mathcal{M} \to \langle \mathbb{P}, \tau_\varphi^{\mathbb{P}} \rangle^\sharp$ be the induced elementary embedding. Let $\mathbb{Q} = j^{-1}(\mathbb{P})$. Let $H = j^{-1}[G]$. Let $y = j^{-1}(\tau_\varphi^{\mathbb{P}})[H]$. So in $V[G]$, $(\mathcal{M}, \mathbb{Q}, H, y, j)$ is a path through T.

Therefore, in $V[G]$, the tree T is illfounded. Hence it is illfounded in V by Δ_1-absoluteness. In V, let $(\mathcal{M}, \mathbb{Q}, H, y, j)$ be such a path. By Fact 4.14, $\mathcal{M}$ is iterable. By Fact 4.16, $\mathcal{M}[H] \models \varphi(y)$ implies $V \models \varphi(y)$. This establishes that $(\exists v)\varphi(v)$ is downward absolute from $V[G]$ to V. This completes the proof. ☐

Definition 4.19. Let I be a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper. If ϖ is the formula from Proposition 4.12, then $\neg(\forall v)\varpi(v)$ is $\mathbf{\Sigma}_3^1$ and can be written as $(\exists v)\zeta(v)$ where ζ is $\mathbf{\Pi}_2^1$. Let μ_E^I be $\tau_\zeta^{\mathbb{P}_I}$ from Fact 4.17.

Definition 4.20. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is proper. Consider the formula "$(\exists y)([\dot x_{\text{gen}}]_E$ is $\Pi_1^1(y))$". Write it as $(\exists y)\psi(y)$. By Fact 4.17, let σ_E^I be $\tau_\psi^{\mathbb{P}_I}$.

Definition 4.21. Suppose I is a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is proper. Let $E \in {\mathbf{\Sigma}_1^1}^{\mathbf{\Delta}_1^1}$. Define $\chi_E^I = \langle \mathbb{P}_I, \mu_E^I, \sigma_E^I \rangle$.

Despite the notation, χ_E^I is not unique since μ_E^I and σ_E^I are not unique.

The following result gives a positive answer to the main question for ${\mathbf{\Sigma}_1^1}^{\mathbf{\Delta}_1^1}$ using sharps for some small sets.

Theorem 4.22. *Suppose I is a σ ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is proper. If for all $x \in {}^\omega 2$, $x^\sharp$ exists and $(\chi_E^I)^\sharp$ exists for all $E \in {\mathbf{\Sigma}_1^1}^{\mathbf{\Delta}_1^1}$, then ${\mathbf{\Sigma}_1^1}^{\mathbf{\Delta}_1^1} \to_I \mathbf{\Delta}_1^1$.*

Proof. Let Θ be sufficiently large and $M \prec H_\Theta$ be countable elementary with $(\chi_E^I)^\sharp \in M$. Note that M thinks $(\chi_E^I)^\sharp$ exists and for all $x \in {}^\omega 2$, $x^\sharp$ exists.

First to show that $(\forall v)(\varpi(v))$ is $\mathbb{P}_I$-generically absolute for M: Since M satisfies all sharps of reals exist, Proposition 4.12 implies "all E-classes are $\mathbf{\Delta}_1^1$" is equivalent to $(\forall v)(\varpi(v))$. The latter is $\mathbf{\Pi}_3^1$ and so its negation is $\mathbf{\Sigma}_3^1$. Since $M \models \langle \mathbb{P}_I, \mu_E^I \rangle^\sharp$ exists, Fact 4.18 implies the statement, $(\forall v)(\varpi(v))$ is absolute between M and $\mathbb{P}_I$ extensions of M. Since M satisfies "all E-classes are $\mathbf{\Delta}_1^1$" and M satisfies all sharps of reals exist, M satisfies $(\forall v)(\varpi(v))$. Therefore, all $\mathbb{P}_I$ extensions of M satisfy the formula $(\forall v)(\varpi(v))$.

Since $\mathbb{P}_I^\sharp$ exists, there exists a $j : L[\mathbb{P}_I] \to L[\mathbb{P}_I]$ with $\mathbb{P}_I \in (V_{\text{crit}(j)})^{L[\mathbb{P}]}$. Therefore, Fact 4.13 implies $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} \dot x_{\text{gen}}^\sharp$ exists . So $M \models 1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} \dot x_{\text{gen}}^\sharp$ exists . By Proposition 4.12,

$$M \models 1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} ([\dot x_{\text{gen}}]_E \text{ is } \mathbf{\Delta}_1^1) \Leftrightarrow \varpi(\dot x_{\text{gen}})$$

Since all $\mathbb{P}_I$ extensions of M satisfy $(\forall v)(\varpi(v))$, all $\mathbb{P}_I$ extensions of M satisfy $[\dot x_{\text{gen}}]_E$ is $\mathbf{\Delta}_1^1$.

By the result of the previous paragraph, $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I}^M (\exists y)\psi(y)$, where ψ is the formula from Definition 4.20. Therefore, $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I}^M \psi(\sigma_E^I)$. Note that $\psi(y)$ is actually $\psi'(\dot x_{\text{gen}}, y)$, where ψ' is $\mathbf{\Pi}_2^1$ with parameters from M asserting that $[\dot x_{\text{gen}}]$ is $\Pi_1^1(y)$. Since $\dot x_{\text{gen}}$ is constructible from $\mathbb{P}_I$, the existence of $\langle \mathbb{P}_I, \dot x_{\text{gen}}, \sigma_E^I \rangle^\sharp$ follows from the existence of $\langle \mathbb{P}_I, \sigma_E^I \rangle^\sharp$. Applying the downward absoluteness of $\mathbf{\Pi}_2^1$ statements from $M[x]$ to $(\langle \mathbb{P}_I, \sigma_E^I \rangle^\sharp)^M[x]$ (where x is any $\mathbb{P}_I$-generic over M real) gives $(\langle \mathbb{P}_I, \sigma_E^I \rangle^\sharp)^M[x] \models \psi'(x, \sigma_E^I[x])$. By Fact 4.15, $(\langle \mathbb{P}_I, \sigma_E^I \rangle^\sharp)^M$ is still iterable. Applying Fact 4.16 (generic Π_2^1-correctness) to $(\langle \mathbb{P}_I, \sigma_E^I \rangle^\sharp)^M[x]$ and V, one has

$V \models \psi'(x, \sigma_E^I[x])$, where x is any $\mathbb{P}_I$-generic over M. So it has been shown that $M[x] \models [x]_E$ is $\Pi_1^1(\sigma_E^I[x])$ and $V \models [x]_E$ is $\Pi_1^1(\sigma_E^I[x])$.

Lemma 4.3 implies that there is some countable α such that for all x which are $\mathbb{P}_I$-generic over M, $[x]_{E_\alpha} = [x]_E$. Therefore if B is an arbitrary I^+ $\mathbf{\Delta}_1^1$ subset and C is the I^+ $\mathbf{\Delta}_1^1$ set of $\mathbb{P}_I$-generic over M reals in B, then $E \upharpoonright C = E_\alpha \upharpoonright C$. $\{E\} \rightarrow_I \mathbf{\Delta}_1^1$. ☐

Since $\mathbb{P}_I$ is a collection of subsets of ${}^\omega\omega$ and both μ_E^I and σ_E^I can be taken to be nice names for reals, χ_E^I is an element of $H_{(2^{\aleph_0})^+}$. Therefore, if there is a measurable or a Ramsey cardinal, then $(\chi_E^I)^\sharp$ will exist.

Corollary 4.23. *If $z^\sharp$ exists for all $z \in H_{(2^{\aleph_0})^+}$, then $\mathbf{\Sigma}_1^{1^{\mathbf{\Delta}_1^1}} \rightarrow_I \mathbf{\Delta}_1^1$ for all σ-ideal I such that $\mathbb{P}_I$ is proper.*

Corollary 4.24. *If there exists a Ramsey cardinal, then $\mathbf{\Sigma}_1^{1^{\mathbf{\Delta}_1^1}} \rightarrow_I \mathbf{\Delta}_1^1$ for all σ-ideal I such that $\mathbb{P}_I$ is proper.*

5. $\mathbf{\Sigma}_1^1$ Equivalence Relations and Some Ideals

Some partial results about the main question for $\mathbf{\Sigma}_1^1$ equivalence relations with all classes $\mathbf{\Delta}_1^1$ will be provided in this and the next section. These are proved using various different techniques and different set theoretic assumptions (usually of lower consistency strength than the full answer of the previous section). These results may be useful in understanding what combination of universes, $\mathbf{\Sigma}_1^1$ equivalence relations, and σ-ideals can not be used to demonstate the consistency of a negative answer to the main question.

In this section, the focus will be on the main question in the case of some classical ideals I with $\mathbb{P}_I$ proper.

Definition 5.1. Let X be a Polish space. Let $I_{\text{countable}} := \{A \subseteq X : |A| \leq \aleph_0\}$. $\mathbb{P}_{I_{\text{countable}}}$ is forcing equivalent to Sacks forcing.

Proposition 5.2. $\mathbf{\Sigma}_1^1 \rightarrow_{I_{countable}} \mathbf{\Delta}_1^1$.

Proof. Let E be any $\mathbf{\Sigma}_1^1$ equivalence relation. (Note that there is no condition on the classes being $\mathbf{\Delta}_1^1$ for this proposition.) Let B be $I_{\text{countable}}^+$ $\mathbf{\Delta}_1^1$ set, i.e. an uncountable $\mathbf{\Delta}_1^1$ set.

Suppose there is some $x \in B$ such that $[x]_E \cap B$ is uncountable. The perfect set property for the $\mathbf{\Sigma}_1^1$ set $[x]_E \cap B$ implies that $[x]_E \cap B$ has a perfect subset C. Then $E \upharpoonright C = \text{ev} \upharpoonright C$. So $\{E\} \rightarrow_{I_{\text{countable}}} \mathbf{\Delta}_1^1$.

Otherwise, $[x]_E \cap B$ is countable for all $x \in B$. $E \upharpoonright B$ is a $\mathbf{\Sigma}_1^1$ equivalence relation with all classes countable. Then $\{E\} \rightarrow_I \mathbf{\Delta}_1^1$ follows from Fact 3.2. ☐

Definition 5.3. Let I_{E_0} denote the σ-ideal σ-generated by the $\mathbf{\Delta}_1^1$ sets on which E_0 is smooth.

Fact 5.4. *$\mathbb{P}_{I_{E_0}}$ is forcing equivalent to Prikry-Silver forcing. Hence $\mathbb{P}_{I_{E_0}}$ is a proper forcing.*

Proof. See [33], Lemma 2.3.37. ☐

Fact 5.5. $\mathbf{\Sigma}_1^1 \rightarrow_{I_{E_0}} \{id, ev, E_0\}$

Proof. See [19], Theorem 7.1.1. ☐

Corollary 5.6. $\mathbf{\Sigma}_1^1 \rightarrow_{I_{E_0}} \mathbf{\Delta}_1^1$.

Proof. This follows immediately from Fact 5.5 since id, ev, and E_0 are all $\mathbf{\Delta}_1^1$ equivalence relations. ☐

Definition 5.7. Let I_{meager} be the σ-ideal σ-generated by the meager subsets of ${}^\omega\omega$ (or more generally any Polish space).

Let I_{null} be the σ-ideal σ-generated by the Lebesgue null subsets of ${}^\omega\omega$.

Kechris communicated to the author the following results concerning the meager ideal. Define a set to be I_{meager} measurable if and only if that set has the Baire property. Define a set to be I_{null} measurable if and only if that set is Lebesgue measurable. First, a well-known result on the additivity of the meager ideal and null ideal under certain types of unions.

Fact 5.8. *Let I be I_{meager} or I_{null}. Let $\{A_\eta\}_{\eta<\xi}$ be a sequence of sets in I. Define a prewellordering $\sqsubseteq$ on $\bigcup_{\eta<\xi} A_\eta$ by: $x \sqsubseteq y$ if and only if the least η such that $x \in A_\eta$ is less than or equal to the least η such that $y \in A_\eta$. If $\sqsubseteq$ is I-measurable (with the version of I defined on the product space), then $\bigcup_{\eta<\xi} A_\eta$ is in I.*

Proof. See [20], Proposition 1.5.1 for a proof. $\square$

Theorem 5.9. *Let I be I_{meager} or I_{null}. If all $\mathbf{\Pi}^1_3$ sets are I measurable, then $\mathbf{\Sigma}^1_1 \overset{\mathbf{\Delta}^1_1}{\to}_I \mathbf{\Delta}^1_1$. Moreover, if E is a $\mathbf{\Sigma}^1_1$ equivalence relation with all classes $\mathbf{\Delta}^1_1$ and B is I^+ $\mathbf{\Delta}^1_1$, then there exists a I^+ $\mathbf{\Delta}^1_1$ $C \subseteq B$ with $B \setminus C \in I$ and $E \restriction C$ is a $\mathbf{\Delta}^1_1$ equivalence relation.*

Proof. (Kechris) For simplicity, assume E is an equivalence relation on ${}^\omega\omega$. Let $B \subseteq {}^\omega\omega$ be I^+ $\mathbf{\Delta}^1_1$. For simplicity, assume $B = {}^\omega\omega$. $({}^\omega\omega)^2 \setminus E$ is a $\mathbf{\Pi}^1_1$ set. Let T be a tree on $\omega \times \omega \times \omega$ such that

$$\neg(x\ E\ y) \Leftrightarrow T^{(x,y)} \text{ is well-founded}$$

For each $\alpha < \omega_1$, let $A_\alpha = \{x : (\forall y)((x,y) \notin E \Rightarrow \text{rk}(T^{(x,y)}) < \alpha)\}$. First, the claim is that for all $x \in {}^\omega\omega$, there exists some $\alpha < \omega_1$ with $x \in A_\alpha$: To see this, fix x and let $L_x = \{(x,y) : y \notin [x]_E\}$. Since $[x]_E$ is $\mathbf{\Delta}^1_1$, L_x is $\mathbf{\Delta}^1_1$. $L_x \subseteq ({}^\omega\omega)^2 \setminus E$. By the boundedness theorem, there exists some $\alpha < \omega_1$ such that $\sup\{\text{rk}(T^{(x,y)}) : (x,y) \in L_x\} < \alpha$. $x \in A_\alpha$. It has been shown that ${}^\omega\omega = \bigcup_{\alpha<\omega_1} A_\alpha$.

The next claim is that there exists some $\alpha < \omega_1$ such that A_α is I^+: Suppose that for all $\alpha < \omega_1$, $A_\alpha \in I$. Note that there is a $\mathbf{\Pi}^1_2$ formula $\Phi(x,c)$ (using the tree T as a parameter) such that if $c \in \text{WO}$, then

$$\Phi(x,c) \Leftrightarrow (\forall y)(\text{rk}(T^{(x,y)}) < \text{ot}(c))$$

Define $\sqsubseteq$ using the sequence $\{A_\alpha : \alpha < \omega_1\}$. Then

$$x \sqsubseteq y \Leftrightarrow (\forall c)(c \in \text{WO} \Rightarrow (\Phi(y,c) \Rightarrow \Phi(x,c)))$$

$\sqsubseteq$ is $\mathbf{\Pi}^1_3$ on ${}^\omega\omega \times {}^\omega\omega$. By Fact 5.8, ${}^\omega\omega = \bigcup_{\alpha<\omega_1} A_\alpha$ is in I. Contradiction.

Choose an $\alpha < \omega_1$ such that A_α is I^+. Since $\mathbf{\Pi}^1_2$ sets are I-measurable, let C be $\mathbf{\Delta}^1_1$ I^+ such that $A_\alpha \triangle C \in I$. Thus $C \setminus A_\alpha \in I$. Since I is the σ-generated by certain $\mathbf{\Delta}^1_1$ sets, there exists a $\mathbf{\Delta}^1_1$ set $D \in I$ such that $C \setminus A_\alpha \subseteq D$. Let $B_0 = C \setminus D$. Note that B_0 is I^+ $\mathbf{\Delta}^1_1$ and $B_0 \subseteq A_\alpha$.

Now suppose $\xi < \omega_1$ and a sequence $\{B_\eta : \eta < \xi\}$ of $\mathbf{\Delta}^1_1$ I^+ sets has been defined with the property that if $\eta_1 \neq \eta_2$ then $B_{\eta_1} \cap B_{\eta_2} \in I$. Let $K_\xi = \bigcup_{\eta<\xi} B_\eta$. Define $A^\xi_\alpha = A_\alpha \setminus K_\xi$. ${}^\omega\omega \setminus K_\xi = \bigcup_{\alpha<\omega_1} A^\xi_\alpha$. If ${}^\omega\omega \setminus K_\xi$ is I^+, then repeating the above procedure produces some I^+ $\mathbf{\Delta}^1_1$ B_ξ with the property that for all $\eta < \xi$, $B_\eta \cap B_\xi \in I$ and for some $\alpha < \omega_1$, $B_\xi \subseteq A^\xi_\alpha \subseteq A_\alpha$.

Observe that for some $\xi < \omega_1$, ${}^\omega\omega \setminus K_\xi$ must be in I. This is because otherwise the construction succeeds in producing an antichain $\{B_\eta : \eta < \omega_1\}$ of cardinality $\aleph_1$ in $\mathbb{P}_I$. However, $\mathbb{P}_I$ has the $\aleph_1$-chain condition. Contradiction.

So choose ξ such that ${}^\omega\omega \setminus K_\xi \in I$. By construction, for each $\eta < \xi$, there is some $\alpha_\eta < \omega_1$ such that $B_\eta \subseteq A^\eta_{\alpha_\eta} \subseteq A_{\alpha_\eta}$. Since $\xi < \omega_1$, there is some $\mu < \omega_1$ such that $\sup\{\alpha_\eta : \eta < \xi\} < \mu$. Then $K_\xi = \bigcup_{\eta<\xi} A_{\alpha_\eta} \subseteq A_\mu$. Hence for all $x, y \in K_\xi$,

$$x\ E\ y \Leftrightarrow \text{rk}(T^{(x,y)}) \geq \mu$$

So K_ξ is I^+ $\mathbf{\Delta}^1_1$ with ${}^\omega\omega \setminus K_\xi \in I$ and $E \restriction K_\xi$ is a $\mathbf{\Delta}^1_1$ equivalence relation. $\square$

Theorem 5.10. *The consistency of* ZFC *implies the consistency of* ZFC *and* $\mathbf{\Sigma}^1_1 \overset{\mathbf{\Delta}^1_1}{\to}_{I_{meager}} \mathbf{\Delta}^1_1$.
The consistency of ZFC + Inaccessible Cardinal *implies the consistency of* ZFC *and* $\mathbf{\Sigma}^1_1 \overset{\mathbf{\Delta}^1_1}{\to}_{I_{null}} \mathbf{\Delta}^1_1$.

Proof. By [31], from a model ZFC, one can obtain a model of ZFC in which all $\text{OD}_{{}^\omega\omega}$ subsets of ${}^\omega\omega$ have the Baire property.

By [32], from a model of ZFC with an inaccessible cardinal, one can obtain a model of ZFC in which all $\text{OD}_{{}^\omega\omega}$ subsets of ${}^\omega\omega$ are Lebesgue measurable.

Then both results follow from Theorem 5.9. $\square$

Let κ be an inaccessible cardinal. $\text{Coll}(\omega, <\kappa)$ denotes the Lévy collapse of κ to ω_1. Since the generic extension of the Lévy collapse of an inaccessible to ω_1 (and the related Solovay's model) appears often in descriptive set theory, the following is worth mentioning:

Corollary 5.11. *Let κ be an inaccessible cardinal in V. Let $G \subseteq Coll(\omega, <\kappa)$ be $Coll(\omega, <\kappa)$-generic over V. Then in $V[G]$, $\mathbf{\Sigma}^1_1 \overset{\mathbf{\Delta}^1_1}{\to}_{I_{meager}} \mathbf{\Delta}^1_1$ and $\mathbf{\Sigma}^1_1 \overset{\mathbf{\Delta}^1_1}{\to}_{I_{null}} \mathbf{\Delta}^1_1$.*

Proof. [32] shows that in this model, all $\mathrm{OD}_{^\omega\omega}$ subsets of $^\omega\omega$ have the Baire property and are Lebesgue measurable. As above, the result follows from Theorem 5.9. $\square$

[31] shows that the existence of an inaccessible cardinal and the statement that all $\mathbf{\Pi}^1_3$ sets are Lebesgue measurable are equiconsistent.

To show that the above statement even for I_{null} is consistent relative to ZFC will require a slight modification of the above proof using a different set theoretic assumption.

Definition 5.12. Let I be a σ-ideal on a Polish space X. $\text{cov}(I)$ is the smallest cardinal κ such that there exists a set $U \subseteq I$ with $\bigcup U = X$ and $|U| = \kappa$.

Proposition 5.13. *Let I be I_{meager} or I_{null}. If all $\mathbf{\Pi}^1_2$ sets are I-measurable and $cov(I) > \aleph_1$, then $\mathbf{\Sigma}^{1\ \mathbf{\Delta}^1_1}_1 \rightarrow_I \mathbf{\Delta}^1_1$.*

Proof. The proof is similar to Theorem 5.9. In this case, one can conclude that for some $\alpha < \omega_1$, A_α is I^+ from the fact that $\text{cov}(I) > \aleph_1$ and $^\omega\omega = \bigcup_{\alpha<\omega_1} A_\alpha$. The I-measurability of $\mathbf{\Pi}^1_2$ sets is needed to find some $C \subseteq A_\alpha$ which is I^+ $\mathbf{\Delta}^1_1$. $\square$

Fact 5.14. *Let I be I_{meager} or I_{null}. $\mathsf{MA} + \neg\mathsf{CH}$ implies all $\mathbf{\Pi}^1_2$ sets are I-measurable and $cov(I) = 2^{\aleph_0} > \aleph_1$.*

Proof. See [26]. $\square$

Theorem 5.15. *The consistency of ZFC implies the consistency of ZFC and $\mathbf{\Sigma}^{1\ \mathbf{\Delta}^1_1}_1 \rightarrow_I \mathbf{\Delta}^1_1$ where I is I_{meager} or I_{null}.*

Proof. The consistency of ZFC implies the consistency of $\mathsf{ZFC} + \mathsf{MA} + \neg\mathsf{CH}$ by a well-known iterated forcing argument. $\square$

6. Thin $\mathbf{\Sigma}^1_1$ Equivalence Relations

Definition 6.1. An equivalence relation E on a Polish space X is thin if and only if there does not exists a perfect set of pairwise E-inequivalent elements.

Let $\mathbf{\Sigma}^{1\ \text{thin}}_1$ denote the class of thin $\mathbf{\Sigma}^1_1$ equivalence relations defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces.

Let $\mathbf{\Sigma}^{1\ \text{thin}\ \mathbf{\Delta}^1_1}_1$ denote the class of thin $\mathbf{\Sigma}^1_1$ equivalence relations with all classes $\mathbf{\Delta}^1_1$ defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces.

Fact 6.2. *Suppose E is a thin $\mathbf{\Sigma}^1_1$ equivalence relation, then E has at most $\aleph_1$ many equivalence classes.*

Proof. See [3]. $\square$

The above fact may suggest that the properness of $\mathbb{P}_I$ should be used with countable support iterations to change covering numbers. It will be shown below that descriptive set theoretic techniques will give a stronger result in just ZFC. However, in the context of proper forcing, the following combinatorial approach is worth mentioning:

Definition 6.3. Let I be a σ-ideal on a Polish space X. $\text{cov}^*(I)$ is the smallest cardinal κ such that there exists some I^+ $\mathbf{\Delta}^1_1$ $B \subseteq X$ and a set $U \subseteq I$ with $|U| = \kappa$ and $B \subseteq \bigcup U$.

Proposition 6.4. *Suppose I is a σ-ideal such that $cov^*(I) > \omega_1$. Then $\mathbf{\Sigma}^{1\ thin\ \mathbf{\Delta}^1_1}_1 \rightarrow_I \{ev\}$.*

Proof. Let $E \in \mathbf{\Sigma}^{1\ \text{thin}\ \mathbf{\Delta}^1_1}_1$. Let $\{C_\alpha : \alpha < \omega_1\}$ enumerate all the E-classes in order type ω_1, using Fact 6.2. Each C_α is $\mathbf{\Delta}^1_1$ since C_α is an equivalence class of E.

Let B be an arbitrary I^+ $\mathbf{\Delta}^1_1$ set. $B = \bigcup_{\alpha<\omega_1} B \cap C_\alpha$. Since $\text{cov}^*(I) > \aleph_1$, there is some α such that $B \cap C_\alpha$ is I^+ $\mathbf{\Delta}^1_1$. Then $E \restriction (B \cap C_\alpha) = \text{ev} \restriction B \cap C_\alpha$. $\square$

Proposition 6.5. *If PFA holds, then for all I such that $\mathbb{P}_I$ is proper, $\mathbf{\Sigma}^{1\ thin\ \mathbf{\Delta}^1_1}_1 \rightarrow_I \{ev\}$.*

Proof. Let B be a I^+ $\mathbf{\Delta}^1_1$ set. Let $U = \{C_\beta : \beta < \omega_1\}$ be a collection of $\mathbf{\Delta}^1_1$ sets in I. $\mathbb{P}_I$ being proper implies that $\mathbb{P}_I \restriction B$ is proper. Let $D_\beta := \{F \in \mathbb{P}_I \restriction B : F \cap C_\beta = \emptyset\}$. D_β is dense in $\mathbb{P}_I \restriction B$. By PFA, there is a filter $G \subseteq \mathbb{P}_I \restriction B$ which is generic for $\{D_\beta : \beta < \omega_1\}$. H constructs a real $x_H \in B$. By genericity, $x_H \notin C_\beta$ for all $\beta < \omega_1$. So U can not cover B. $\text{cov}^*(I) > \aleph_1$. The result follows from Proposition 6.4. $\square$

The results are unsatisfactory in several ways. Models of PFA satisfy $\neg$CH and this was an essential fact since the proof used $\text{cov}^*(I) > \aleph_1$. Definability of the equivalence relation was not used in any deep way. The core of the proofs was combinatorial, using $\text{cov}^*(I) > \aleph_1$.

The rest of this section provides results addressing the main question for thin $\mathbf{\Sigma}^1_1$ equivalence relations which rely on definability properities of these equivalence relations. The best validation of the definability approach to thin $\mathbf{\Sigma}^1_1$ equivalence is that a stronger result will be proved with weaker assumptions (just ZFC).

Fact 6.6. *Let E be a thin $\mathbf{\Sigma}^1_1$ equivalence relation on a Polish space X. Let $\mathbb{P}$ be a forcing. Suppose $\tau \in V^{\mathbb{P}}$ is such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \tau \in X$. Then there is a dense set D^E_τ such that for all $p \in D^E_\tau$, $(p,p) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau_{left}\ E\ \tau_{right}$, where τ_{left} and τ_{right} denote the $\mathbb{P}\times\mathbb{P}$ names for the evaluation of τ according to the left and right generic for $\mathbb{P}$, respectively, coming from a generic for $\mathbb{P}\times\mathbb{P}$.*

Proof. This is due to Silver. See [4], Lemma 2.1 or the proof of [15], Theorem 2.3.

A sketch of the result is provided:

Suppose not. Then there exists some $u \in \mathbb{P}$ such that for all $q \leq_{\mathbb{P}} u$, $(q,q) \not\Vdash_{\mathbb{P}\times\mathbb{P}} \tau_{\text{left}}\ E\ \tau_{\text{right}}$. Hence, there is some u such that for all $q \leq_{\mathbb{P}} u$, there exists $q_0, q_1 \leq_{\mathbb{P}} q$ with $(q_0,q_1) \Vdash_{\mathbb{P}\times\mathbb{P}} \neg(\tau_{\text{left}}\ E\ \tau_{\text{right}})$.

Suppose E is a thin $\Sigma^1_1(z)$ equivalence relation. Let Θ be some large ordinal such that V_Θ reflects the necessary statements to perform the proof below:

Let $N \prec V_\Theta$ be a countable elementary substructure with $z, \mathbb{P}, u, \tau \in N$. Let M be the Mostowski collapse of N and $\pi : N \to M$ be the Mostowski collapsing map. One may assume that for all $x \in X$, $\text{tc}(x) \subseteq \omega$. So π does not move reals or elements of X. In particular $\pi(z) = z$. Let $\mathbb{Q} = \pi(\mathbb{P})$, $v = \pi(u)$, and $\sigma = \pi(\tau)$. By elementarity, M satisfies that for all $q \leq_{\mathbb{Q}} v$, there exists $q_0, q_1 \leq_{\mathbb{Q}} q$ such that $(q_0,q_1) \Vdash_{\mathbb{Q}\times\mathbb{Q}} \neg(\sigma_{\text{left}}\ E\ \sigma_{\text{right}})$.

Let $(D_n : n \in \omega)$ enumerate all the dense open sets in $\mathbb{Q}\times\mathbb{Q}$ of M. One may assume that $D_{n+1} \subseteq D_n$, by replacing D_n with $E_n = \bigcap_{m\leq n} D_m$. Next, a function $f : {}^{<\omega}2 \to \mathbb{Q}$ will be constructed with the following properties:

(1) If $s \subseteq t$, then $f(t) \leq_{\mathbb{Q}} f(s)$.

(2) For all $n \in \omega$, if $|s| = |t| = n$ and $s \neq t$, then $(f(s), f(t)) \in D_n$.

(3) For all $s \in {}^{<\omega}2$, $(f(s0), f(s1)) \Vdash_{\mathbb{Q}\times\mathbb{Q}} \neg(\sigma_{\text{left}}\ E\ \sigma_{\text{right}})$.

To construct this f: Let $f(\emptyset) = v$.

Suppose for all $s \in {}^n2$, $f(s)$ has been constructed with the above properties. For each $s \in {}^n2$, find some $q^{s0}, q^{s1} \leq (f(s), f(s))$ such that $(q^{s0}, q^{s1}) \Vdash_{\mathbb{Q}\times\mathbb{Q}} \neg(\sigma_{\text{left}}\ E\ \sigma_{\text{right}})$. Using the fact that D_{n+1} is dense open, find $\{r^t : t \in {}^{n+1}2\}$ with the property that for all $t \in {}^{n+1}2$, $r^t \leq_{\mathbb{Q}} q^t$ and for all $a, b \in {}^{n+1}2$ with $a \neq b$, $(r^a, r^b) \in D_{n+1}$. For $t \in {}^{n+1}2$, define $f(t) = r^t$.

For each $x \in {}^\omega 2$, let $G_x := \{p \in \mathbb{Q} : (\exists n)(f(x \restriction n) \leq_{\mathbb{Q}} p)\}$. If $x, y \in {}^\omega 2$ and $x \neq y$, then $G_x \times G_y$ is $\mathbb{Q}\times\mathbb{Q}$-generic over M, using (2) and the assumption that for all $n \in \omega$, $D_{n+1} \subseteq D_n$. So let n be largest such that $x \restriction n = y \restriction n$. Let $s = x \restriction n$. Without loss of generality, suppose $x(n) = 0$ and $y(n) = 1$. Then $f(s0) \in G_x$ and $f(s1) \in G_y$. Also $(f(s0), f(s1)) \Vdash_{\mathbb{Q}\times\mathbb{Q}} \neg(\sigma_{\text{left}}\ E\ \sigma_{\text{right}})$. By the forcing theorem applied in M, $M[G_x][G_y] \models \neg(\sigma[G_x]\ E\ \sigma[G_y])$. By Mostowski absoluteness, $V \models \neg(\sigma[G_x]\ E\ \sigma[G_y])$.

Define $\Phi : {}^\omega 2 \to X$ by $\Phi(x) = \sigma[G_x]$. By an appropriate coding, Φ is a $\mathbf{\Delta}^1_1$ function. $\Phi[{}^\omega 2]$ is a $\mathbf{\Sigma}^1_1$ set of pairwise disjoint E-inequivalent elements. By the perfect set property for $\mathbf{\Sigma}^1_1$ sets, there is a perfect set of pairwise E-inequivalent elements. This contradicts E being a thin equivalence relation. ☐

Fact 6.7. *Let E be a thin $\mathbf{\Sigma}^1_1$ equivalence relation on a Polish space X. Let $\mathbb{P}$ be some forcing and $\tau \in V^{\mathbb{P}}$ be such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \tau \in X$. Suppose $p \in D^E_\tau$. Let $M \prec H_\Theta$ be a countable elementary substructure with Θ sufficiently large and $\mathbb{P}, p, \tau \in M$. Then for all $G, H \in V$ such that $p \in G$, $p \in H$, and G and H are $\mathbb{P}$-generic over M, $V \models \tau[G]\ E\ \tau[H]$.*

Proof. This is due to Silver. See [4], Lemma 2.4.

Suppose G and H are any two such generics. Let K be such that it is $\mathbb{P}$-generic over $M[G][H]$. Then $M[G][K] \models \tau[G]\ E\ \tau[K]$ and $M[H][K] \models \tau[H]\ E\ \tau[K]$. By Mostowski absoluteness, $M[G][H][K] \models \tau[G]\ E\ \tau[H]$. By Mostowski absoluteness again, $V \models \tau[G]\ E\ \tau[H]$. ☐

Theorem 6.8. *$\mathbf{\Sigma}^1_1$ ^{thin} $\to_I \{ev\}$ whenever I is a σ-ideal such that $\mathbb{P}_I$ is proper.*

Proof. Let $B \in \mathbb{P}_I$. Since $D^E_{\dot{x}_{\text{gen}}}$ from Fact 6.7 is dense, there exists some $B' \leq_{\mathbb{P}_I} B$ such that $B' \in D^E_{\dot{x}_{\text{gen}}}$. So $(B', B') \Vdash_{\mathbb{P}_I\times\mathbb{P}_I} (\dot{x}_{\text{gen}})\ E\ (\dot{x}_{\text{gen}})$. Let $M \prec H_\Theta$ with Θ sufficiently large and $\mathbb{P}, B' \in M$. By Fact 2.4, the set $C \subseteq B'$ of $\mathbb{P}_I$-generic over M reals is an I^+ $\mathbf{\Delta}^1_1$ set. For $x \in C$, if G_x denotes the $\mathbb{P}_I$-generic over M filter constructed from x, then $\dot{x}_{\text{gen}}[G_x] = x$. Note that for all $x \in C$, $B' \in G_x$. By Fact 6.7, for all $x, y \in C$, $V \models x \; E \; y$. Hence $E \upharpoonright C = \text{ev} \upharpoonright C$. ☐

Note that in this result, E does not need to be an equivalence relation with all $\mathbf{\Delta}^1_1$ classes.

7. Positive Answer for $\mathbf{\Pi}^1_1$ Equivalence Relations

The variant of Question 2.7 for $\mathbf{\Pi}^1_1$ equivalence relations can be phrased as follows:

Question 7.1. Let $\mathbf{\Pi}^{1\ \mathbf{\Delta}^1_1}_1$ be the class of $\mathbf{\Pi}^1_1$ equivalence relations with all classes $\mathbf{\Delta}^1_1$ defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces. If I is a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper, then does $\mathbf{\Pi}^{1\ \mathbf{\Delta}^1_1}_1 \to_I \mathbf{\Delta}^1_1$ hold?

A positive answer for the $\mathbf{\Pi}^1_1$ case follows from the same assumptions as the main question for $\mathbf{\Sigma}^1_1$ in almost the exact same manner as above:

Lemma 7.2. *Let E be a $\Pi^1_1(z)$ equivalence relation on ${}^\omega\omega$. Then there exists $\mathbf{\Delta}^1_1$ relations E_α, for $\alpha < \omega_1$, with the property that if $\alpha < \beta$, then $E_\alpha \subseteq E_\beta$, $E = \bigcup_{\alpha<\omega_1} E_\alpha$, and there exists a club set $C \subseteq \omega_1$ such that for all $\alpha \in C$, E_α is an equivalence relation.*

Proof. Let T be a z-recursive tree on $\omega \times \omega \times \omega$ such that $(x, y) \in E \Leftrightarrow T^{(x,y)}$ is wellfounded. For each $\alpha < \omega_1$, define $E_\alpha := \{(x, y) : \text{rk}(T^{(x,y)}) < \alpha\}$. Each E_α is $\mathbf{\Delta}^1_1$. If $\alpha < \beta$, $E_\alpha \subseteq E_\beta$. $E = \bigcup_{\alpha<\omega_1} E_\alpha$.

Let C be the set of all α such that E_α is an equivalence relation. Increasing union of equivalence relations are equivalence relations so C is closed. Fix $\alpha < \omega_1$. The set $D = \{(x, x) : x \in {}^\omega\omega\}$ is Σ^1_1. So by the boundedness theorem, there exist some $\delta < \omega_1$ such that $\text{rk}(T^{(x,x)}) < \delta$ for all $x \in {}^\omega\omega$. Let $\beta_0 = \max\{\alpha, \delta\}$. Suppose β_n has been defined. The set $G = \{(x, y) : (y, x) \in E_{\beta_n}\}$ is $\mathbf{\Sigma}^1_1$. By the boundedness theorem, there exists some $\beta' > \beta_n$ such that for all $(x, y) \in G$, $\text{rk}(T^{(x,y)}) < \beta'$. The set $H = \{(x, z) : (\exists y)((x, y) \in E_{\beta_n} \wedge (y, z) \in E_{\beta_n}\}$ is $\mathbf{\Sigma}^1_1$. Again by the boundedness theorem, there exists some $\beta_{n+1} > \beta'$ such that for all $(x, z) \in H$, $\text{rk}(T^{(x,z)}) < \beta_{n+1}$. One has constructed an increasing sequence $\{\beta_n : n \in \omega\}$. Let $\beta = \sup\{\beta_n : n \in \omega\}$. Then E_β is an equivalence relation. C is unbounded. ☐

Lemma 7.3. *Let E be a $\Pi^1_1(z)$ equivalence relation. Let $x, y \in {}^\omega\omega$ be such that $[x]_E$ is $\Sigma^1_1(y)$. Let δ be an ordinal such that $\omega_1^{y\oplus z} \leq \delta$ and E_δ is a equivalence relation. Then $[x]_E = [x]_{E_\delta}$.*

Proof. Define $E' \subseteq ({}^\omega\omega)^2$ by

$$a \; E' \; b \Leftrightarrow (a \in [x]_E \wedge b \in [x]_E) \vee (a = b)$$

E' is $\Sigma^1_1(y)$. $E' \subseteq E$. By the effective boundedness theorem, there exists a $\alpha < \omega_1^{y\oplus z} \leq \delta$ such that for all $(x, y) \in E'$, $\text{rk}(T^{(x,y)}) < \alpha$. Hence $E' \subseteq E_\alpha$.

Since $E_\delta \subseteq E$, $[x]_{E_\delta} \subseteq [x]_E$. Also $[x]_E = [x]_{E'} \subseteq [x]_{E_\alpha} \subseteq [x]_{E_\delta}$. Therefore, $[x]_E = [x]_{E_\delta}$. ☐

Note that x is not used as a parameter in the above lemma. This is in contrast to Lemma 4.2. This observation will be used later. (See Proposition 10.6.)

Lemma 7.4. *Let E be a $\Pi^1_1(z)$ equivalence relation. Let $\mathbb{P}$ be a forcing in M which adds a generic real. Choose Θ to be a regular cardinal greater than $|\mathbb{P}|^+$. Let $M \prec H_\Theta$ be a countable elementary substructure with $z, \mathbb{P} \in M$ and contains $|\mathbb{P}|^+$. Suppose for all g which are $\mathbb{P}$-generic over M, there exists a $y \in M[g]$ such that $V \models [g]_E$ is $\Sigma^1_1(y)$. Then there exists a countable ordinal α such that for all $\mathbb{P}$-generic over M reals g, $[g]_E = [g]_{E_\alpha}$.*

Proof. The proof is almost the same as the proof for Lemma 4.3 using Lemma 7.3 in place of Lemma 4.2. ☐

These previous results can be used to give a positive answer for a specific Π^1_1 equivalence relation in ZFC.

Example 7.5. Let H be an equivalence relation on ${}^\omega\omega$ defined by $x \; H \; y$ if and only if $x \in L_{\omega_1^y}(y) \wedge y \in L_{\omega_1^x}(x)$.

H is a Π^1_1 equivalence relation with all classes countable. H is the equivalence relation of hyperarithmetic equivalence.

If I is a σ-ideal on ${}^\omega\omega$ with $\mathbb{P}_I$ proper, then $\{H\} \to_I \mathbf{\Delta}^1_1$.

Proof. Fix B an I^+ $\mathbf{\Delta}^1_1$ set. Choose $M \prec H_\Theta$ with Θ sufficiently large and $B, \mathbb{P}_I \in M$. By Fact 2.4, let $C \subseteq B$ be the set of $\mathbb{P}_I$-generic over M elements in B. Let $x \in C$. ω_1^x is a countable ordinal in $M[x]$. In $M[x]$, $L_{\omega_1^x}(x)$ is countable. $[x]_E^{M[x]} \subseteq L_{\omega_1^x}(x)$. Therefore, in $M[x]$, there is a function $f : \omega \to {}^\omega\omega$ such that f enumerates $[x]_H^{M[x]}$. By absoluteness, $[x]_H = [x]_H^{M[x]}$. So $[x]_H$ is $\Delta^1_1(f)$ and $f \in M[x]$. By Lemma 7.4, there is some countable ordinal α such that $[x]_E = [x]_{E_\alpha}$ for all $x \in C$. So $E \restriction C = E_\alpha \restriction C$. $E \restriction C$ is $\mathbf{\Delta}^1_1$. $\square$

Definition 7.6. Let E be a $\Pi^1_1(z)$ equivalence relation. Define the set D by

$$(x, T) \in D \Leftrightarrow (T \text{ is a tree on } \omega \times \omega) \wedge (\forall y)(y \; E \; x \Leftrightarrow T^y \notin \text{WF})$$

D is $\Pi^1_2(z)$.

Theorem 7.7. *Assume all $\mathbf{\Pi}^1_2$ sets are universally Baire and $\mathbf{\Pi}^1_4$-generic absoluteness holds. Let I be a σ-ideal such that $\mathbb{P}_I$ is proper. Then ${\mathbf{\Pi}^1_1}^{\mathbf{\Delta}^1_1} \to_I \mathbf{\Delta}^1_1$.*

Proof. The proof is the same as Theorem 4.8 with the required change. $\square$

A similar argument using iterable structures as in the $\mathbf{\Sigma}^1_1$ case yields a positive answer from a more precise assumption with lower consistency strength.

Proposition 7.8. *Let E be a $\Pi^1_1(z)$ equivalence relation. There is a $\Pi^1_3(z)$ formula $\varpi(v)$ in free variable v such that:*

Let $x \in {}^\omega\omega$. If $(x \oplus z)^\sharp$ exists, then the statement "$[x]_E$ is $\mathbf{\Delta}^1_1$" is equivalent to $\varpi(x)$.

Assume for all $r \in {}^\omega\omega$, $r^\sharp$ exists. The statement "all E-classes are $\mathbf{\Delta}^1_1$" is equivalent to $(\forall x)\varpi(x)$. In particular, this statement is $\Pi^1_3(z)$.

Proof. Assume for simplicity, E is a Π^1_1 equivalence relation on ${}^\omega\omega$. Let T be a tree on $\omega \times \omega \times \omega$ such that

$$(x, y) \in E \Leftrightarrow T^{x,y} \text{ is wellfounded}$$

Let $\varpi(v)$ be the statement:

$$(\forall y)(y = v^\sharp \Rightarrow \text{“}1_{\text{Coll}(\omega,<c_1)} \Vdash_{\text{Coll}(\omega,<c_1)} (\exists c)(c \in \text{WO} \wedge (\forall y)((y \; E \; v) \Rightarrow \text{rk}(T^{v,y}) < \text{ot}(c)))\text{”} \in y)$$

The rest of the argument is the same as in Proposition 4.12. $\square$

Definition 7.9. Let I be a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper. Let μ^I_E be $\tau^{\mathbb{P}_I}_{\neg\varpi}$ from Fact 4.17.

Definition 7.10. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is proper. Consider the formula "$(\exists y)([\dot{x}_{\text{gen}}]_E$ is $\Sigma^1_1(y))$". Write it as $(\exists y)\psi(y)$. By Fact 4.17, let σ^I_E be $\tau^{\mathbb{P}_I}_\psi$.

Definition 7.11. Suppose I is a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is proper. Let $E \in {\mathbf{\Pi}^1_1}^{\mathbf{\Delta}^1_1}$. Define $\chi^I_E = \langle \mathbb{P}_I, \mu^{\mathbb{P}_I}_E, \sigma^{\mathbb{P}_I}_E \rangle$.

Theorem 7.12. *Suppose I is a σ ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is proper. If for all $x \in {}^\omega 2$, $x^\sharp$ exists and $(\chi^I_E)^\sharp$ exists for all $E \in {\mathbf{\Pi}^1_1}^{\mathbf{\Delta}^1_1}$, then ${\mathbf{\Pi}^1_1}^{\mathbf{\Delta}^1_1} \to_I \mathbf{\Delta}^1_1$.*

Proof. This is similar to Theorem 4.22. $\square$

Corollary 7.13. *If $z^\sharp$ exists for all $z \in H_{(2^{\aleph_0})^+}$, then ${\mathbf{\Pi}^1_1}^{\mathbf{\Delta}^1_1} \to_I \mathbf{\Delta}^1_1$ for all σ-ideal I such that $\mathbb{P}_I$ is proper.*

8. $\mathbf{\Pi}^1_1$ Equivalence Relations with Thin or Countable Classes

The preservation of the statement "all classes are $\mathbf{\Delta}^1_1$" played an important role in the consistency results above. Next, one will consider $\mathbf{\Pi}^1_1$ equivalence relations which are very sensitive to set theoretic assumptions and generic extensions.

Definition 8.1. Let X be a Polish space. $A \subseteq X$ is thin if and only if it does not contain a perfect set.

Fact 8.2. *For each $z \in {}^\omega 2$, define $Q_z := \{x \in {}^\omega 2 : x \in L_{\omega_1^{z\oplus x}}(z)\}$. Q_z is the largest thin $\Pi^1_1(z)$ set in the sense that if S is a thin $\Pi^1_1(z)$ set, then $S \subseteq Q_z$. Moreover, for each $\alpha < \omega_1^{L[z]}$, there exists some $x \in Q_z$ such that $\alpha < \omega_1^x$. Therefore, if $\omega_1^{L[z]} = \omega_1$, then Q_z is an uncountable thin $\Pi^1_1(z)$ set. It is consistent that $\mathbf{\Pi}^1_1$ sets do not have the perfect set property.*

Proof. See [24], pages 83-87. One will give the $\Pi^1_1(z)$ definition to get a better understanding of what Q_z is:

$$x \in Q_z \Leftrightarrow (\forall M)((M \text{ is an } \omega\text{-model of } \mathsf{KP} \wedge z \in M \wedge x \in M) \Rightarrow M \models x \in L[z])$$

So $x \in Q_z$ if and only if $L_{\omega_1^{z\oplus x}}(z \oplus x) = L_{\omega_1^{z\oplus x}}(z)$. Or put another way, the smallest admissible set containing $z \oplus x$ is a model of $V = L[z]$. Certainly $Q_z \subseteq L[z]$. So Q_z can also be thought of as the set of reals that appear in $L[z]$ very quickly in the sense that $x \in Q_z$ if and only if the first ordinal α such that $L_\alpha(z \oplus x)$ is admissible is also the first z-admissible ordinal α such that $x \in L_\alpha[z]$. □

Now one can give a simple example of an equivalence relation E, a model of ZFC, and forcing which does not preserve the statement "all E classes are $\mathbf{\Delta}^1_1$". Note that this statement is $\mathbf{\Pi}^1_4$ so it will be preserved if the universe satisfies $\mathbf{\Pi}^1_4$-generic absoluteness. The desired example will necessarily have to reside in a universe with weak large cardinals.

Definition 8.3. ω_1 is inaccessible to reals if and only if for all $x \in {}^\omega\omega$, $L[x] \models (\omega_1^V$ is inaccessible) if and only if for all $x \in {}^\omega\omega$, $\omega_1^{L[x]} < \omega_1$.

Proposition 8.4. *Let E and F be equivalence relations on $({}^\omega\omega)^2$ defined by*

$$(a,x)\ E\ (b,y) \Leftrightarrow (a = b) \wedge (x, y \in Q_a \vee x = y)$$

$$(a,x)\ F\ (b,y) \Leftrightarrow (a = b) \wedge (x, y \notin Q_a \vee x = y)$$

E is a Π^1_1 equivalence relation and F is a Σ^1_1 equivalence relation.

Let κ be an inaccessible cardinal in L. Let $G \subseteq Coll(\omega, <\kappa)$ be $Coll(\omega, <\kappa)$-generic over L. Then $L[G] \models$ E and F have all classes $\mathbf{\Delta}^1_1$. Let $g \subseteq Coll(\omega,\kappa)$ be $Coll(\omega,\kappa)$-generic over $L[G]$, then $L[G][g] \models$ not all E and F classes are $\mathbf{\Delta}^1_1$.

Proof. The formula provided in the proof of Fact 8.2 shows that the formula "$x \in Q_z$" is Π^1_1 in variables x and z. From this, it follows that E and F are Π^1_1 and Σ^1_1, respectively.

In $L[G]$, ω_1 is inaccessible to reals ([28] Theorem 8.20). For each (a,b), $[(a,b)]_E$ is either a singleton or in bijection with Q_a. Since $Q_a \subseteq ({}^\omega\omega)^{L[a]}$, in all cases, $[(a,b)]_E$ is countable and hence $\mathbf{\Delta}^1_1$. F-classes are then singletons or complements of countable sets. All F-classes are $\mathbf{\Delta}^1_1$.

$\text{Coll}(\omega, <\kappa) * \dot{\text{Coll}}(\omega, \check{\kappa})$ is a forcing (in L) of size κ which collapses κ to ω. Such forcings are forcing equivalent to $\text{Coll}(\omega,\kappa)$ by [16] Lemma 26.7. Let $h \subseteq \text{Coll}(\omega,\kappa)$ which is $\text{Coll}(\omega,\kappa)$-generic over L with $L[h] = L[G][g]$. $L[G][g] \models \omega_1^{L[h]} = \omega_1$. ω_1 is not inaccessible to reals in $L[G][g]$. Moreover, $[(h,h)]_E$ is not $\mathbf{\Delta}^1_1$ in $L[G][g]$ as it is an uncountable thin set and the perfect set property holds for $\mathbf{\Sigma}^1_1$ sets. Similarly, F has a class which is not $\mathbf{\Delta}^1_1$. □

In the previous example, in $L[G]$, $\text{Coll}(\omega, \omega_1^{L[G]}) = \text{Coll}(\omega,\kappa)$ is not a proper forcing. One may ask whether there is an $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$ equivalence relation with all classes $\mathbf{\Delta}^1_1$ and a proper forcing coming from a σ-ideal on a Polish space such that in the induced generic extension, the statement that "all classes are $\mathbf{\Delta}^1_1$" is false. Sy-David Friedman's forcing to code subsets of ω_1 is an $\aleph_1$-c.c. forcing which can be repesented as an idealized forcing which (like in the proof of the above proposition) adds a real r such that $L[A][r] = L[r]$. The two equivalence relations from the above proposition can be used with this forcing to give a similar result. See Section 10 for more details about this forcing.

For $\mathbf{\Sigma}^1_1$ equivalence relations with all classes countable, Proposition 3.2 shows that the main question has a positive answer without additional set theoretic assumptions. There were two important aspects of the proof. First, the countability of all classes of a $\mathbf{\Sigma}^1_1$ equivalence relation is $\mathbf{\Sigma}^1_1$ and hence remains true in all generic extensions. This fact is used to give an enumeration f of $[x]_E$ in $M[x]$. Secondly, the statement that f enumerates $[x]_E$ is $\mathbf{\Pi}^1_1$ and hence absolute between (the countable model) $M[x]$ and V.

One can ask the same question for $\mathbf{\Pi}^1_1$ equivalence relations with all classes countable. However, the above proof can not be applied. First, the countability of all classes of a $\mathbf{\Pi}^1_1$ equivalence relation is $\mathbf{\Pi}^1_4$. Secondly, the statement that some function f enumerates $[x]_E$ is $\mathbf{\Pi}^1_2$; hence, it does not necessarily persist from $M[x]$ to V.

The $\mathbf{\Pi}^1_1$ equivalence relations where these issues are most perceptible are the equivalence relations E with all classes countable but for some x, $L[x] \models [x]_E$ is uncountable . It is not provable that all E-classes are countable; however, all the E-classes are thin.

Proposition 8.5. *Let A be a $\mathbf{\Pi}^1_1$ set. The statement "A is thin" is $\mathbf{\Pi}^1_2$. Let E be a $\mathbf{\Pi}^1_1$ equivalence relation. The statement "all E-classes are thin" is $\mathbf{\Pi}^1_2$. Both of these statements are absolute to generic extensions.*

Proof.

$$(\forall T)(T \text{ is perfect tree } \Rightarrow ((\exists x)((\forall n)(x \upharpoonright n \in T) \wedge x \notin A)))$$

$$(\forall x)(\forall T)(T \text{ is perfect tree } \Rightarrow ((\exists y)((\forall n)(y \upharpoonright n \in T) \wedge \neg(x\ E\ y))))$$

These two $\mathbf{\Pi}^1_2$ formulas are equivalent to "A is thin" and "all E classes are thin", respectively. ☐

Definition 8.6. Let $\mathbf{\Pi}^{1\ \aleph_0}_1$ denote the class of all $\mathbf{\Pi}^1_1$ equivalence relations with all classes countable defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces. Let $\mathbf{\Pi}^{1\ \text{thin}}_1$ denote the class of all $\mathbf{\Pi}^1_1$ equivalence relations with all classes thin defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces.

Theorem 8.7. *If $\omega_1^L < \omega_1$, then $\Pi^{1\ thin}_1 \rightarrow_{I_{meager}} \mathbf{\Delta}^1_1$.*

Proof. Let E be a Π^1_1 equivalence relation. Fix a non-meager $\mathbf{\Delta}^1_1$ set B. Let $\mathbb{C}$ denote Cohen forcing, i.e., finite partial functions from ω into 2. Let U be the set of all constructible dense subsets of $\mathbb{C}$. Since $\omega_1^L < \omega_1$, $|U| = \aleph_0$.

For a sufficiently large cardinal Θ, let $M \prec H_\Theta$ be a countable elementary substructure with $B, \mathbb{P}_{I_{\text{meager}}}, \omega_1^L, U \in M$, $\omega_1^L \subseteq M$, and $U \subseteq M$. By Fact 2.4, let C be the set of $\mathbb{P}_{I_{\text{meager}}}$-generic over M reals in B.

Take $x \in C$. Since Cohen forcing $\mathbb{C}$ and $\mathbb{P}_{I_{\text{meager}}}$ are forcing equivalent and $U \subseteq M$, x is also $\mathbb{C}$-generic over L. Since $\mathbb{C}$ satisfies the $\aleph_1$-chain condition, $\omega_1^{L[x]} = \omega_1^L < \omega_1$. Since ω_1^L is countable in M, $L_{\omega_1^{L[x]}}[x] = L_{\omega_1^L}[x] \subseteq M[x]$ and is countable there. Since $[x]_E$ is thin, $[x]_E \subseteq L_{\omega_1^{L[x]}}[x]$. In $M[x]$, there is an enumeration $f : \omega \to ([x]_E)^{M[x]}$. The claim is that $([x]_E)^{M[x]} = [x]_E$: since $([x]_E)^V \subseteq L_{\omega_1^{L[x]}}[x] \subseteq M[x]$, $M[x] \models y\ E\ x \Leftrightarrow (L[x])^{M[x]} \models y\ E\ x \Leftrightarrow L[x] \models y\ E\ x \Leftrightarrow V \models y\ E\ x$, by Mostowski absoluteness.

Therefore, in V, $[x]_E$ is $\Delta^1_1(f)$ and $f \in M[x]$. By Lemma 7.4, there is some countable $\alpha < \omega_1$, such that $E \upharpoonright C = E_\alpha \upharpoonright C$. The latter is $\mathbf{\Delta}^1_1$. ☐

If ω_1 is inaccessible to reals, then $\mathbf{\Pi}^{1\ \text{thin}}_1 = \mathbf{\Pi}^{1\ \aleph_0}_1$. Familiar models that satisfy ω_1 is inaccessible to reals include generic extensions of the Lévy collapse of an inaccessible cardinal to ω_1. Next, one will consider the main question for $\mathbf{\Pi}^{1\ \aleph_0}_1$ in models of this type and obtain some improved consistency results.

The main large cardinal useful here is the remarkable cardinal isolated in [30] to understand absoluteness for proper forcing in $L(\mathbb{R})$. It is a fairly weak large cardinal. Its existence is consistent relative to ω-Erdös cardinals. If $0^\sharp$ exists, then all Silver's indiscernibles are remarkable cardinals in L. Also if κ is a remarkable cardinal, then κ is a remarkable cardinal in L.

Definition 8.8. ([30] Definition 1.1) A cardinal κ is a remarkable cardinal if and only if for all regular cardinals $\theta > \kappa$, there exists $M, N, \pi, \sigma, \bar{\kappa}$ and $\bar{\theta}$ such that the following holds:

(i) M and N are countable transitive sets.
(ii) $\pi : M \to H_\theta$ is an elementary embedding.
(iii) $\pi(\bar{\kappa}) = \kappa$.
(iv) $\sigma : M \to N$ is an elementary embedding with $\text{crit}(\sigma) = \bar{\kappa}$.
(v) $\bar{\theta} = \text{ON} \cap M$, $\sigma(\bar{\kappa}) > \bar{\theta}$, and $N \models \theta$ is a regular cardinal.
(vi) $M \in N$ and $N \models M = H_\theta$.

Fact 8.9. *(Schindler) Let κ be a remarkable cardinal in L. Let $G \subseteq Coll(\omega, < \kappa)$ be $Coll(\omega, < \kappa)$-generic over L. Let $\mathbb{P} \in L[G]$ be a proper forcing. Let $H \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over $L[G]$. If $x \in ({}^\omega\omega)^{L[G][H]}$, then there exists a forcing $\mathbb{Q} \in L_\kappa$ and a $K \subseteq \mathbb{Q}$ in $L[G][H]$ which is $\mathbb{Q}$-generic over L and $x \in L[K]$.*

Proof. See [30], Lemma 2.1. ☐

Theorem 8.10. *Let κ be a remarkable cardinal in L. Let $G \subseteq Coll(\omega, < \kappa)$ be $Coll(\omega, < \kappa)$ generic over L. In $L[G]$, if I is a σ-ideal such that $\mathbb{P}_I$ is proper, then $\mathbf{\Pi}^1_1{}^{\aleph_0} \to_I \mathbf{\Delta}^1_1$.*

Proof. Working in $L[G]$, let E be an equivalence relation in $\mathbf{\Pi}^1_1{}^{\aleph_0}$. For simplicity, assume E is Π^1_1 (otherwise one should include the parameter defining E in all the discussions below). In particular, all E-classes are thin, and this statement will be absolute to all generic extensions.

Let B be an I^+ $\mathbf{\Delta}^1_1$ set. Let $M \prec H_\Theta$ be a countable elementary substructure with Θ a sufficiently large cardinal and $B, \mathbb{P}_I, G \in M$. $H_\Theta = L_\Theta[G]$. Therefore, $M = L^M[G]$. Note that from the point of view of M, G is L^M generic for $\text{Coll}(\omega, < \kappa)^M$. Using Fact 2.4, let $C \subseteq B$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic over M reals in B.

Fix $x \in C$. Applying Fact 8.9 in $M[x] = L^M[G][x]$, there exist some $\mathbb{Q} \in (L_\kappa)^M$ and $K \subseteq \mathbb{Q}$ in $M[x]$ which is $\mathbb{Q}$-generic over L^M such that $x \in L[K]$. Since M satisfies $\mathbb{Q} \in L_\kappa$ and κ is a remarkable cardinal (in particular inaccessible) in L, M thinks that $\mathscr{P}^L(\mathbb{Q}) \in L_\kappa$. Since $M = L[G]$, $M \models \mathscr{P}^L(\mathbb{Q})$ is countable. Let $f : \omega \to \mathscr{P}^L(\mathbb{Q})$ be a function in M such that M thinks it surjects onto $\mathscr{P}^L(\mathbb{Q})$. Since $M \prec (H_\Theta)^{L[G]}$, f really is a surjection onto $\mathscr{P}^L(\mathbb{Q})$ in the real universe $L[G]$. This establishes that $\mathscr{P}^L(\mathbb{Q}) \subseteq M$. In particular, $\mathscr{P}^L(\mathbb{Q}) \subseteq L^M$. This and the fact that K is generic over L^M imply that K is $\mathbb{Q}$-generic over the real L. Since $\mathbb{Q} \in L_\kappa$, all cardinals of L greater than $|\mathbb{Q}|$ are preserved in $L[K]$. Therefore, $\omega_1^{L[x]} \leq \omega_1^{L[K]} \leq (|\mathbb{Q}|^+)^L$. Since $\mathbb{Q} \in M$ and $M \models \mathbb{Q} \in L$, there is some ordinal α such that $M \models L \models |\mathbb{Q}|^+ = \alpha$. Because $M \prec H_\Theta$, the real universe $L[G]$ satisfies $L \models |\mathbb{Q}|^+ = \alpha$. This establishes that $(|\mathbb{Q}|^+)^L \in M$. Since $\mathbb{Q} \in L_\kappa$ and κ is inaccessible, $(|\mathbb{Q}|^+)^L < \kappa$. Since $M = L^M[G]$, $(|\mathbb{Q}|^+)^L$ is a countable ordinal in M. As shown above, $\omega_1^{L[x]} < (|\mathbb{Q}|^+)^L$, so in M, $\omega_1^{L[x]}$ is countable. Since $[x]_E$ is thin, $[x]_E \subseteq ({}^\omega\omega)^{L[x]}$. As $\omega_1^{L[x]}$ is countable in $M[x]$, $M[x] \models [x]_E$ is countable. There exists some surjection $h : \omega \to ([x]_E)^{M[x]}$. The claim is that $([x]_E)^{L[G]} = ([x]_E)^{M[x]}$: since $([x]_E)^{L[G]} \subseteq L_{\omega_1^{L[x]}}[x] \subseteq M[x]$, $M[x] \models y \ E \ x \Leftrightarrow (L[x])^{M[x]} \models y \ E \ x \Leftrightarrow (L[x])^{L[G]} \models y \ E \ x \Leftrightarrow L[G] \models y \ E \ x$, by Mostowski absoluteness.

Therefore, in $L[G]$, $[x]_E$ is $\Delta^1_1(h)$ and $h \in M[x]$. By Lemma 7.4, there is a countable $\alpha < \omega_1^{L[G]}$, such that $E \restriction C = E_\alpha \restriction C$. $E_\alpha \restriction C$ is $\mathbf{\Delta}^1_1$. $\square$

It is now known that much more holds: in the model $L[G]$ of Theorem 8.10, the main question has a positive answer for all equivalence relations $E \in L(\mathbb{R})$ with all classes $\mathbf{\Delta}^1_1$. See [6]. This shows that the consistency strength of a remarkable cardinal is an upper bound on the consistency strength of a positive answer to the main question for projective equivalence relations with all classes $\mathbf{\Delta}^1_1$.

Using some well-known results of Kunen, a similar proof shows that the consistency of $\mathbf{\Pi}^1_1{}^{\aleph_0} \to_I \mathbf{\Delta}^1_1$ for I such that $\mathbb{P}_I$ is $\aleph_1$-c.c. follows from the consistency of a weakly compact cardinal.

Fact 8.11. *(Kunen) Let κ be a weakly compact cardinal. Let $\mathbb{P}$ be a κ-c.c. forcing. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. If $x \in H_\kappa^{V[G]}$, then there exists a forcing $\mathbb{Q} \in V_\kappa$ and a $K \subseteq \mathbb{Q}$ which is generic over V such that $x \in V[K]$.*

Proof. This is due to Kunen. See [14], Lemma 5.3 for a proof. $\square$

Theorem 8.12. *Let κ be a weakly compact cardinal in L. Let $G \subseteq Coll(\omega, < \kappa)$ be $Coll(\omega, < \kappa)$-generic over L. In $L[G]$, if I is a σ-ideal such that $\mathbb{P}_I$ is $\aleph_1$-c.c., then $\mathbf{\Pi}^1_1{}^{\aleph_0} \to_I \mathbf{\Delta}^1_1$.*

Proof. Let $\dot{\mathbb{P}}_I$ be a name for $\mathbb{P}_I$ in $L[G]$. $\text{Coll}(\omega, < \kappa)$ satisfies the κ-chain condition. Since $\aleph_1^{L[G]} = \kappa$, for some $p \in G$, $p \Vdash_{\text{Coll}(\omega, <\kappa)} \dot{\mathbb{P}}_I$ satisfies the $\check{\kappa}$-chain condition. By considering the forcing of conditions below p, one may as well assume $p = 1_{\text{Coll}(\omega, <\kappa)}$. Then $\text{Coll}(\omega, < \kappa) * \dot{\mathbb{P}}_I$ satisfies the κ-chain condition. Now use Fact 8.11 and finish the proof much like in Theorem 8.10. $\square$

9. $\mathbf{\Delta}^1_2$ Equivalence Relations with all Classes $\mathbf{\Delta}^1_1$

One can ask the same question for $\mathbf{\Delta}^1_2$ equivalence relations with all classes $\mathbf{\Delta}^1_1$: If E is a $\mathbf{\Delta}^1_2$ equivalence relation with all classes $\mathbf{\Delta}^1_1$ and I is a σ-ideal such that $\mathbb{P}_I$ is proper, does $\{E\} \to_I \mathbf{\Delta}^1_1$ hold?

It will be shown that in L there is a $\mathbf{\Delta}^1_2$ equivalence relation E_L such that $\{E_L\} \to_I \mathbf{\Delta}^1_1$ does not hold for any σ-ideal I.

Definition 9.1. Let E_L be the equivalence relation defined on ${}^\omega 2$ by $x \; E_L \; y$ if and only if

$$(\forall A)((A \text{ is a well-founded model of } \mathsf{KP}+\mathsf{V}=\mathsf{L}+\mathsf{INF}) \Rightarrow (x \in A \Leftrightarrow y \in A))$$

where INF asserts that ω exists. E_L is a Π^1_2 equivalence relation.

Here A is considered as a structure with domain ω. As A thinks ω exists, there is an isomorphic copy of ω in A. The statement "$x \in A$" should be understood using this copy of ω in A.

Rather than $\mathsf{KP}+\mathsf{V}=\mathsf{L}$, one could also use some $\Upsilon+\mathsf{V}=\mathsf{L}$ where Υ is a large enough fragment of ZFC to prove all the familiar properties about perfect sets and constructibility needed below. If one is willing to assume that there exists a transitive model of ZFC, then one can replace the above with $\mathsf{ZFC}+\mathsf{V}=\mathsf{L}$ and be in the familiar setting.

Definition 9.2. Assume $V = L$. Let $\iota : {}^\omega 2 \to \omega_1$ be the function such that $\iota(x)$ is the smallest admissible ordinal α such that $x \in L_\alpha$.

Proposition 9.3. *For all $x, y \in {}^\omega 2$, $x \; E_L \; y$ if and only if $\iota(x) = \iota(y)$.*

Proof. Assume $\iota(x) = \iota(y)$. Let A be a wellfounded model of $\mathsf{KP}+\mathsf{V}=\mathsf{L}$ such that $x \in A$. There is some β such that L_β is the Mostowski collapse of A. L_β is transitive and satisfies KP, so it is an admissible set. β is an admissible ordinal. $\iota(x) \leq \beta$. $y \in L_{\iota(y)} = L_{\iota(x)} \subseteq L_\beta$. So $y \in A$. Hence $x \in A$ implies $y \in A$. By a symmetric argument, $y \in A$ implies $x \in A$. $x \; E_L \; y$.

Assume $x \; E_L \; y$. Suppose $\alpha < \omega_1$ with $L_\alpha \models \mathsf{KP}$ and $x \in L_\alpha$. Since L_α is countable, there is a countable structure A with domain ω isomorphic to L_α. $A \models \mathsf{KP}$, A is an ω-model, and $x \in A$. $x \; E_L \; y$ implies that $y \in A$. Therefore, $y \in L_\alpha$. Hence $\iota(x) \leq \iota(y)$. By a symmetric argument, $\iota(y) \leq \iota(x)$. $\iota(x) = \iota(y)$. ☐

Earlier drafts of this paper only asserted that E_L was Π^1_2. Drucker observed that a very similar equivalence relation to E_L was actually Δ^1_2:

Proposition 9.4. *(Drucker) E_L is Δ^1_2.*

Proof. The claim is that

$$x \; E_L \; y \Leftrightarrow H_{\aleph_1} \models (\exists M)((M \text{ is transitive}) \wedge (x, y \in M) \wedge (M \models \mathsf{KP}+\mathsf{V}=\mathsf{L}) \wedge (M \models \psi(x,y)))$$

where

$$\psi(x,y) \Leftrightarrow (\forall A)((A \text{ is transitive} \wedge A \models \mathsf{KP}+\mathsf{V}=\mathsf{L}) \Rightarrow (x \in A \Leftrightarrow y \in A))$$

To see this: $(\Rightarrow)$ By Proposition 9.3, $\iota(x) = \iota(y)$. Then $H_{\aleph_1}$ satisfies the above formula using $L_{\iota(x)}$.

$(\Leftarrow)$ Suppose $\neg(x \; E_L \; y)$. Let M witness the negation of the statement from Defintion 9.1. Without loss of generality, $\iota(x) < \iota(y)$. By Δ_1 absoluteness, if $H_{\aleph_1}$ thinks M is transitive and satisfies $\mathsf{KP}+\mathsf{V}=\mathsf{L}$, then M really is transitive and satisfies $\mathsf{KP}+\mathsf{V}+\mathsf{L}$. So $M = L_\alpha$ for some $\alpha < \omega_1$. Since $x, y \in M = L_\alpha$, $\alpha \geq \iota(y)$. Then $M \models \neg(\psi(x,y))$ since $L_{\iota(x)} \in L_\alpha = M$, $x \in L_{\iota(x)}$, and $y \notin L_{\iota(x)}$.

$\psi(x,y)$ is a first order formula in the language of set theory. First order satisfaction is Δ_1. The above shows that $x \; E_L \; y$ is equivalent to a formula which is Σ_1 over $H_{\aleph_1}$. Hence E_L is Σ^1_2. ☐

Assuming $\mathsf{V}=\mathsf{L}$, Proposition 9.3 associates each E_L class with a countable ordinals. This suggests that E_L is thin. However, the complexity of the statement that a particular $\boldsymbol{\Delta}^1_2$ equivalence relation is thin is beyond the scope of Shoenfield absoluteness. Therefore the usual argument of passing to a forcing extension satisfying $\neg\mathsf{CH}$ will not work. Morever, E_L looks quite different in models that do not satisfy $\mathsf{V}=\mathsf{L}$. Thinness will be proved more directly.

The following fact will be useful. It implies that if $\alpha < \beta$ are admissible ordinals and a new real appears in L_β which was not in L_α, then L_α is countable from the view of L_β.

Fact 9.5. *If $\omega < \alpha < \beta$ are admissible ordinals and $({}^\omega 2)^{L_\beta} \not\subseteq L_\alpha$, then there is an $f \in L_\beta$ such that $f : \omega \to \alpha$ is a surjection. In particular, $L_\beta \models |L_\alpha| = \aleph_0$.*

Proof. This is essentially a result of Putnam. Below, a brief sketch of the proof is given using some elementary fine structure theory. (See [17], [29], or [10].)

Note that if α is admissible, then $\omega \cdot \alpha = \alpha$. [17] Lemma 2.15 shows that $L_\alpha = J_\alpha$, if α is admissible.

Now suppose $\alpha < \beta$ are admissible ordinals. Since $({}^\omega 2)^{J_\beta} \not\subseteq L_\alpha$, there is some $x \in \mathscr{P}(\omega)$ such that $x \in J_\beta$ and $x \notin J_\alpha$. Then there is some $\alpha < \gamma < \beta$ and some $n \in \omega$ such that x is Σ_n definable over J_γ but not in

J_γ. [17] Lemma 3.4 (i) shows that all J_γ are Σ_n-uniformizable for all n. Then [17] Lemma 3.1 can be applied to show that there is a Σ_n in J_γ surjection f of ω onto J_γ. f is definable in J_γ and so $f \in J_{\gamma+1} \subseteq J_\beta$. Since $J_\alpha \subseteq J_\gamma$, using this f, one can construct a surjection in J_β from ω onto J_α. □

Lemma 9.6. *Suppose α is an ordinal such that there exists an $x \in {}^\omega 2$ with $\iota(x) = \alpha$, then there exists a greatest $\beta < \alpha$ such that there exists a $y \in {}^\omega 2$ with $\iota(y) = \beta$.*

Proof. Fix an x such that $\iota(x) = \alpha$. If the result was not true, then there exists a sequences of reals $(x_n : n \in \omega)$ such that $\iota(x_n) < \alpha$ and $\alpha = \lim_{n\in\omega} \iota(x_n)$. $L_{\iota(x)} = \bigcup_{n\in\omega} L_{\iota(x_n)}$. $x \in L_{\iota(x)}$. This implies $x \in L_{\iota(x_n)}$ for some $n \in \omega$. This contradicts $\iota(x) = \alpha$ being the smallest admissible ordinal γ such that $x \in L_\gamma$. □

Proposition 9.7. $(\mathsf{V} = \mathsf{L})$ *E_L is a thin equivalence relation.*

Proof. Let $T \subseteq {}^{<\omega}2$ be an arbitrary perfect tree. Let $\alpha = \iota(T)$. L_α satisfies that there are no functions from ω taking reals as images which enumerate all paths through T. By Lemma 9.6, let $\beta < \alpha$ be greatest such that there is a y with $\iota(y) = \beta$. By Fact 9.5, $L_\alpha \models |L_\beta| = \aleph_0$. However, since L_α satisfies no function from ω into the reals enumerates the paths through T, there exists $v, w \in L_\alpha$ such that in L_α, v and w are paths through T and $v, w \notin L_\beta$. By the choice of β, $\iota(v) = \iota(w) = \alpha$. By Proposition 9.3, $v \; E_L \; w$. By Δ_1-absoluteness, $v, w \in [T]$. It has been shown that every perfect set has E_L equivalent elements. □

Proposition 9.8. *If E is a thin equivalence relation with all classes countable, then for any σ-ideal I, $\{E\} \rightarrow_I \boldsymbol{\Delta}^1_1$ fails.*

Proof. Suppose there exists some $\boldsymbol{\Delta}^1_1$ I^+ B such that $E \restriction B$ is $\boldsymbol{\Delta}^1_1$. By Silver's Dichotomy for $\boldsymbol{\Pi}^1_1$ equivalence relations, either $E \restriction B$ has countably many classes or a perfect set of pairwise E-inequivalent elements. The former is not possible since this would imply the I^+ set B is a countable union of countable sets. The latter is also not possible since E is thin. Contradiction. □

Theorem 9.9. *$(\mathsf{V} = \mathsf{L})$ For any σ-ideal I on ${}^\omega 2$, $\{E_L\} \rightarrow_I \boldsymbol{\Delta}^1_1$ fails.*

In particular in L, ${\boldsymbol{\Delta}^1_2}^{\boldsymbol{\Delta}^1_1} \rightarrow_I \boldsymbol{\Delta}^1_1$ for σ-ideal I with $\mathbb{P}_I$ proper is not true. (${\boldsymbol{\Delta}^1_2}^{\boldsymbol{\Delta}^1_1}$ is the class of $\boldsymbol{\Delta}^1_2$ equivalence relation with all classes $\boldsymbol{\Delta}^1_1$.)

Proof. E_L is thin and has all classes countable. Use Proposition 9.8. □

A positive answer to the main question for $\boldsymbol{\Delta}^1_2$ equivalence relations with all classes $\boldsymbol{\Delta}^1_1$ is now known to follow from large cardinals. In fact, [8] has shown that if there is a measureable cardinal above infinity many Woodin cardinals then a positive answer holds for all equivalence relations in $L(\mathbb{R})$ with all classes $\boldsymbol{\Delta}^1_1$.

10. Conclusion

This last section will put the results of this paper into perspective. Some questions will be raised and some speculations will be made.

Large cardinal assumptions were used throughout the paper to obtain a positive answer to the main question in its various forms. In the most general case for $\boldsymbol{\Sigma}^1_1$ or $\boldsymbol{\Pi}^1_1$ equivalence relations with all $\boldsymbol{\Delta}^1_1$ classes, iterability assumptions were used to get a positive answer. Iterability is a fairly strong large cardinal assumption: for example, it requires the universe to transcend L in a way set forcing extensions can never do.

However, this paper leaves open the possibility that even the most general form of this question for ${\boldsymbol{\Sigma}^1_1}^{\boldsymbol{\Delta}^1_1}$ and ${\boldsymbol{\Pi}^1_1}^{\boldsymbol{\Delta}^1_1}$ could be provable in just ZFC.

Question 10.1. Is it consistent (relative large cardinals) that there is a σ-ideal I on a Polish space with $\mathbb{P}_I$ proper and $E \in {\boldsymbol{\Sigma}^1_1}^{\boldsymbol{\Delta}^1_1}$ such that $\{E\} \rightarrow_I \boldsymbol{\Delta}^1_1$ is false?

Same question for ${\boldsymbol{\Pi}^1_1}^{\boldsymbol{\Delta}^1_1}$.

The results of this paper provide limitations to any attempt to produce a counterexample to a positive answer to the main question.

The results of the paper seem to suggest a universe with few and very weak large cardinals is the ideal place to consider finding a counterexample. For example, Theorems 4.22 and 7.12 show that any universe that has sharps for sets in $H_{(2^{\aleph_0})^+}$ will always give a positive answer to the main question.

The following is perhaps the main open question:

Question 10.2. Is there a negative answer to the main question for the $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$ case in L?

Cohen forcing ($\mathbb{P}_{I_{\text{meager}}}$) is perhaps the simpliest of all forcings. This paper leaves open the possibility that Cohen forcing in L could be used to produce a counterexample to the main question.

Question 10.3. Can Cohen forcing (meager ideal) be used with some $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$ equivalence relation with all classes $\mathbf{\Delta}^1_1$ to produce a counterexample to the main question?

Propositions 5.2 and 5.4 show that the ideal of countable sets (Sacks forcing) and the E_0-ideal (Prikry-Silver forcing) can never be used to produce a counterexample to the main question in the $\mathbf{\Sigma}^1_1$ case.

One of the most common forcing extensions in descriptive set theory is the extension by the (gentle) Lévy collapse $\text{Coll}(\omega, < \kappa)$, where κ is some inaccessible cardinal. Here there is a partial answer to Question 10.3: Corollary 5.11 shows that the meager ideal and null ideal can not be used in an extension by the Lévy collapse of an inaccessible to produce a counterexample to the main question in the $\mathbf{\Sigma}^1_1$ case. Moreover, Fact 5.14 implies that these two ideals cannot be used for a counterexample if $\mathsf{MA} + \neg\mathsf{CH}$ holds.

Propositions 3.2 and 3.1 assert that $\mathbf{\Sigma}^1_1$ equivalence relations with all classes countable or are $\mathbf{\Delta}^1_1$ reducible to orbit equivalence relations of Polish group actions cannot be used to show the consistency of a negative answer. One may suspect that an unusual $\mathbf{\Sigma}^1_1{}^{\mathbf{\Delta}^1_1}$ equivalence relation may be necessary. Thin equivalence relations include somewhat unusual objects such as F_{ω_1}, E_{ω_1}, and any potential counterexamples to Vaught's conjecture. However, Theorem 6.8 shows that thin $\mathbf{\Sigma}^1_1$ equivalence relations have the strongest form of canonization in the sense that one of their classes is in I^+.

It seems that one has reached an impasse in regard to the main question for $\mathbf{\Sigma}^1_1{}^{\mathbf{\Delta}^1_1}$. There is a lack of interesting examples of $\mathbf{\Sigma}^1_1{}^{\mathbf{\Delta}^1_1}$ equivalence relations which may be useful for producing a consistency result for a negative answer to the main question for $\mathbf{\Sigma}^1_1{}^{\mathbf{\Delta}^1_1}$.

Here is where $\mathbf{\Pi}^1_1{}^{\mathbf{\Delta}^1_1}$ becomes much more interesting and provides a possible path forward. What appears to be promising is that $\mathbf{\Pi}^1_1$ equivalence relations seem to be much more suspectible to set theoretic assumptions.

One difficulty in producing the appropriate type of $\mathbf{\Sigma}^1_1$ equivalence relation is the requirement that all classes be $\mathbf{\Delta}^1_1$. In the $\mathbf{\Pi}^1_1$ case, one situation in which this requirement is easily obtained is by considering $\mathbf{\Pi}^1_1$ equivalence relations with all classes thin and assume ω_1 is inaccessible to reals, i.e., the class $\mathbf{\Pi}^1_1{}^{\text{thin}}$.

Even in this case, one must still limit the universe to one in which only weak large cardinals exist: The easiest way to obtain ω_1 is inaccessible to real is via a Lévy collapse. Theorem 8.10 shows that this attempt will never work if one uses a Lévy collapse extension of a remarkable cardinal. Moreover, Theorem 8.12 shows that using $\mathbf{\Pi}^1_1{}^{\text{thin}}$ with a $\aleph_1$-c.c. forcing will never work in a Lévy collapse extension of a weakly compact cardinal.

A closer look at the proofs of Lemmas 7.3 and 7.4 shows the following:

Definition 10.4. Let $E \in \mathbf{\Pi}^1_1{}^{\mathbf{\Delta}^1_1}$. Let $r(x) = \min\{\omega_1^z : [x]_E \text{ is } \Sigma^1_1(z)\}$.

Proposition 10.5. *Let $E \in \mathbf{\Pi}^1_1{}^{\mathbf{\Delta}^1_1}$ and I be a σ-ideal such that $\mathbb{P}_I$ is proper. Suppose, for all $B \in \mathbb{P}_I$, there exists some $C \subseteq B$ with $C \in \mathbb{P}_I$ and $\sup\{r(x) : x \in C\} < \omega_1$. Then $\{E\} \to_I \mathbf{\Delta}^1_1$.*

Therefore, any counterexample to a positive answer for the main question for $\mathbf{\Pi}^1_1{}^{\mathbf{\Delta}^1_1}$ must violate the hypothesis of this proposition. The next result gives a hypothetical condition under which this happens:

Proposition 10.6. *Suppose ω_1 is inaccessible to reals. Let I be a σ-ideal on a Polish space such that $\mathbb{P}_I$ is proper and whenever g is $\mathbb{P}_I$-generic over V, $V[g] = L[g]$. Let $E \in \Pi^1_1{}^{\text{thin}}$ with the property that for all x, $L[x] \models [x]_E$ is uncountable thin. Then for all $C \in \mathbb{P}_I$, $\sup\{r(x) : x \in C\} = \omega_1$.*

Proof. The first claim is that $[x]_E$ can not be $\Delta^1_1(z)$ for any z such that $\omega_1^z < \omega_1^{L[x]}$. (Note that ω_1^z refers to the least z-admissible ordinal and $\omega_1^{L[x]}$ is the least uncountable cardinal of $L[x]$.)

Suppose otherwise: $[x]_E$ is $\Sigma^1_1(z)$ and $\omega_1^z < \omega_1^{L[x]}$. As in Lemma 7.3, define

$$a\ E'\ b \Leftrightarrow (a \in [x]_E \wedge b \in [x]_E) \vee (a = b)$$

E' is $\Sigma^1_1(z)$. $E' \subseteq E$. By the effective bounding theorem, there are some $\alpha < \omega_1^z$ such that $E' \subseteq E_\alpha$. Now, applying Lemma 7.2 in $L[x]$ and the fact that $\alpha < \omega_1^z < \omega_1^{L[x]}$, there exists some β such that $\alpha < \beta < \omega_1^{L[x]}$ such that E_β is an equivalence relation. Using the argument in Lemma 7.3, $[x]_E = [x]_{E_\beta}$. E_β is $\boldsymbol{\Delta}^1_1(c)$ for any $c \in {}^\omega 2$ such that $\text{ot}(c) = \beta$. Since $\beta < \omega_1^{L[x]}$, there exists such a $c \in L[x]$. Hence $[x]_{E_\beta}$ is $\Delta^1_1(x, c)$.

$$V \models (\forall a)(a\ E\ x \Leftrightarrow a\ E_\beta\ x)$$

Since $x, c \in L[x]$ and this statement is $\Pi^1_2(x, c)$, by Schoenfield absoluteness

$$L[x] \models (\forall a)(a\ E\ x \Leftrightarrow y\ E_\beta\ x)$$

So $L[x] \models [x]_E$ is $\boldsymbol{\Delta}^1_1$. However, the assumption was that $L[x] \models [x]_E$ is uncountable thin. ZFC proves that no $\boldsymbol{\Delta}^1_1$ set can be uncountable thin. Contradiction. This proves the claim.

So now let $\alpha < \omega_1$. Let $M \prec H_\Theta$ with $\alpha \subseteq M$ and $C, \mathbb{P}_I \in M$. Note that $\omega_1^M \geq \alpha$. Let $x \in C$ be $\mathbb{P}_I$-generic over M. Then $M[x] \models \omega_1^{L[x]} = \omega_1^{M[x]} = \omega_1^M \geq \alpha$, using the fact that $\mathbb{P}_I$ has the property $V[g] = L[g]$, wherever g is $\mathbb{P}_I$ generic over V. Certainly, the real $(\omega_1^{L[x]})^V$ is greater than or equal to $(\omega_1^{L[x]})^M \geq \alpha$. So $\omega_1^{L[x]} \geq \alpha$. By the claim above, $r(x) \geq \alpha$. Hence $\sup\{r(x) : x \in C\} = \omega_1$. □

Note that if V satisfies ω_1 is inaccessible to reals and $V[g] = L[g]$ whenever g is $\mathbb{P}_I$-generic over V, then "ω_1 is inaccessible to reals" is not preserved into the extension $V[g] = L[g]$. Compare this to what happens in the $\text{Coll}(\omega, <\kappa)$ extension of L when κ is a remarkable cardinal in L (see Theorem 8.10).

Given this result, the natural questions are whether such an ideal exists and whether such a Π^1_1 $^{\text{thin}}$ equivalence relation exists.

First consider the following: Suppose $\kappa \in L$ and κ is not Mahlo in L. Let $G \subseteq \text{Coll}(\omega, <\kappa)$. In $L[G]$, $\omega_1^{L[G]}$ is not Mahlo and $L[G]$ satisfies ω_1 is inaccessible to reals. By [28] Exercise 8.7, there is an $A \subseteq \omega_1$ in $L[G]$, which is reshaped, i.e., for all $\xi < \omega_1$, $L[A \cap \xi] \models |\xi| = \aleph_0$. Since $L[A] \subseteq L[G]$, $\omega_1^{L[A]} \leq \omega_1^{L[G]}$. Since A is reshaped, $L[A] \models \omega_1^{L[A]} \geq \omega_1^{L[G]}$. So $\omega_1^{L[A]} = \omega_1^{L[G]}$. Since $L[G]$ satisfies ω_1 is inaccessible to reals, $L[A]$ also satisfies ω_1 is inaccessible to reals.

In [13] Section 1, it is shown that in $L[A]$, where A is a reshaped subset of ω_1, there is an $\aleph_1$-c.c. forcing which adds a real g such that $L[A][g] = L[g]$. This forcing consists of perfect trees. By [34] Corollary 2.1.5, there is a σ-ideal I_F such that $\mathbb{P}_{I_F}$ is forcing equivalent to Sy-David Friedman's forcing to code subsets of ω_1. In $L[A]$, I_F would be a σ-ideal that satisfies the property of Proposition 10.6.

It is not known whether $\sup\{r(x) : x \in C\} = \omega_1$ for all I^+ set C is enough for a negative answer to the main question for $\boldsymbol{\Pi}^1_1$ $^{\text{thin}}$. It could be possible that there is a C such that for all $x \in C$, $[x]_E$ is very complicated as x ranges over C, but C consists of pairwise E-inequivalent elements (or even C is a single E-class).

In L, Jensen's minimal nonconstructible Δ^1_3 real forcing (see [18] and [16], Chapter 28) is also a forcing consisting of perfect trees. Again by [34] Corollary 2.1.5, there is a σ-ideal I_J such that $\mathbb{P}_{I_J}$ is forcing equivalent to Jensen's forcing. $\mathbb{P}_{I_J}$ is $\aleph_1$-c.c. by [16] Lemma 28.4. Moreover, by [16] Corollary 28.6, if g, h are $\mathbb{P}_{I_J}$-generic over L, then $g \times h$ is $\mathbb{P}_{I_J} \times \mathbb{P}_{I_J}$ generic over L. Hence, below any B such that $B \Vdash_{\mathbb{P}_{I_J}} (\dot{x}_{\text{gen}})_{\text{left}}\ E (\dot{x}_{\text{gen}})_{\text{right}}$ (or $B \Vdash_{\mathbb{P}_{I_J}} \neg((\dot{x}_{\text{gen}})_{\text{left}}\ E\ (\dot{x}_{\text{gen}})_{\text{right}})$), if C is the I^+ set of $\mathbb{P}_{I_J}$-generic real over M in B (for some $M \prec H_\Theta$), then B consists of pairwise E-inequivalent (or pairwise E-equivalent) reals. But of course, this example does not satisfy all of the conditions of Proposition 10.6.

It is not known whether the $\boldsymbol{\Pi}^1_1$ $^{\text{thin}}$ equivalence relations needed in Proposition 10.6 exist.

Question 10.7. Let κ be inaccessible but not Mahlo in L. Suppose $G \subseteq \text{Coll}(\omega, <\kappa)$ be generic over L. Let $A \subseteq \omega_1$ with $A \in L[G]$ be a reshaped subset of ω_1. Then is there a Π^1_1 equivalence relation E such that for all $x \in ({}^\omega\omega)^{L[A]}$, $L[x] \models [x]_E$ is uncountable thin?

This leads to an interesting related question about whether it is possible to partition ${}^\omega\omega$ in a Π^1_1 way into $\boldsymbol{\Pi}^1_1$ pieces that are all uncountable thin:

Question 10.8. In L, is there a Π^1_1 equivalence relation E such that $L \models (\forall x)([x]_E$ is uncountable thin$)$?

Sy-David Friedman has communicated to the author a solution to this last question. See the appendix below for more information.

11. Appendix

This appendix includes some remarks of Sy-David Friedman.

Sy-David Friedman and Törnquist, using some ideas of Miller and Conley, have given a solution to Question 10.8.

Theorem 11.1. *(Friedman, Törnquist) In L, there exists a $\mathbf{\Pi}^1_1$ equivalence relation E such that $L \models (\forall x)([x]_E$ is uncountable thin).*

Proof. E will be an equivalence relation on $\mathbb{R}$. Consider $\mathbb{R}$ with its usual $\mathbb{Q}$-vector space structure. By [21] Exercise 19.2 (i), let C be a perfect $\mathbf{\Pi}^0_1$ $\mathbb{Q}$-linearly independent set of reals. Let $P \subseteq C$ be an uncountable thin $\mathbf{\Pi}^1_1$ subset. Let $\langle C \rangle$ and $\langle P \rangle$ denote the additive subgroups of $\mathbb{R}$ generated by C and P, respectively.

Since C consists of $\mathbb{Q}$-linearly independent reals, each element of $\langle C \rangle$ has a unique representation as $\mathbb{Z}$-linear combinations of elements of C. By Lusin-Novikov (countable section) uniformization, $\langle C \rangle$ is $\mathbf{\Delta}^1_1$. Also by Lusin-Novikov, there is a $\mathbf{\Delta}^1_1$ function Φ on $\mathbb{R}$ such that if $r \in \langle C \rangle$, then $\Phi(r)$ is a representation of r as a $\mathbb{Z}$-linear combination of elements of C, and if $r \notin \langle C \rangle$, then $\Phi(r)$ is some default value.

Then $\langle P \rangle$ has the following definition: $r \in \langle P \rangle$ if and only if $r \in \langle C \rangle$ and $\Phi(r)$ consists of only elements from P. The latter is $\mathbf{\Pi}^1_1$. Hence $\langle P \rangle$ is a coanalytic subgroup of $\mathbb{R}$.

By definition, $\langle P \rangle$ is the set of $\mathbb{Z}$-linear combinations of elements of P. Since P is thin, by Mansfield-Solovay, P consists entirely of constructible reals. In particular, in any forcing extension $L[G]$ of L, $P^L = P^{L[G]}$. So, $\langle P \rangle^{L[G]}$ consists of $\mathbb{Z}$-linear combinations of elements of $P^{L[G]} = P^L$. Hence, $\langle P \rangle^{L[G]} = \langle P \rangle^L$. If $\langle P \rangle^L$ had a perfect subset, then by Schoenfield's absoluteness, $\langle P \rangle^{L[G]}$ would have a perfect subset. If G was generic for a forcing which makes $(2^{\aleph_0})^{L[G]} > \aleph_1^L$, then $|\langle P \rangle|^{L[G]} = (2^{\aleph_0})^{L[G]} > \aleph_1^L = |\langle P \rangle|^L$. This contradicts $\langle P \rangle^{L[G]} = \langle P \rangle$. This shows that in L, $\langle P \rangle$ is uncountable thin.

Let E be the coset equivalence relation of $\mathbb{R}/\langle P \rangle$: $r \; E \; s \Leftrightarrow (r - s) \in \langle P \rangle$. E is $\mathbf{\Pi}^1_1$. For all r, $[r]_E$ is in bijection with $\langle P \rangle$. Hence $[r]_E$ is uncountable thin. □

At the time of asking Question 10.8, there was hope that any natural constructibly coded $\mathbf{\Pi}^1_1$ equivalence relation which witnessed a positive answer to Question 10.8 would also serve as a witness to a positive answer to Question 10.7.

Unfortunately, the equivalence relation E of Theorem 11.1 does not work. The definition of E has a particular constructibly coded thin $\mathbf{\Pi}^1_1$ group built into it. E, as a coset relation, copies this thin uncountable (in L) set throughout the reals. Now suppose V is some universe such that $\omega_1^L < \omega_1^V$. In V, choose some $z \in \mathbb{R}$ such that $L[z] \models \omega_1^L < \omega_1$. Since $[z]_E$ is in bijection with $\langle P \rangle$ (which is in bijection with ω_1^L), $L[z] \models [z]_E$ is countable.

It seems any possible solution to Question 10.7 will need to be defined without using any explicit definition of a thin $\mathbf{\Pi}^1_1$ set.

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