ORDINAL DEFINABILITY AND COMBINATORICS OF EQUIVALENCE RELATIONS

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Abstract. Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation coded in HOD. E has an ordinal definable equivalence class without any ordinal definable elements if and only if HOD $\models E$ is unpinned.

$\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$ proves E-class section uniformization when E is a $\mathbf{\Sigma}^1_1$ equivalence relation on $\mathbb{R}$ which is pinned in every transitive model of ZFC containing the real which codes E: Suppose R is a relation on $\mathbb{R}$ such that each section $R_x = \{y : (x, y) \in R\}$ is an E-class, then there is a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $R(x, f(x))$.

$\mathsf{ZF} + \mathsf{AD}$ proves that $\mathbb{R} \times \kappa$ is Jónsson whenever κ is an ordinal: For every function $f : [\mathbb{R} \times \kappa]^{\leq\omega}_{=} \to \mathbb{R} \times \kappa$, there is an $A \subseteq \mathbb{R} \times \kappa$ with A in bijection with $\mathbb{R} \times \kappa$ and $f[[A]^{\leq\omega}_{=}] \neq \mathbb{R} \times \kappa$.

1. Introduction

The questions of concern here are problems of independent interests that appeared during the study of the Jónsson property for nonwellorderable sets under the axiom of determinacy.

Let $N \in \omega \cup \{\omega\}$ and X be some set. Define $[X]^N_{=} = \{x \in {}^N X : (\forall i, j < N)(i \neq j \Rightarrow x(i) \neq x(j))\}$ and $[X]^{\leq\omega}_{=} = \bigcup_{n\in\omega} [X]^n_{=}$. Let $\approx$ denote the relation of being in bijection. Define $\mathscr{P}_N(X) = \{Y \subseteq X : Y \approx N\}$ and $\mathscr{P}_{<\omega}(X) = \bigcup_{n\in\omega} \mathscr{P}_n(X)$.

An N-Jonsson function for X is a function $f : [X]^N_{=} \to X$ so that for all $Y \subseteq X$ with $Y \approx X$, $f[[Y]^N_{=}] = X$. A function $f : [X]^{\leq\omega}_{=} \to X$ is a Jónsson function if and only if for all $Y \subseteq X$ with $Y \approx X$, $f[[Y]^{\leq\omega}_{=}] = X$. A set X has the Jónsson property if and only if there are no Jónsson functions for X.

The classical study of the Jónsson property involved wellordered sets. For wellordered sets X, Jónsson functions for X are formulated using $\mathscr{P}_N(X)$ rather than $[X]^N_{=}$. Under AC, the following results are known: [4] showed that every infinite set has an ω-Jónsson function. The existence of such a function is also where Kunen's proof of the Kunen's inconsistency uses AC. The existence of a cardinal with the Jónsson property implies $0^\sharp$ exists. Results of Erdős and Hajnal (see [3] and [4]) imply that under CH, $2^{\aleph_0}$ is not Jónsson. Hence $\mathbb{R}$ is not Jónsson under CH. On the other hand, real valued measurable cardinals are Jónsson (see [3] Corollary 11.1). Solovay showed it is consistent relative to a measurable cardinal that $2^{\aleph_0}$ is real valued measurable. Hence it is consistent relative to a measurable cardinal that $\mathbb{R}$ is Jónsson.

Using the axiom of determinacy AD, [15] showed that $\aleph_n$ is Jónsson for each $n \in \omega$. [7] showed that every cardinal $\kappa < \Theta$ is Jónsson under $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$. In fact, Woodin showed that $\mathsf{ZF} + \mathsf{AD}^+$ can prove every cardinal $\kappa < \Theta$ is Jónsson.

Under AD, there are sets which cannot be wellordered. Some important examples are quotients of $\mathbf{\Delta}^1_1$ equivalence relations such as $=$, E_0, E_1, E_2, and E_3 (see Definition 2.15). Holshouser and Jackson (see [6] and [5]) showed that $\mathbb{R}$ has the Jónsson property and there are no 2-Jónsson functions for $\mathbb{R}/E_0$ under AD. [2] showed that under AD, there is a 3-Jónsson function for $\mathbb{R}/E_0$. Results from [2] seem to suggest that $\mathbb{R}/E_1$, $\mathbb{R}/E_2$, and $\mathbb{R}/E_3$ do not have that Jónsson property, but no Jónsson functions for these quotients have yet to be constructed.

For the $\mathbf{\Delta}^1_1$ equivalence relations mentioned above, various dichotomy theorems assert the significance of these equivalence relations in the degree structure of $\mathbf{\Delta}^1_1$ equivalence relations under $\mathbf{\Delta}^1_1$ reducibility. The proofs of these dichotomy results give specific combinatorial structures to sets A such that $E \leq_{\mathbf{\Delta}^1_1} E \restriction A$, when E is one of the $\mathbf{\Delta}^1_1$ equivalence relations above. For example, if $A \subseteq \mathbb{R}$ is $\mathbf{\Sigma}^1_1$ and $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \restriction A$, then A contains an E_0-tree (a perfect tree with very specific symmetry conditions; see [2] Definition 5.2).

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Similarly, if $A \subseteq \mathbb{R}$ is $\boldsymbol{\Sigma}^1_1$ and $E_2 \leq_{\boldsymbol{\Delta}^1_1} E_2 \restriction A$, then A contains an E_2-tree (a perfect tree with certain summability conditions; see [2] Fact 14.14).

The following describes the techniques from [2] for investigating the Jónsson property for $\mathbb{R}/E_0$: To study functions $f : [\mathbb{R}/E_0]^2_= \to \mathbb{R}/E_0$, one would like to lift f to a function $F : \mathbb{R}^2 \to \mathbb{R}$ with the property that for all $(x_1, x_2) \in \mathbb{R}^2$, $[F(x_1, x_2)]_{E_0} = f([x_1]_{E_0}, [x_2]_{E_0})$. Such a function F is called a lift of f. Then one tries to produce an E_0-tree on which the collapse of F misses elements of $\mathbb{R}/E_0$. On the other hand, using the specific combinatorial structure of E_0-trees, one can define a map $F : \mathbb{R}^3 \to \mathbb{R}$ which is E_0-invariant and given any real x, there is a triple (x_1, x_2, x_3) of E_0-unrelated reals so that $F(x_1, x_2, x_3) \; E_0 \; x$. The collapse of F would then be a 3-Jónsson map.

As described in the above example, the existence of lifts of functions from $\mathbb{R}/E \to \mathbb{R}/E'$, where E and E' are equivalence relations on $\mathbb{R}$, seems to be useful in the study of functions on quotients. The existence of a lift is an immediate consequence of uniformization. $\mathsf{AD}_\mathbb{R}$ has full uniformization. Moreover, a lift of a function $f : \mathbb{R}/E \to \mathbb{R}/E'$ requires only uniformization for relations whose sections are E'-classes. Woodin showed that countable section uniformization holds in AD^+. Thus lifts exist for functions into $\mathbb{R}/E_0$ under AD^+. Moreover for the purpose of showing that there are no 2-Jónsson functions for $\mathbb{R}/E_0$, AD alone has a sufficient uniformization: Let $f : [\mathbb{R}/E_0]^2_= \to \mathbb{R}/E_0$. One can apply comeager uniformization (which holds in just AD) to find a function $F : C \to \mathbb{R}$, where $C \subseteq \mathbb{R}^2$ is comeager, which lifts f on C. Then the 2-Mycielski property for E_0 shows that there is a set A such that $E_0 \leq_{\boldsymbol{\Delta}^1_1} E_0 \restriction A$ and $\{(x_1, x_2) \in A^2 : \neg(x_1 \; E_0 \; x_2)\} \subseteq C$. (See [2] Definition 2.11 for the definition of the Mycielski property.) This roughly implies that F lifts f on a set whose quotient by E_0 has cardinality $\mathbb{R}/E_0$. However, [2] showed that except for $=$ which has the full Mycielski property, a very limited amount of the Mycielski property holds for the other equivalence relations of interest.

Motivated by this question of E-class section uniformization, Zapletal asked a related question: Does every ordinal definable E_2 equivalence class contain an ordinal definable real, under $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$? He informed the author that the equivalence relation $=^+$, defined on ${}^\omega\mathbb{R}$ as equality of range, has ordinal definable classes with no ordinal definable elements assuming $\mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$, and that this phenomenon can be viewed as a consequence of the unpinnedness of $=^+$. He asked then whether pinnedness can be used to characterize those $\boldsymbol{\Delta}^1_1$ equivalence relations with ordinal definable equivalence classes without any ordinal definable elements.

For countable equivalence relations, Zapletal's question has a positive answer under AD^+: Under AD^+, every ordinal definable countable set of reals contains only ordinal definable elements. The proof of this can be found within the proof of Woodin's countable section enumeration under AD^+, which states that for every relation R with countable sections there is a function that takes x to a wellordering of the section R_x. The main idea is to consider the canonical wellordering of R_x in $\mathrm{HOD}_S^{L[S,x,z]}$ as z ranges over a Turing cone of reals and S is some set of ordinals from an ∞-Borel code for R. (See [14] for the proof.) This implies that under AD^+, every ordinal definable E class contains only ordinal definable elements if E is an equivalence relation with all countable classes defined using only ordinal parameters.

The determinacy assumptions are important for these questions since [11] showed that in a forcing extension of the constructible universe L, there is an ordinal definable E_0 equivalence class with no ordinal definable elements. Similar examples are given in [12] which showed that in a forcing extension of L, there are definable relations with each section an E_0-class but have no uniformizations which are ordinal definable in a real.

Section 2 will show roughly that in $L(\mathbb{R}) \models \mathsf{AD}$, if a Σ^1_1 equivalence relation E has an OD equivalence class without any OD elements, then HOD must think that E is unpinned:

Theorem 2.12 *Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let T be a set of ordinals. Let E be an equivalence relation which is $\Sigma^1_1(s)$ for some $s \in \mathrm{HOD}_T$ and let A be an OD_T E-class. If A has no OD_T elements, then $\mathrm{HOD}_T \models E$ is unpinned.*

Models of $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$ are considered natural models of AD^+. If $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R})$ satisfies this theory. Woodin, [1] Corollary 3.2, has shown that if $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$ holds, then either there is a set of ordinals J so that $V = L(J, \mathbb{R})$ or else $V \models \mathsf{AD}_\mathbb{R}$.

The proof of this theorem uses the idea of taking ultraproducts of $\text{HOD}_S^{L[S,z]}$ (where the Turing degree of z serves as the index and S is a set of ordinals) using Martin's Turing cone measure. This technique appears in Woodin's proof that sets of reals have ∞-Borel codes in $L(\mathbb{R})$ when $L(\mathbb{R}) \models \mathsf{AD}$ as exposited in [16] Claim 1.6.

Theorem 2.13 ($\mathsf{ZF} + \mathsf{AD}^+$) *Let E be a $\mathbf{\Sigma}_1^1$ equivalence relation defined in HOD_R, where R is some set. Suppose $\text{HOD}_R \models E$ is unpinned. Then there is an OD_R E-class with no OD_R elements.*

These two results together give a very succient answer to Zapletal's question in natural models of AD^+:

Corollary 2.14 *Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let E be a $\mathbf{\Sigma}_1^1$ equivalence relation coded in* HOD. *E has an* OD *E-class with no* OD *elements if and only if* $\text{HOD} \models E$ *is unpinned.*

Many important examples of pinned $\mathbf{\Delta}_1^1$ equivalence relations include $=$, E_0, E_1, E_2, smooth, hyperfinite, and hypersmooth equivalence relations.

Using the previous theorem, one obtains E-class section uniformization for equivalence relations satisfying some definable pinnedness condition. This is particular useful when the equivalence relations are provably pinned:

Theorem 3.1 *Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. If E is a $\mathbf{\Sigma}_1^1$ equivalence relation which is pinned in every transitive model of* ZFC *containing the real that codes E, then every relation R whose sections are all E-classes can be uniformized.*

As a consequence, every function $f : \mathbb{R}/E \to \mathbb{R}/F$ has a lift under $\mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$ when F is $=$, E_0, E_1, E_2, smooth, hyperfinite, essentially countable, or hypersmooth.

Section 4.1 will study the Jónsson property of some nonwellorderable sets. Holshouser and Jackson have shown that $\mathbb{R} \times \kappa$ for any $\kappa < \Theta$ has the Jónsson property. They use that $\mathbb{R}$ and all ordinals $\kappa < \Theta$ have the Jónsson property. A natural question would be whether $\mathbb{R} \times \kappa$ is Jónsson for all ordinals κ. The proof that $\mathbb{R}$ is Jónsson has a clear flavor of classical descriptive set theory since it uses comeagerness, continuity, the Mycielski property, and fusions of perfect trees. The proof that ordinals $\kappa < \Theta$ is Jónsson have a somewhat different flavor. A related question would be whether the Jónsson property for κ is relevant to showing $\mathbb{R} \times \kappa$ is Jónsson. Does there exists a more classical proof that $\mathbb{R} \times \kappa$ is Jónsson? It will be shown that:

Theorem 4.15 ($\mathsf{ZF} + \mathsf{AD}$) *For any ordinal κ, $\mathbb{R} \times \kappa$ has the Jónsson property.*

Whether or not κ is Jónsson does not appear in the proof of the above theorem. This result is proved while investigating the Jónsson property for wellordered disjoint unions $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ where each E_α is an equivalence relation with all classes countable and $\mathbb{R}/E_\alpha \approx \mathbb{R}$. The techniques have a very classical flavor using results about lengths of wellordered sequences of reals, additivity of the meager ideal, comeager uniformization, and fusions of perfect trees. There are also some discussions about the cardinality of $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$. However, it remains open whether $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ has the Jónsson property.

This section concludes by producing a 6-Jónsson function for $(\mathbb{R}/E_0) \times \kappa$ for any $\kappa < \Theta$ under AD. This shows that $(\mathbb{R}/E_0) \times \kappa$ for $\kappa < \Theta$ is not Jónsson under AD.

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2. Ordinal Definable Equivalence Classes

V will denote the universe of set theory in consideration. If M is a model of set theory and A is some concept given by some formula, then A^M will denote the relativization of that formula inside M. If a concept A is unrelativized, then it is assumed to mean A^V, although it may be written A^V for emphasis. $\mathbb{R}$ will denote ${}^\omega\omega$, the Baire space, consisting of functions from ω to ω with its usual metric. (Although it may sometimes denote ${}^\omega 2$, the Cantor space.) The elements of $\mathbb{R}$ will be called reals.

If X is a set, then OD_X denotes the class of sets which are ordinal definable using X as a parameter. HOD_X is the collection of sets which are hereditarily ordinal definable from X. $\mathrm{HOD}_X \models \mathsf{ZFC}$ and has a canonical global wellordering definable using X.

Fact 2.1. *(Vopěnka) Suppose S is a set of ordinals. Let $x \in \mathbb{R}$.*

In $L[S,x]$, let $\mathbb{P}$ denote the forcing of nonempty OD_S subsets of $\mathbb{R}$ ordered by $\subseteq$. Using the canonical S-definable bijection of OD_S subsets onto ON, let $\mathbb{O}_S \in \mathrm{HOD}_S$ be the forcing that results by transferring $\mathbb{P}$ onto ON using this map.

Then there is a $G \in L[S,x]$, which is $\mathbb{O}_S$-generic over HOD_S, so that $L[S,x] = \mathrm{HOD}_S[G] = \mathrm{HOD}_S[x]$.

Proof. See [8] Theorem 15.46. □

Definition 2.2. Let $X \subseteq \mathbb{R}$, S be a set of ordinals, and φ be a formula in the language of set theory. (S,φ) is an ∞-Borel code for X if and only if for all $x \in \mathbb{R}$, $x \in X \Leftrightarrow L[S,x] \models \varphi(S,x)$.

Definition 2.3. ([18] Section 9.1) AD^+ consists of the following:
(1) $\mathrm{DC}_{\mathbb{R}}$.
(2) Every $A \subseteq \mathbb{R}$ has an ∞-Borel code.
(3) For all $\lambda < \Theta$, $A \subseteq \mathbb{R}$, and continuous function $\pi : {}^\omega\lambda \to \mathbb{R}$, $\pi^{-1}[A]$ is determined.

(λ is given the discrete topology. Θ is the supremum of the ordinals which are surjective images of $\mathbb{R}$. Games with moves from λ are defined the same way as the more familiar games on ω.)

Definition 2.4. ([19]) Let E be an equivalence relation on $\mathbb{R}$. Let $\mathbb{P}$ be a forcing. Let τ be a $\mathbb{P}$-name.

Let $\tau_{\text{left}}, \tau_{\text{right}}$ be the canonical $\mathbb{P}\times\mathbb{P}$-names with the property that τ_{left} and τ_{right} are evaluated according to τ using the left and right $\mathbb{P}$-generic filters, respectively, coming from a $\mathbb{P}\times\mathbb{P}$-generic filter.

τ is an E-pinned name if and only if $1_{\mathbb{P}\times\mathbb{P}} \Vdash_{\mathbb{P}\times\mathbb{P}} \tau_{\text{left}}\ E\ \tau_{\text{right}}$.

An E-pinned name τ is an E-trivial name if and only if there is some $x \in \mathbb{R}$ so that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \tau\ E\ \check{x}$.

E is a pinned equivalence relation if and only if all forcings $\mathbb{P}$, every E-pinned $\mathbb{P}$-names is E-trivial.

Pinnedness is more accurately a property of a fixed definition for the equivalence relation E (which is to be used to interpret E in generic extensions). This paper is concern only with $\mathbf{\Sigma}^1_1$ equivalence relations and such equivalence relations are always defined as the projection of certains trees on $\omega\times\omega\times\omega$.

Definition 2.5. Let $\leq_T$ denote the Turing reducibility relation on ${}^\omega\omega$. For $x, y \in {}^\omega\omega$, let $x \equiv_T y$ if and only if $x \leq_T y$ and $y \leq_T x$. A Turing degree is a $\equiv_T$ equivalence class. If $x, y \in {}^\omega\omega$, then define $[x]_{\equiv_T} \leq_T [y]_{\equiv_T}$ if and only if $x \leq_T y$.

Let $\mathcal{D}$ denote the set of Turing degrees. A Turing cone with base $C \in \mathcal{D}$ is the set $\{D \in \mathcal{D} : C \leq_T D\}$. Define Martin's measure $\mathcal{U}$ by: for $A \in \mathscr{P}(\mathcal{D})$, $A \in \mathcal{U}$ if and only if A contains a Turing cone.

Under AD, the Martin's measure is a countably complete ultrafilter on $\mathcal{D}$.

Definition 2.6. ($\mathsf{ZF}+\mathsf{AD}$) Let T be some set. Let $\mathcal{H}$ be a (usually proper class) function on $\mathcal{D}$ which is definable using only T and ordinals as parameters and takes each X to some transitive class. Assume that there is some (usually proper class) function $\mathcal{R}$ definable using only T and ordinals as parameters so that for each $X \in \mathcal{D}$, $\mathcal{R}(X)$ is a wellordering of $\mathcal{H}(X)$.

Let $M^T_{\mathcal{H},\mathcal{R}}$ denote the collection of OD_T functions on $\mathcal{D}$ taking each $X \in \mathcal{D}$ to an element in $\mathcal{H}(X)$. For $F, G \in M^T_{\mathcal{H},\mathcal{R}}$, let $F \sim G$ if and only if $\{X \in \mathcal{D} : F(X) = G(X)\} \in \mathcal{U}$.

Let $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$ denote the collection of equivalence classes of $M^T_{\mathcal{H},\mathcal{R}}$ under $\sim$. Define $[F]_\sim \in [G]_\sim$ if and only if $\{X \in \mathcal{D} : F(X) \in G(X)\} \in \mathcal{U}$.

Fact 2.7. ($\mathsf{ZF}+\mathsf{AD}$) *$\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$ is a T-definable class consisting of OD_T elements. Using the T-definable bijection of OD_T and ON, $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$ is isomorphic to a class inside HOD_T. $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$ is well-founded; hence, it can be considered as a transitive structure inside HOD_T.*

The Łoś's theorem holds for $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$: Suppose $F_0, ..., F_{k-1} \in M^T_{\mathcal{H},\mathcal{R}}$ and φ is a formula of $\{\dot\in\}$, then $\mathcal{M}^T_{\mathcal{H},\mathcal{R}} \models \varphi([F_0]_\sim, ..., [F_k]_\sim)$ if and only $\{X \in \mathcal{D} : \mathcal{H}(X) \models \varphi(F_0(X), ..., F_{k-1}(X))\} \in \mathcal{U}$.

For each $\alpha < \omega_1$, let $c_\alpha : \mathcal{D} \to \{\alpha\}$ be the constant function taking value α. The class $[c_\alpha]_\sim$ represents the ordinal α in $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$.

For each $r \in \mathbb{R}$ which is OD_T and belongs to $\mathcal{H}(X)$ for a cone of $X \in \mathcal{D}$, define the function $c_r : \mathcal{D} \to \{\emptyset, r\}$ by $c_r(X) = r$ if $r \in \mathcal{H}(X)$ and $c_r(X) = \emptyset$ if otherwise. Then $[c_r]_\sim$ represents r in $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$.

Proof. $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$ is a structure in OD_T since $M^T_{\mathcal{H},\mathcal{R}} \subseteq \mathrm{OD}_T$. Note the $\in$ relation of $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$ is definable from T. Using the the canonical bijection of OD_T and ON, one can transfer $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$ and its $\in$-relation onto ON. This new isomorphic structure consists entirely of ordinals and hence elements of HOD_T.

Let $F \in M^T_{\mathcal{H},\mathcal{R}}$. Suppose $[F]_\sim$ is not wellfounded. There is some set $X \subseteq \{[G]_\sim : [G]_\sim \in [F]_\sim\}$ without an $\in^{\mathcal{M}^T_{\mathcal{H},\mathcal{R}}}$-minimal element. Let $L(0)$ be the OD_T-least function G so that $[G]_\sim \in X$. Suppose $L(n)$ has been defined. Let $L(n+1)$ be the OD_T-least function G so that $[G]_\sim \in X$ and $[G]_\sim \in [L(n)]_\sim$. Let $A_n = \{x \in \mathcal{D} : L(n+1)(x) \in L(n)(x)\}$. Each $A_n \in \mathcal{U}$. Since $\mathcal{U}$ is countably complete, $\bigcap_{n\in\omega} A_n \neq \emptyset$. Let $x \in \bigcap_{n\in\omega} A_n$. Then $\langle L(n)(x) : n \in \omega\rangle$ is an $\in$-decreasing sequence in V. Contradiction. $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$ is well-founded. Using the Mostowski collapse, one may consider $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$ as a transitive structure inside of HOD_T.

The proof of Łoś's theorem is by induction on formula complexity: The result holds for the atomic formulas by definition. Assume the result holds for φ and ψ, then the result holds for $\neg\varphi$ and $\varphi \wedge \psi$ by the usual arguments. (Note the case involving $\neg$ requires that $\mathcal{U}$ is an ultrafilter.) Suppose the result has been shown for φ. If $\mathcal{M}^T_{\mathcal{H},\mathcal{R}} \models (\exists x)\varphi(x, [F_0]_\sim, ..., [F_{k-1}]_\sim)$, then there exists some $G \in M^T_{\mathcal{H},\mathcal{R}}$ so that $\mathcal{M}^T_{\mathcal{H},\mathcal{R}} \models \varphi([G]_\sim, [F_0]_\sim, ..., [F_{k-1}]_\sim)$. Using the induction hypothesis, $\{X \in \mathcal{D} : \mathcal{H}(X) \models (\exists x)\varphi(x, F_0(X), ..., F_{k-1}(X))\} \in \mathcal{U}$. Suppose $\{X \in \mathcal{D} : (\exists x)\varphi(x, F_0(X), ..., F_{k-1}(X))\} \in \mathcal{U}$. Define G on $\mathcal{D}$ by letting $G(X)$ be the $\mathcal{R}(X)$-least element z of $\mathcal{H}(X)$ such that $\mathcal{H}(X) \models \varphi(z, F_0(X), ..., F_{k-1}(X))$ if such an element exists and $\emptyset$ otherwise. G is OD_T and so belongs to $M^T_{\mathcal{H},\mathcal{R}}$. By the induction hypothesis, $\mathcal{M}^T_{\mathcal{H},\mathcal{R}} \models \varphi([G]_\sim, [F_0]_\sim, ..., [F_{k-1}]_\sim)$. Therefore, $\mathcal{M}^T_{\mathcal{H},\mathcal{R}} \models (\exists x)\varphi(x, [F_0]_\sim, ..., [F_{k-1}]_\sim)$. This completes the sketch of Łoś's theorem.

Suppose $[F]_\sim \in [c_\alpha]_\sim$. Let $A = \{X \in \mathcal{D} : F(X) \in \alpha\}$. $A \in \mathcal{U}$. Let $A_\beta = \{X \in \mathcal{D} : F(X) = \beta\}$. $\mathcal{A} = \bigcup_{\beta<\alpha} A_\beta$. Since $\mathcal{U}$ is countably complete and α is countable, there is some $\beta < \alpha$ so that $A_\beta \in \mathcal{U}$. Then $c_\beta \sim F$. This shows that $[c_\alpha]_\sim$ represents α in $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$ when $\alpha < \omega_1$. ☐

Fact 2.8. *(Woodin, [1] Theorem 3.4) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let T be a set of ordinals. A set $X \subseteq \mathbb{R}$ which is OD_T has an ∞-Borel code (S, φ) which is OD_T.*

Fact 2.9. *(Woodin, [1] Theorem 2.18) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let T be a set of ordinals. There is some set of ordinals $\mathbb{X}$ so that $\mathrm{HOD}_T = L[\mathbb{X}]$. (Note that $\mathbb{X}$ is OD_T.)*

In the case of $L(\mathbb{R})$ and $T = \emptyset$, the set $\mathbb{X}$ can be taken to be $\mathbb{P}^\omega$ which is the direct limit indexed by $n \in \omega$ of Vopěnka forcing on $\mathbb{R}^n$. This follows from Woodin's result that $L(\mathbb{R})$ is a symmetric collapse extension of its HOD. One can find an exposition of this result in [16].

Fact 2.10. *(Woodin, [1] Section 2.2) Assuming $\mathsf{ZF} + \mathsf{AD}^+$, $\prod_{X\in\mathcal{D}} \mathrm{ON}/\mathcal{U}$ is wellfounded.*

Assume AD^+, the wellfoundedness of $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$ can also be proved from Fact 2.10. For the question of Zapletal, one will need to form an ultraproduct of the form $\mathcal{M}^T_{\mathcal{H},\mathcal{R}}$ so that all the reals of HOD belong to this ultraproduct.

Fact 2.11. *Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let T be a set of ordinals. Let $\mathbb{X}$ be a set of ordinals as given by Fact 2.9, so that $\mathrm{HOD}_T = L[\mathbb{X}]$. For each $X \in \mathcal{D}$, let $\mathcal{H}(X) = \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$ and $\mathcal{R}(X)$ be the canonical wellordering of $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. Then $\mathcal{M}^{\mathbb{X}}_{\mathcal{H},\mathcal{R}}$ is wellfounded, $\mathcal{M}^{\mathbb{X}}_{\mathcal{H},\mathcal{R}} \subseteq \mathrm{HOD}_T$, and $\mathbb{R}^{\mathrm{HOD}_T} \subseteq \mathcal{M}^T_{\mathcal{R},\mathcal{H}}$.*

Proof. Note that $\mathbb{X}$ is OD_T. Observe that for all $X \in \mathcal{D}$, $\mathrm{HOD}_T = L[\mathbb{X}] \subseteq \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. So if $r \in \mathrm{HOD}_T$, then $r \in \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. The function c_r is $\mathrm{OD}_{\mathbb{X}}$ and belongs to $M^{\mathbb{X}}_{\mathcal{H},\mathcal{R}}$. This result now follows from Fact 2.7. ☐

Theorem 2.12. *Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let T be a set of ordinals. Let E be an equivalence relation which is $\Sigma^1_1(s)$ for some $s \in \mathrm{HOD}_T$ and let A be an OD_T E-class. If A has no OD_T elements, then $\mathrm{HOD}_T \models E$ is unpinned.*

Proof. For simplicity, let $T = \emptyset$. By Fact 2.9, let $\mathbb{X}$ be a set of ordinals so that $\mathrm{HOD} = L[\mathbb{X}]$. By Fact 2.8, A has an ∞-Borel code in $\mathrm{HOD} = L[\mathbb{X}]$. Modifying $\mathbb{X}$ by including an ordinal if necessary, one may as well assume that there is some formula φ so that $(\mathbb{X}, \varphi)$ forms an ∞-Borel code for A.

Recall that E is $\Sigma^1_1(s)$ means there is some s-recursive tree T on $\omega \times \omega \times \omega$ so that $x \mathrel{E} y$ if and only if $L[s, x, y] \models T^{x,y}$ is illfounded, where $T^{x,y} = \{u : (x \upharpoonright |u|, y \upharpoonright |u|, u) \in T\}$. In this way, E is ∞-Borel with a code that is a subset of ω.

Suppose $y \geq_T x$ for some $x \in A$. By Fact 2.1, there is some $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},y]}$-name $\tau \in \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},y]}$ and some $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},y]}$-generic over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},y]}$ filter $G \in L[\mathbb{X},y]$ so that $\tau[G] = x$ and $L[\mathbb{X},y] = \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},y]}[G]$. Since $V \models L[\mathbb{X},x] \models \varphi(\mathbb{X},x)$, $L[\mathbb{X},y] \models L[\mathbb{X},x] \models \varphi(\mathbb{X},x)$. Since $L[\mathbb{X},y] = \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},y]}[G]$, one has $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},y]}[G] \models L[\mathbb{X},x] \models \varphi(\mathbb{X},x)$. There is some $q \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},y]}$ so that $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},y]} \models q \Vdash_{\mathbb{O}_{\mathbb{X}}} L[\check{\mathbb{X}},\tau] \models \varphi(\check{\mathbb{X}},\tau)$. Let q_y and τ_y be the $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},y]}$-least such q and τ with the above properties. In order to satisfy the technical requirement of using the largest condition of the forcing in the definition of pinnedness, let $\mathbb{U}_y = \{p \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},y]} : p \leq_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},y]}} q_y\}$, $\leq_{\mathbb{U}_y} = \leq_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},y]}} \upharpoonright \mathbb{U}_y$, and $1_{\mathbb{U}_y} = q_y$. If y does not Turing compute any element of A, then one can just let $\mathbb{U}_y$ and τ_y be $\emptyset$.

If $x \equiv_T y$, $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},x]} = \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},y]}$ and their canonical global wellorderings are the same. This shows that $\mathbb{U}_x = \mathbb{U}_y$ and $\tau_x = \tau_y$. If $X \in \mathcal{D}$ and $x \in X$, let $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]} = \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},x]}$, $\mathbb{U}_X = \mathbb{U}_x$, and $\tau_X = \tau_x$. For $X \in \mathcal{D}$, let $\mathcal{H}(X) = \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$ and $R(X)$ be the canonical global wellordering of $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. For $X \in \mathcal{D}$, define $\Phi_{\mathbb{U}}(X) = \mathbb{U}_X$ and $\Phi_\tau(X) = \tau_X$. Let $\mathcal{M} = \mathcal{M}_{\mathcal{H},\mathcal{R}}^{\mathbb{X}}$. Note that $\Phi_{\mathbb{U}}, \Phi_\tau \in M_{\mathcal{H},\mathcal{R}}^{\mathbb{X}}$. Let $\mathbb{U} = [\Phi_{\mathbb{U}}]_\sim$ and $\tau = [\Phi_\tau]_\sim$. Let $c_{\mathbb{X}}$ be the constant function taking value $\mathbb{X}$. Note that $c_{\mathbb{X}} \in M_{\mathcal{H},\mathcal{R}}^{\mathbb{X}}$. Let $\mathbb{X}^\infty = [c_{\mathbb{X}}]_\sim$. As in Fact 2.7, $\mathcal{M}$ will be identified as a transitive class in HOD^V. Thus $\mathbb{U}$, τ, and $\mathbb{X}^\infty$ belong to HOD^V.

By Łoś's theorem, $\mathcal{M}$ is a model of ZFC, $\mathbb{U}$ is some forcing, τ is some $\mathbb{U}$-name adding a real, $\mathbb{X}^\infty$ is a set of ordinals, and $\mathcal{M} \models 1_{\mathbb{U}} \Vdash_{\mathbb{U}} L[\mathbb{X}^\infty,\tau] \models \varphi(\mathbb{X}^\infty,\tau)$.

Claim 1:

$$\mathcal{M} \models 1_{\mathbb{U}\times\mathbb{U}} \Vdash_{\mathbb{U}\times\mathbb{U}} (\forall x)(\forall y)((L[\mathbb{X}^\infty,x] \models \varphi(\mathbb{X}^\infty,x) \wedge L[\mathbb{X}^\infty,y] \models \varphi(\mathbb{X}^\infty,y)) \Rightarrow x\ E\ y)$$

(Note that the ultraproduct moves $\mathbb{X}$ to $\mathbb{X}^\infty$. However, E as a $\Sigma_1^1(s)$ equivalence relation has the real s as its ∞-Borel code. The constant function c_s taking value s belongs to $M_{\mathcal{H},\mathcal{R}}^{\mathbb{X}}$. In $\mathcal{M}$, $[c_s]_\sim$ represents s. That is, s is not moved by the ultraproduct. Hence it is appropriate to continue to denote E by E in $\mathcal{M}$ as it is still the same $\mathbf{\Sigma}_1^1$ equivalence relation.)

To see the claim: Fix some $z \in A$. By Łoś's theorem, it suffices to prove that for all $r \geq_T z$:

$$\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},r]} \models 1_{\mathbb{U}_r\times\mathbb{U}_r} \Vdash_{\mathbb{U}_r\times\mathbb{U}_r} (\forall x)(\forall y)((L[\mathbb{X},x] \models \varphi(\mathbb{X},x) \wedge L[\mathbb{X},y] \models \varphi(\mathbb{X},y)) \Rightarrow x\ E\ y)$$

Fix some $(p,q) \in \mathbb{U}_r \times \mathbb{U}_r$. Since $L[\mathbb{X},r] \models \mathsf{AC}$ and $V \models \mathsf{AD}$, ω_1^V is inaccessible in $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},r]}$. Hence $\mathbb{U}_r \times \mathbb{U}_r$ and its power set in $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},r]}$ are countable in V. There exists $G \times H \in V$ containing (p,q) which is $\mathbb{U}_r \times \mathbb{U}_r$-generic over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},r]}$. Since $G \times H \in V$, all sets of $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},r]}[G\times H]$ belong to V. Let x and y be reals of $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},r]}[G\times H]$ so that $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},r]}[G\times H] \models L[\mathbb{X},x] \models \varphi(\mathbb{X},x) \wedge L[\mathbb{X},y] \models \varphi(\mathbb{X},y)$. Then $V \models L[\mathbb{X},x] \models \varphi(\mathbb{X},x) \wedge L[\mathbb{X},y] \models \varphi(\mathbb{X},y)$. Since $(\mathbb{X},\varphi)$ is an ∞-Borel code for A in V, $x \in A$ and $y \in A$. Since A is an E-class, $x\ E\ y$. By Mostowski absoluteness, $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},r]}[G\times H] \models x\ E\ y$. This shows that $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},r]}[G\times H]$ satisfies the formula behind the above forcing relation. Since $G \times H$ is generic, there is some $(p',q') \leq_{\mathbb{U}_r\times\mathbb{U}_r} (p,q)$ so that in $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},r]}$, (p',q') forces that formula. Since (p,q) was arbitrary, this establishes the claim.

Claim 2:

$$\mathcal{M} \models 1_{\mathbb{U}\times\mathbb{U}} \Vdash_{\mathbb{U}\times\mathbb{U}} (\forall x)(\forall y)((L[\mathbb{X}^\infty,x] \models \varphi(\mathbb{X}^\infty,x) \wedge x\ E\ y) \Rightarrow L[\mathbb{X}^\infty,y] \models \varphi(\mathbb{X}^\infty,y))$$

The proof essentially uses the same idea as Claim 1.

Now to show that $\mathbb{U}$ and τ witness that E is unpinned in HOD^V:

First to show that τ is an E-pinned name in HOD^V: Let $G \times H$ be any $\mathbb{U}\times\mathbb{U}$-generic filter over HOD^V. Since $\mathcal{M} \subseteq \mathrm{HOD}^V$, if G and H are generic over HOD^V, then G and H are generic over $\mathcal{M}$. By the forcing theorem, $\mathcal{M}[G] \models L[\mathbb{X}^\infty,\tau[G]] \models \varphi(\mathbb{X}^\infty,\tau[G])$ and $\mathcal{M}[H] \models L[\mathbb{X}^\infty,\tau[H]] \models \varphi(\mathbb{X}^\infty,\tau[H])$. By Claim 1, $\mathcal{M}[G\times H] \models \tau[G]\ E\ \tau[H]$. By Mostowski absoluteness, $\mathrm{HOD}^V[G\times H] \models \tau[G]\ E\ \tau[H]$. Since $G \times H$ was arbitrary, $\mathrm{HOD}^V \models 1_{\mathbb{U}\times\mathbb{U}} \Vdash_{\mathbb{U}\times\mathbb{U}} \tau_{\text{left}}\ E\ \tau_{\text{right}}$. This shows that τ is an E-pinned $\mathbb{U}$-name in HOD^V.

Finally, to show that τ is not E-trivial: Suppose there is some $x \in \mathrm{HOD}^V$ so that $\mathrm{HOD}^V \models 1_{\mathbb{U}} \Vdash_{\mathbb{U}} \tau\ E\ \check{x}$. Let $G \subseteq \mathbb{U}$ be a $\mathbb{U}$-generic over HOD^V filter. Then $\mathrm{HOD}^V[G] \models \tau[G]\ E\ x$. By Mostowski absoluteness, $\mathcal{M}[G] \models \tau[G]\ E\ x$. G is also generic over $\mathcal{M}$. By the forcing theorem, $\mathcal{M}[G] \models L[\mathbb{X}^\infty,\tau[G]] \models \varphi(\mathbb{X}^\infty,\tau[G])$. Since $x \in \mathrm{HOD}^V$, Fact 2.11 and Fact 2.7 imply that $[c_x]_\sim$ represents x in $\mathcal{M}$. By Claim 2 applied in $\mathcal{M}[G\times H]$ where H is any $\mathbb{U}$-generic filter over $\mathcal{M}[G]$, $\mathcal{M}[G] \models L[\mathbb{X}^\infty,x] \models \varphi(\mathbb{X}^\infty,x)$. Thus $\mathcal{M} \models L[[c_{\mathbb{X}}]_\sim,[c_x]_\sim] \models$

$\varphi([c_{\mathbb{X}}]_\sim, [c_x]_\sim)$. By Łoś's theorem, for a Turing cone of X's (such that $x \in \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$), $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]} \models L[\mathbb{X},x] \models \varphi(\mathbb{X},x)$. This implies $V \models L[\mathbb{X},x] \models \varphi(\mathbb{X},x)$. $V \models x \in A$ since $(\mathbb{X},\varphi)$ is the ∞-Borel code for A in V. This contradicts the assumption that A has no OD elements.

This completes the proof. □

Theorem 2.13. ($\mathsf{ZF}+\mathsf{AD}$) *Let E be a $\mathbf{\Sigma}_1^1$ equivalence relation defined in HOD_R, where R is some set. Suppose $\mathrm{HOD}_R \models E$ is unpinned. Then there is an OD_R E-class with no OD_R elements.*

Proof. Since $\mathrm{HOD}_R \models E$ is unpinned, there exists some forcing $\mathbb{P} \in \mathrm{HOD}_R$ and $\mathbb{P}$-name $\sigma \in \mathrm{HOD}_R$ so that within HOD_R, $\mathbb{P}$ and σ witness that E is not pinned.

Inside HOD_R (which models AC), let N be an elementary substructure of some large enough rank initial segment of HOD_R with the property that (1) N contains the code for E, (2) $\mathbb{R} \subseteq N$, (3) $\mathbb{P},\sigma \in N$, and (4) N has cardinality $|\mathbb{R}|$. Let M be the Mostowski collapse of N. Let $\mathbb{Q}$ and τ be the image of $\mathbb{P}$ and σ under the Mostowski collapse map. As E is $\mathbf{\Sigma}_1^1$, the code for E is a tree on $\omega\times\omega\times\omega$ whose projection is E. So a code for E is merely a subset of ω. Hence the Mostowski collapse map does not move the code for E. Note that $|M|^V = |\mathbb{R}^{\mathrm{HOD}_R^V}|^V = \aleph_0$ since AD holds. Hence there are generics for $\mathbb{Q}$ over M that lie in V.

Suppose G and H are two generic filters for $\mathbb{Q}$ over M which belong to V. Since $M[G]$ and $M[H]$ are countable in V, one can construct a generic filter $J \in V$ so that $G\times J$ and $H \times J$ are generic filters for $\mathbb{Q}\times\mathbb{Q}$. By elementarity, $M \models \tau$ is E-pinned. Thus $M[G\times J] \models \tau[G]\ E\ \tau[J]$ and $M[H\times J] \models \tau[H]\ E\ \tau[J]$. By Mostowski absoluteness, $\tau[G]\ E\ \tau[J]$ and $\tau[H]\ E\ \tau[J]$ holds in V. Since E is an equivalence relation, $\tau[G]\ E\ \tau[H]$. This shows that whenever G and H are $\mathbb{Q}$-generic filters over M that belong to V (but may not be mutually generic), $\tau[G]\ E\ \tau[H]$.

$M \models \tau$ is not E-trivial by elementarity. Since $\mathbb{R}^{\mathrm{HOD}_R} \subseteq M$, for any $G \subseteq \mathbb{Q}$ which is $\mathbb{Q}$-generic over M and any $x \in \mathbb{R}^{\mathrm{HOD}_R}$, $M[G] \models \neg(\tau[G]\ E\ x)$. By absoluteness, if $G \in V$, then $\neg(\tau[G]\ E\ x)$.

In V, let A be the set of $x \in \mathbb{R}$ so that there exists some $G \subseteq \mathbb{Q}$ which is $\mathbb{Q}$-generic over M and $x\ E\ \tau[G]$. Since $\mathbb{Q},\tau \in M$ and $M \in \mathrm{HOD}_R$, A is OD_R. By the discussion of the above two paragraphs, A is a single E-class and has no elements of OD_R.

Note that the only consequence of AD that is used is that there is no uncountable wellordered set of reals. □

The following answers the question of Zapletal.

Corollary 2.14. *Assume $\mathsf{ZF}+\mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let E be a $\mathbf{\Sigma}_1^1$ equivalence relation coded in HOD. E has an OD E-class with no OD elements if and only if $\mathrm{HOD} \models E$ is unpinned.*

The rest of this section will give some examples.

Definition 2.15. The following are some important $\mathbf{\Delta}_1^1$ equivalence relations.

Let $=$ denote the identity equivalence relation on $\mathbb{R}$.

Let $=^+$ denote the Friedman-Stanley jump of $=$ which is defined on ${}^\omega\mathbb{R}$ by $x =^+ y$ if and only if $\{x(n) : n \in \omega\} = \{y(n) : n \in \omega\}$. ($=^+$ is equality of range.)

Let E_0 be the equivalence relation on $\mathbb{R}$ (or ${}^\omega 2$) defined by $x\ E_0\ y$ if and only if $(\exists k)(\forall n \geq k)(x(n) = y(n))$.

Let E_1 be the equivalence relation on ${}^\omega\mathbb{R}$ defined by $x\ E_1\ y$ if and only $(\exists k)(\forall n \geq k)(x(n) = y(n))$.

Let E_2 be the equivalence relation on ${}^\omega 2$ defined by $x\ E_2\ y$ if and only if $\sum\{\frac{1}{n} : n \in x \triangle y\} < \infty$, where $\triangle$ denotes the symmetric difference operation.

Fact 2.16. *The equivalence relations $=$, E_0, E_1, and E_2 are pinned $\mathbf{\Delta}_1^1$ equivalence relations. Every $\mathbf{\Delta}_1^1$ equivalence relation with countable classes is pinned. Every smooth, hyperfinite, essentially countable, or hypersmooth equivalence relation is pinned.*

The equivalence relation $=^+$ is unpinned.

Proof. See [10] Chapter 11.

The Solovay product lemma states: Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcings. Suppose $G\times H$ is $\mathbb{P}\times\mathbb{Q}$-generic over V. Then $V[G] \cap V[H] = V$.

From the Solovay product lemma, it follows that $=$, E_0, and E_1 are pinned equivalence relations.

If $E \leq_{\mathbf{\Delta}_1^1} F$ and F is pinned, then E is also pinned. This implies that smooth, hyperfinite, and hypersmooth equivalence relations are pinned.

[10] Theorem 17.1.3 (iii) states that $\mathbf{\Delta}_1^1$ equivalence relations with all classes $\mathbf{\Sigma}_3^0$ are pinned. This implies that E_2 and every $\mathbf{\Delta}_1^1$ equivalence relation with countable classes are pinned. Therefore, essentially countable equivalence relations are pinned.

Let $\mathbb{Q} = \text{Coll}(\omega, \mathbb{R})$. Let τ be the name for the generic surjection of ω onto $\mathbb{R}$. $\mathbb{Q}$ and τ witness that $=^+$ is unpinned since if τ was forced to be $=^+$ related to a ground model element, then $\mathbb{R}$ would be countable in the ground model. $\square$

Example 2.17. The proof above that $=^+$ is unpinned can be used to produce an OD $=^+$-class with no OD elements assuming $(\mathscr{P}(\mathbb{R}))^{\text{HOD}}$ is countable.

Let $\mathbb{Q} = \text{Coll}(\omega, \mathbb{R})$ and τ be the generic surjection of ω onto $\mathbb{R}$ as defined inside of HOD. (Note that τ is an $=^+$-pinned name.) By the assumption, there exists $\mathbb{Q}$-generics over HOD in V. Let A be the collection of $x \in {}^\omega\mathbb{R}$ such that there exist some $G \subseteq \mathbb{Q}$ which is $\mathbb{Q}$-generic over HOD and $x =^+ \tau[G]$. A is an OD $=^+$ equivalence class. A cannot contain any OD elements for otherwise HOD would think $\mathbb{R}^{\text{HOD}}$ is countable.

3. Equivalence Class Section Uniformization and Lifting

Theorem 3.1. *Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. *Let T be a set of ordinals. Let E be a $\mathbf{\Sigma}_1^1$ equivalence relation coded in* HOD_T. *Suppose E is pinned in* $\text{HOD}_{T,x}$ *for all $x \in \mathbb{R}$. Let $R \subseteq \mathbb{R} \times \mathbb{R}$ be* OD_T *and have the property that for all $x \in \mathbb{R}$, $R_x = \{y : R(x, y)\}$ is an E-class. Then there is a function $F : \mathbb{R} \to \mathbb{R}$ which is* OD_T *and uniformizes R: that is, for all $x \in \mathbb{R}$, $R(x, F(x))$.*

If E is a $\mathbf{\Sigma}_1^1$ equivalence relation which is pinned in every transitive model of ZFC *containing the real that codes E, then every relation R whose sections are E-classes can be uniformized. (For example, E could be any of the pinned equivalence relations from Fact 2.16.)*

Proof. Under these assumptions , for each $x \in \mathbb{R}$, R_x is an $\text{OD}_{T,x}$ E-class. Since $\text{HOD}_{T,x} \models E$ is unpinned, Theorem 2.12 implies that R_x must have an $\text{OD}_{T,x}$ element. For each $x \in \mathbb{R}$, let $F(x)$ be the least element of $\text{HOD}_{T,x}$ under the canonical global wellordering of $\text{HOD}_{T,x}$ which belongs to R_x. F is an OD_T uniformization of R.

For the second statement, under AD^+, any such relation R has an ∞-Borel code (S, φ). By modifying S if necessary, one may assume that HOD_S contains a code for E as a $\mathbf{\Sigma}_1^1$ set. By the hypothesis, E is pinned in every $\text{HOD}_{S,x}$, where $x \in \mathbb{R}$. The second statement follows from the first statement. $\square$

[19] has shown that if E is a $\mathbf{\Delta}_1^1$ equivalence relation coded in some transitive model M and N is some transitive model with $M \subseteq N$, then E is pinned in M if and only if E is pinned in N. Therefore, in the first statement of Theorem 3.1, it suffices just to have $\text{HOD}_T \models E$ is pinned, when E is a $\mathbf{\Delta}_1^1$ equivalence relation.

However [19] also shows that, in general, pinnedness for $\mathbf{\Sigma}_1^1$ equivalence relation is not absolute by producing a pinned $\mathbf{\Sigma}_1^1$ equivalence relation in L which is unpinned in a forcing extension of L. However, in the present situation, one is concerned with models of the form HOD_T^V and $\text{HOD}_{T,x}^V$ where V is a model of determinacy. Possible more can be said in such settings. This suggests the following question.

Question 3.2. In the first statement of Theorem 3.1, can the condition that E is pinned in $\text{HOD}_{T,x}$ for all $x \in \mathbb{R}$ be replace by just E is pinned in HOD_T when E is a $\mathbf{\Sigma}_1^1$ equivalence relation coded in HOD_T?

Regardless, most natural examples are $\mathbf{\Delta}_1^1$. Moreover, for most of the natural examples, pinnedness is provable in ZFC.

Definition 3.3. Let E be an equivalence relation on some set X. Let F be an equivalence relation on some set Y. Let $n \in \omega$. Let $f : (X/E)^n \to (Y/F)$ be some function. A function $F : X^n \to Y$ is a lift of f if and only if for all $x_0, ..., x_{n-1} \in X$, $[F(x_0, ..., x_{n-1})]_F = f([x_0]_E, ..., [x_{n-1}]_E)$.

Corollary 3.4. *Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. *Suppose E is an equivalence relation on $\mathbb{R}$. Suppose F is a $\mathbf{\Sigma}_1^1$ equivalence relation on $\mathbb{R}$ which is pinned in every transitive models of* ZFC *containing the real that codes F. For all $n \in \omega$, every function $f : (\mathbb{R}/E)^n \to (\mathbb{R}/F)$ has a lift.*

In particular, this lifting property holds when F is E_0, E_1, E_2, smooth, hyperfinite, essentially countable, or hypersmooth.

Proof. Define the relation $R(x_0, ..., x_{n-1}, y)$ if and only if $y \in f([x_0]_E, ..., [x_{n-1}]_E)$. For each $(x_0, ..., x_{n-1}) \in \mathbb{R}^n$, $R_{(x_0,...,x_{n-1})} = f([x_0]_E, ..., [x_{n-1}]_E)$, which is an F-class. By assumption, F is pinned in every model of ZFC containing the real that codes F. Theorem 3.1 implies that R has a uniformizing function G. G is a lift of f. □

Example 3.5. Under $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$, every relation can be uniformized. Hence, E-class section uniformization and lifting for E holds for every equivalence relation E on $\mathbb{R}$. However $\mathsf{ZF} + \mathsf{AD}^+$ is not able to prove E-class section uniformization when E is an unpinned equivalence relation. The following is an example.

Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$.

Define $R(x, y)$ if and only if y is not OD_x. R has no uniformizing function: Suppose $f : \mathbb{R} \to \mathbb{R}$ uniformized R. Since $V = L(\mathbb{R})$, every set of reals is ordinal definable from some real. Thus f is OD_z for some $z \in \mathbb{R}$. Hence $f(z)$ is OD_z. However, $R(z, f(z))$ implies that $f(z)$ is not OD_z. Contradiction.

Define $S(x, y)$ if and only if $\{y_n : n \in \omega\} = \mathbb{R}^{\mathrm{HOD}_x}$, where $y_n \in \mathbb{R}$ denotes the n^{th} section of y under some coding of pairs of integers by integers. If $S(x, y)$, then $y \notin \mathrm{OD}_x$ for otherwise $\mathbb{R}^{\mathrm{HOD}_x}$ would be countable in HOD_x. Since $S \subseteq R$ and R has no uniformization, S also has no uniformization.

Every instance of F-class section uniformization gives a lift of a function from $f : \mathbb{R} \to (\mathbb{R}/F)$. Therefore, failure of F-class section uniformization is a failure of lifting for F. However, the more interesting instance of the lifting property involving function of the form $f : (\mathbb{R}/F) \to (\mathbb{R}/F)$. Zapletal informed the author of an example:

Example 3.6. (Zapletal) Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$. There is a $f : ({}^\omega\mathbb{R}/ =^+) \to ({}^\omega\mathbb{R}/ =^+)$ which does not have a lift.

Define f as follows: Let $b \in {}^\omega\mathbb{R}$ be such that for all $n \in \omega$, $b(n)$ is the constant 0 function. Let $C \in ({}^\omega\mathbb{R}/ =^+)$. If there is some $x \in \mathbb{R}$ so that C is the $=^+$ equivalence class of enumerations of $[x]_T$ (the Turing degree of x), then $f(C)$ is the $=^+$ equivalence class of enumerations of $\mathbb{R}^{\mathrm{HOD}_x}$. Note that $f(C)$ does not depend on x. If C is the not the $=^+$ equivalence class of enumerations of any Turing degree, then let $f(C) = [b]_{=^+} = \{b\}$.

Now suppose that f has a lift $F : {}^\omega\mathbb{R} \to {}^\omega\mathbb{R}$. Since $V = L(\mathbb{R})$, F is OD_z for some $z \in \mathbb{R}$. Since $[z]_T \subseteq \mathrm{HOD}_z$ and HOD_z thinks $[z]_T$ is countable, there is a $c \in ({}^\omega\mathbb{R})^{\mathrm{HOD}_z}$ such that c enumerates $[z]_T$. Thus $F(c) \in \mathrm{HOD}_z$. Since F is a lift of f, $F(c) \in f([c]_{=^+})$. By definition, $F(c) \in {}^\omega\mathbb{R}$ is an enumeration of $\mathbb{R}^{\mathrm{HOD}_z}$. Then HOD_z would think its own set of reals are countable. Contradiction.

4. Jónsson Property

Definition 4.1. Let X be a set and $n \in \omega$. Let E be an equivalance relation on X. Let $[X]^n_E = \{(x_0, ..., x_{n-1}) \in {}^nX : (\forall i < n)(\forall j < n)(i \neq j \Rightarrow \neg(x_i \; E \; x_j))\}$. Let $[X]^{<\omega}_E = \bigcup_{n\in\omega}[X]^n_E$.

A set X has the Jónsson property if and only if for all functions $f : [X]^{<\omega}_= \to X$, there is some $Y \subseteq X$ with $Y \approx X$ and $f[[Y]^{<\omega}_=] \neq X$. (The symbol $\approx$ is the relation of being in bijection.)

For $n < \omega$, an n-Jónsson function for X is a map $f : [X]^n_= \to X$ so that for all $Y \subseteq X$ with $Y \approx X$, $f[[X]^n_=] = X$.

Fact 4.2. *Under* $\mathsf{ZF} + \mathsf{AD}$,

([5] *and* [6]*)* $\mathbb{R}$ *has the Jónsson property.*

([2]*) There is a* 3*-Jónsson function for* $\mathbb{R}/E_0$*. Hence* $\mathbb{R}/E_0$ *does not have the Jónsson property.*

For the rest of this section, $\mathbb{R}$ will refer to ${}^\omega 2$, the set of infinite binary sequences.

Definition 4.3. A nonempty subset p of ${}^{<\omega}2$ is a tree if and only if for all $s \in p$ and $t \subseteq s$, $t \in p$. A tree p is a perfect tree if and only if for all $s \in p$, there is a $t \supseteq s$ so that $t\hat{}0, t\hat{}1 \in p$.

Let $\mathbb{S}$ be the set of all perfect trees. Let $\leq_{\mathbb{S}} = \subseteq$.

Let $p \in \mathbb{S}$. A node $s \in p$ is a split node if and only if $s\hat{}0, s\hat{}1 \in p$. A node $s \in p$ is a split of p if and only if $s \upharpoonright (|s| - 1)$ is a split node of p. For $n \in \omega$, s is an n-split of p if and only if s is a $\subseteq$-minimal element of p with exactly n-many proper initial segments which are split nodes of p.

Let $\mathrm{split}^n(p)$ denote the set of n-splits of p. Note that $|\mathrm{split}^n(p)| = 2^n$ and $\mathrm{split}^0(p) = \{\emptyset\}$.

If $p, q \in \mathbb{S}$, define $p \leq^n_{\mathbb{S}} q$ if and only if $p \leq_{\mathbb{S}} q$ and $\mathrm{split}^n(p) = \mathrm{split}^n(q)$.

If $p \in \mathbb{S}$ and $s \in p$, then define $p_s = \{t \in p : t \subseteq s \vee s \subseteq t\}$.

Let $p \in \mathbb{S}$. Let Λ be defined as follows:

(i) $\Lambda(p, \emptyset) = \emptyset$.
(ii) Suppose $\Lambda(p, s)$ has been defined for all $s \in {}^n2$. Fix an $s \in {}^n2$ and $i \in 2$. Let $t \supseteq \Lambda(p, s)$ be the minimal split node of p extending $\Lambda(p, s)$. Let $\Lambda(p, s\hat{}i) = t\hat{}i$.

Let $\Xi(p, s) = p_{\Lambda(p,s)}$.

Fact 4.4. *A fusion sequence is a sequence $\langle p_n : n \in \omega\rangle$ in $\mathbb{S}$ so that for all $n \in \omega$, $p_{n+1} \leq^n_{\mathbb{S}} p_n$. Let $p_\omega = \bigcap_{n\in\omega} p_n$. Then $p_\omega \in \mathbb{S}$ and is called the fusion of the above fusion sequence.*

Fact 4.5. *Suppose $p \in \mathbb{S}$. Let $\langle r_n : n \in \omega\rangle$ be a sequence of positive integers. Let $\langle f_n : n \in \omega\rangle$ be a sequence such that for all $n \in \omega$, $f_n : [[p]]^{r_n}_{=} \to {}^\omega\mathbb{R}$ is a continuous function. Then there is some $q \leq_{\mathbb{S}} p$ and $z \in \mathbb{R}$ so that for all $m, n \in \omega$ and $y \in f_n[[[q]]^{r_n}_{=}]$, $z \neq y(m)$.*

Proof. Let $B : \omega \to \omega \times \omega$ be a surjection with the property that the inverse image of any (e, g) is infinite.

Objects $\langle z_n : n \in \omega\rangle$ and $\langle q_n : n \in \omega\rangle$ will be built with the following properties.
(I) For each $n \in \omega$, $z_n \in {}^{<\omega}2$ and $z_n \subsetneq z_{n+1}$. For each $n \in \omega$, $q_n \in \mathbb{S}$, $q_n \leq_{\mathbb{S}} p$, and $q_{n+1} \leq^n_{\mathbb{S}} q_n$.
(II) For each $n \in \omega$, suppose $B(n) = (e, g)$. Then for each sequence $(\sigma_1, ..., \sigma_{r_e})$ of pairwise distinct strings in ${}^n2$, there is some $\tau \in {}^{<\omega}2$ so that for all y with

$$y \in f_e[[\Xi(q_{n+1}, \sigma_1)] \times ... \times [\Xi(q_{n+1}, \sigma_{r_e})]]$$

$y(g) \in N_\tau$ and z_{n+1} and τ are incompatible.

Suppose these objects can be constructed. Then $\langle q_n : n \in \omega\rangle$ forms a fusion sequence. By Fact 4.4, $q = \bigcap_{n\in\omega} q_n$ is a perfect tree. Let $z = \bigcup_{n\in\omega} z_n$. Let $e, g \in \omega$. Suppose $(x_1, ..., x_{r_e}) \in [[q]]^{r_e}_{=}$. By the assumption on B, there is some n large enough so that $B(n) = (e, g)$ and there are pairwise distinct strings $\sigma_1, ..., \sigma_{r_e} \in {}^n2$ with $\Lambda(q, \sigma_1) \subset x_1$, ..., $\Lambda(q, \sigma_{r_e}) \subset x_{r_e}$. Then by (II), z_{n+1} is not an initial segment of $y(g)$. Hence $y(g) \neq z$.

It remains to construct these objects.

Let $z_0 = \emptyset$ and $q_0 = p$.

Suppose q_n and z_n have been constructed. Suppose that $B(n) = (e, g)$. Enumerate all the r_e-tuples of distincts strings in ${}^n2$ as $(\sigma^0_1, ..., \sigma^0_{r_e})$, ..., $(\sigma^M_1, ..., \sigma^M_{r_e})$ for some $M \in \omega$.

Let $s_0 = q_n$. Let $\ell_0 = z_n$. Suppose s_k and ℓ_k have been defined for some fixed $k \leq M$. For each $1 \leq i \leq r_e$, let $c_i = \sigma^k_i\hat{}\bar{0}$. Let $d_i = \bigcup_{n<\omega} \Lambda(r_k, c_i \upharpoonright n)$. By the continuity of f_e on $[[p]]^{r_e}_{=}$, there is some $N > n$ so that for all

$$y \in f_e[[\Xi(s_k, c_1 \upharpoonright N)] \times ... \times [\Xi(s_k, c_{r_e} \upharpoonright N)]],$$

$f_e(d_1, ..., d_{r_e})(g) \upharpoonright |\ell_k + 1| \subseteq y(g)$. Define $\ell_{k+1} = \ell_k\hat{}(1 - f_e(d_1, ..., d_{r_e})(g)(|\ell_k|))$, that is ℓ_{k+1} extends ℓ_k by one using the opposite of the value of $f_e(d_1, ..., d_{r_e})(g)(|\ell_k|)$. Let $s_{k+1} \leq^n_{\mathbb{S}} s_k$ be such that for all $\sigma \in {}^n2$, if $\sigma = \sigma^k_i$ for some $1 \leq i \leq r_e$, then $\Xi(s_{k+1}, \sigma) = \Xi(s_k, c_i \upharpoonright N)$ and if σ is otherwise, then $\Xi(s_{k+1}, \sigma) = \Xi(s_k, \sigma)$.

Finally, let $q_{n+1} = s_{M+1}$ and $z_{n+1} = \ell_{M+1}$. This completes the construction. □

Fact 4.6. *Let δ be an ordinal. Let $\langle A_\alpha : \alpha < \delta\rangle$ be a sequence of meager subsets of $\mathbb{R}$. Define a prewellordering on $\bigcup_{\alpha<\delta} A_\alpha$ by $x \preceq y$ if and only if the least ordinal ξ such that $x \in A_\xi$ is less or equal to the least ordinal ξ such that $y \in A_\xi$. Assume that $\preceq$ as a subset of $\mathbb{R} \times \mathbb{R}$ has the Baire property. Then $\bigcup_{\alpha<\delta} A_\alpha$ is meager.*

($\mathsf{ZF} + \mathsf{AD}$) *Every wellordered union of meager sets is meager.*

Proof. See [13]. The second statement follows from the fact that every subset of $\mathbb{R}\times\mathbb{R}$ has the Baire property under AD. □

Fact 4.7. *(Mycielski) Suppose $\langle C_n : n \in \omega\rangle$ is a sequence so that each C_n is a comeager subset of $\mathbb{R}^n$. Then there is a perfect tree p so that for all $n \in \omega$, $[[p]]^n_{=} \subseteq C_n$.*

Fact 4.8. ($\mathsf{ZF} + \mathsf{AD}$) *(Comeager uniformization) Let $R \subseteq \mathbb{R} \times \mathbb{R}$ be a relation. Then there is a comeager set $C \subseteq \mathbb{R}$ and a function $f : C \to \mathbb{R}$ so that for all $x \in C$, $R(x, f(x))$.*

Fact 4.9. ($\mathsf{ZF} + \mathsf{AD}$) *Let E be an equivalence relation on $\mathbb{R}$ with all classes countable and $\mathbb{R}/E \approx \mathbb{R}$. Let p be perfect tree. Then $[p]/E \approx \mathbb{R}$.*

Proof. Note that $[p]/E$ injects into $\mathbb{R}/E$ by inclusion. Composing with the bijection then shows that $[p]/E$ injects into $\mathbb{R}$. Let $\Phi : \mathbb{R}/E \to \mathbb{R}$ be a bijection. Since E has only countable classes and countable unions of countable sets are countable under AD, $[p]/E$ is an uncountable set. Hence $\Phi[[p]/E]$ is an uncountable subset of $\mathbb{R}$. By the perfect set property, there is some perfect tree q so that $[q] \subseteq \Phi[[p]/E]$. Φ^{-1} injects $[q]$ into $[p]/E$. Hence $\mathbb{R}$ injects into $[p]/E$. By Cantor-Schröder-Bernstein, $[p]/E \approx \mathbb{R}$. □

Fact 4.10. $(\mathsf{ZF} + \mathsf{AD})$ *Let $A \subseteq \mathbb{R}$. Let $\preceq$ be a prewellordering on A. For each $x \in A$, let $[x]_\preceq = \{y : x \preceq y \wedge y \preceq x\}$. If for all $x \in A$, $[x]_\preceq$ is countable, then A is countable.*

Proof. If A is not countable, then by the perfect set property, there is some perfect tree p so that $[p] \subseteq A$. Using the notation from Definition 4.3, define $x \sqsubseteq y$ if and only if $\bigcup_{n\in\omega} \Lambda(p, x \restriction n) \preceq \bigcup_{n\in\omega} \Lambda(p, y \restriction n)$. Then $\sqsubseteq$ is a prewellordering on $\mathbb{R}$ so that for each $x \in \mathbb{R}$, $[x]_\sqsubseteq$ is countable. Let β be the length of $\sqsubseteq$. For each $\alpha < \beta$, let A_α be the prewellordering class of $\sqsubseteq$ with rank α. $\bigcup_{\alpha<\beta} A_\alpha = \mathbb{R}$ and each A_α is countable (and hence meager). This is not possible by Fact 4.6. □

Question 4.11. (Holshouser-Jackson) $(\mathsf{ZF} + \mathsf{AD})$ Let κ be an ordinal. Let $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of equivalence relations on $\mathbb{R}$ with all classes countable so that $\mathbb{R}/E_\alpha \approx \mathbb{R}$. Does the disjoint union $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ have the Jónsson property?

Note that one is not given a sequence of bijections $\langle \Phi_\alpha : \alpha < \kappa\rangle$ witnessing $\mathbb{R}/E_\alpha \approx \mathbb{R}$. With such a sequence of bijections, one can construct a bijection witnessing $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha \approx \mathbb{R} \times \kappa$. In this case, Theorem 4.15 below would imply $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ has the Jónsson property. The following is an interesting question.

Question 4.12. (Holshouser-Jackson) $(\mathsf{ZF} + \mathsf{AD})$ Let κ be an ordinal. Let $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of equivalence relations on $\mathbb{R}$ with all classes countable so that $\mathbb{R}/E_\alpha \approx \mathbb{R}$. Is $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha \approx \mathbb{R} \times \kappa$?

The following theorem gives some information concerning the Jónsson property.

Theorem 4.13. $(\mathsf{ZF} + \mathsf{AD})$ *Let κ be an ordinal. Let $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of equivalence relation on $\mathbb{R}$ with all classes countable. Let $f : [\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha]^{\leq\omega}_= \to \bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$. Then there is some perfect tree p so that $f[[\bigsqcup_{\alpha<\kappa}[p]/E_\alpha]^{\leq\omega}_=] \neq \bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$.*

Proof. Let E be the equivalence relation on $\mathbb{R}\times\kappa$ defined by: $(x, \alpha)\ E\ (y, \beta)$ if and only if $\alpha = \beta$ and $x\ E_\alpha\ y$. Then $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ is in bijection with the quotient $(\mathbb{R} \times \kappa)/E$. In the following f will be considered as a function taking values in $(\mathbb{R} \times \kappa)/E$.

Let X be the collection of surjections $\sigma : \{1, ..., n\} \to \{1, ..., m\}$ where $1 \leq m \leq n$ are integers. For all $\sigma \in X$, let $n(\sigma) = n$ and $m(\sigma) = m$, i.e. $n(\sigma)$ and $m(\sigma)$ indicate the domain and range of σ, respectively.

For each $\sigma \in X$, define $A^\sigma \subseteq \mathbb{R}^{m(\sigma)} \times \mathbb{R}$ by

$$(x_1, ..., x_{m(\sigma)}, y) \in A^\sigma \Leftrightarrow (\exists\alpha_1)...(\exists\alpha_{n(\sigma)})(\exists\beta)([(y, \beta)]_E = f([(x_{\sigma(1)}, \alpha_1)]_E, ..., [(x_{\sigma(n(\sigma))}, \alpha_{n(\sigma)})]_E))$$

In the following, fix a wellordering of $n(\sigma)$-tuples of ordinals. For each $(x_1, ..., x_{m(\sigma)})$, the elements of $A^\sigma_{(x_1,...,x_{m(\sigma)})}$ can be prewellordered as follows: $y_0 \sqsubseteq y_1$ if and only if the least $(\alpha_1, ..., \alpha_{n(\sigma)})$ such that there exists (a unique) β with

$$(y_1, \beta) \in f([(x_{\sigma(1)}, \alpha_1)]_E, ..., [(x_{\sigma(n(\sigma))}, \alpha_{n(\sigma)})]_E)$$

is less than or equal to the least $(\alpha_1, ..., \alpha_{n(\sigma)})$ such that there exists (a unique) β with

$$(y_2, \beta) \in f([(x_{\sigma(1)}, \alpha_1)]_E, ..., [(x_{\sigma(n(\sigma))}, \alpha_{n(\sigma)})]_E).$$

Let $y \in A^\sigma_{(x_1,...,x_{m(\sigma)})}$. Let $(\alpha_1, ..., \alpha_{n(\sigma)})$ be the least $n(\sigma)$-tuple of ordinals such that for some (unique) β, $(y, \beta) \in f([(x_{\sigma(1)}, \alpha_1)]_E, ..., [(x_{\sigma(n(\sigma))}, \alpha_{n(\sigma)})]_E)$. Then $[y]_\sqsubseteq \subseteq \pi_1[f([(x_{\sigma(1)}, \alpha_1)]_E, ..., [(x_{\sigma(n(\sigma))}, \alpha_{n(\sigma)})]_E)]$, where $\pi_1 : \mathbb{R} \times \kappa \to \mathbb{R}$ is the projection onto the first coordinate. Note $f([(x_{\sigma(1)}, \alpha_1), ..., [(x_{\sigma(n(\sigma))}, \alpha_{n(\sigma)})]_E)$ is contained inside of $\mathbb{R} \times \{\beta\}$ for some β. Since E_β is an equivalence relation with countable classes, $\pi_1[f([(x_{\sigma(1)}, \alpha_1), ..., [(x_{\sigma(n(\sigma))}, \alpha_{n(\sigma)})]_E)]$ is countable. It has been shown that each $\sqsubseteq$-prewellordering class is countable. By Fact 4.10, $A^\sigma_{(x_1,...,x_{m(\sigma)})}$ is countable for all $(x_1, ..., x_{m(\sigma)})$.

Fix $\sigma \in X$. Let $R^\sigma \subseteq \mathbb{R}^{m(\sigma)} \times {}^\omega\mathbb{R}$ be defined by $(x, h) \in R^\sigma$ if and only if $h : \omega \to A^\sigma_x$ is a surjection. For all $x \in \mathbb{R}^{\sigma(m)}$, $R^\sigma_x \neq \emptyset$ since A^σ_x is countable. By comeager uniformization (Fact 4.8), there is some comeager set $C^\sigma \subseteq \mathbb{R}^{m(\sigma)}$ and some function $H^\sigma : C^\sigma \to {}^\omega\mathbb{R}$ so that $R^\sigma(x, H^\sigma(x))$ for all $x \in C^\sigma$. Using AD, one may assume that H^σ is continuous on C^σ by choosing a smaller comeager set if necessary.

By the results of Mycielski, Fact 4.7, there is some perfect tree p so that $[[p]]^{m(\sigma)}_= \subseteq C^\sigma$ for all $\sigma \in X$. Note that for all $\sigma \in X$, $H^\sigma \restriction [[p]]^{m(\sigma)}_=$ is a continuous function with the property that for all $x \in [[p]]^{m(\sigma)}_=$,

$H^\sigma(x) \in {}^\omega\mathbb{R}$ enumerates A^σ_x. By Fact 4.5, there is some $q \leq_{\mathbb{S}} p$ and some $z \in \mathbb{R}$ so that for all $\sigma \in X$, $j \in \omega$, and $(x_1, ..., x_{m(\sigma)}) \in [[q]]^{m(\sigma)}_=$, $H^\sigma(x)(j) \neq z$.

Now suppose $(r_1, \alpha_1), ..., (r_n, \alpha_n) \in [q] \times \kappa$ are such that $([(r_1, \alpha_1)]_E, ..., [(r_n, \alpha_n)]_E) \in [\bigsqcup_{\alpha<\kappa}[q]/E_\alpha]^n_=$. There is some $m \leq n$, $(x_1, ..., x_m) \in [[q]]^m_=$, and surjection $\sigma : \{1, ..., n\} \to \{1, ..., m\}$ so that $(r_1, ..., r_n) = (x_{\sigma(1)}, ..., x_{\sigma(n)})$. Then $z \notin A^\sigma_{(x_1,...,x_m)}$ implies that $(z, \beta) \notin f([(r_1, \alpha_1)]_E, ..., [(r_n, \alpha_n)]_E)$ for all $\beta < \kappa$.

This shows that $f[[\bigsqcup_{\alpha<\kappa}[q]/E_\alpha]^{\leq\omega}_=] \neq \bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$. ☐

Let p be the perfect tree given by Theorem 4.13. Assume that each $\mathbb{R}/E_\alpha \approx \mathbb{R}$. By Fact 4.9, each $[p]/E_\alpha \cong \mathbb{R}$. If $\bigsqcup_{\alpha<\kappa}[p]/E_\alpha \approx \bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$, then Theorem 4.13 would imply $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ has the Jónsson property. This suggests the following natural question.

Question 4.14. ($\mathsf{ZF} + \mathsf{AD}$) Let κ be an ordinal. Let $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of equivalence relations on $\mathbb{R}$ with all classes countable and $\mathbb{R}/E_\alpha \approx \mathbb{R}$ for each $\alpha < \kappa$. Let p be a perfect tree. Is $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha \approx \bigsqcup_{\alpha<\kappa}[p]/E_\alpha$?

When all the E_α's are the identity equivalence relation, $=$, then one can exhibit the desired bijection. This gives the following result.

Theorem 4.15. ($\mathsf{ZF} + \mathsf{AD}$) *For any ordinal κ, $\mathbb{R} \times \kappa$ has the Jónsson property.*

Proof. Let $\langle E_\alpha : \alpha < \omega\rangle$ be a sequence where each E_α is the identity equivalence relation, $=$, on $\mathbb{R}$. Note that $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha \approx \mathbb{R} \times \kappa$. Apply Theorem 4.13 to this sequence. For any perfect tree p, $\bigsqcup_{\alpha<\kappa}[p]/E_\alpha \approx \bigsqcup_{\alpha<\kappa}[p] \approx \mathbb{R} \times \kappa$. ☐

Many of the results above are trivial if the sequence $\langle E_\alpha : \alpha < \kappa\rangle$ is accompanied by a sequence $\langle \Phi_\alpha : \alpha < \kappa\rangle$ where each $\Phi_\alpha : \mathbb{R}/E_\alpha \to \mathbb{R}$ is a bijection. A natural question would be to construct an example $\langle E_\alpha : \alpha < \kappa\rangle$ such that for each $\alpha < \kappa$, $\mathbb{R}/E_\alpha \approx \mathbb{R}$ but there does not exists a sequence $\langle \Phi_\alpha : \alpha < \kappa\rangle$ which uniformly witnesses these bijections exist. Also, is the condition that each E_α be an equivalence relation with all classes countable necessary in Question 4.12 and 4.14? The following example of Holshouser-Jackson answers these questions.

Example 4.16. Fix some recursive coding of binary relations on ω by reals. Let WO denote the collection of reals that code wellorderings on ω. For $\alpha < \omega_1$, let WO_α denote the reals coding wellorderings of ordertype α. For $\alpha < \omega_1$, let E_α be the equivalence relation on $\mathbb{R}$ defined by $x \; E_\alpha \; y$ if and only if $(x = y) \vee (x \notin \mathrm{WO}_\alpha \wedge y \notin \mathrm{WO}_\alpha)$. For each $\alpha < \omega_1$, E_α is $\mathbf{\Delta}^1_1$ bireducible to $=$. Hence $\mathbb{R} \approx \mathbb{R}/E_\alpha$.

For each $\alpha < \omega_1$, if $x \in \mathrm{WO}_\alpha$, identify $[x]_{E_\alpha} = \{x\}$ with x. For each $\alpha < \omega_1$, $\mathbb{R} \setminus \mathrm{WO}_\alpha$ is a single E_α equivalence class. Identify it with α. Under this identification, one has a bijection of $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ with $\mathrm{WO} \sqcup \omega_1 \approx \mathbb{R} \sqcup \omega_1$.

$\mathbb{R} \sqcup \omega_1$ is not in bijection with $\mathbb{R} \times \omega_1$: Suppose $\Phi : \mathbb{R} \sqcup \omega_1 \to \mathbb{R} \times \omega_1$ is a bijection. $\pi_1[\Phi[\omega_1]]$ can be wellordered using Φ and the wellordering on ω_1. ($\pi_1 : \mathbb{R} \times \omega_1 \to \mathbb{R}$ is the projection onto the first coordinate.) Under AD, there is no uncountable sequence of distinct reals; hence, $\pi_1[\Phi[\omega_1]]$ is countable. Let $r \in \mathbb{R}$ such that $r \notin \pi_1[\Phi[\omega_1]]$. $\Phi^{-1}[\{r\} \times \omega_1] \subseteq \mathbb{R}$. But $\Phi^{-1}[\{r\} \times \omega_1]$ can be wellordered. This would give an uncountable sequence of distinct reals in $\mathbb{R}$. Contradiction.

As mentioned above, if $\langle E_\alpha : \alpha < \omega_1\rangle$ was accompanied by a sequence of bijections $\langle \Phi_\alpha : \alpha < \omega_1\rangle$, then one can construction a bijection between $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ and $\mathbb{R} \times \omega_1$. Thus, there cannot be such a sequence of bijections under AD.

Note that E_α has exactly one uncountable class. This example shows Question 4.12 has a negative answer without the condition that each E_α has all countable classes.

Let p be a perfect tree such that $[p] \subseteq \mathbb{R} \setminus \mathrm{WO}$. Then $\bigsqcup_{\alpha<\omega_1}[p]/E_\alpha \approx \omega_1$. ω_1 is not in bijection with $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha \approx \mathbb{R} \sqcup \omega_1$. Hence Question 4.14 has a negative answer if all the equivalence relations do not have all classes countable.

If all the equivalence relations in $\langle E_\alpha : \alpha < \kappa\rangle$ have all classes countable and $\mathbb{R} \approx \mathbb{R}/E_\alpha$, then $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ contains a subset which is in bijection with $\mathbb{R} \sqcup \omega_1$ but itself is not in bijection with $\mathbb{R} \sqcup \omega_1$.

Fact 4.17. ($\mathsf{ZF} + \mathsf{AD}$) *Let κ be an uncountable ordinal. Let $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of equivalence relations on $\mathbb{R}$ so that for each $\alpha < \kappa$, E_α has all classes countable and $\mathbb{R} \approx \mathbb{R}/E_\alpha$. Then $\mathbb{R} \sqcup \kappa$ injects into $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$, but $\mathbb{R} \sqcup \kappa$ is not in bijection with $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$.*

Proof. Let $\bar{0} : \omega \to \{0\}$, be the constant 0 function. For each $\alpha < \kappa$, identify $[\bar{0}]_{E_\alpha}$ with α. Let $\Phi : \mathbb{R} \to (\mathbb{R}/E_0) \setminus [\bar{0}]_{E_0}$ be a bijection. Identify $\Phi(r)$ with r. Using this identification, there is a subset of $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ which is in bijection with $\mathbb{R} \sqcup \kappa$.

Suppose there is a bijection $\Phi : \mathbb{R} \sqcup \kappa \to \bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$. $\bigcup_{\alpha<\kappa} \Phi(\alpha)$ can be prewellordered by $x \sqsubseteq y$ if and only if the least α such that $x \in \Phi(\alpha)$ is less than or equal to the least α such that $y \in \Phi(\alpha)$. Each $\sqsubseteq$-class is countable. Fact 4.10 implies that $\bigcup_{\alpha<\kappa} \Phi(\alpha)$ is countable. Let $r \in \mathbb{R}$ with $r \notin \bigcup_{\alpha<\kappa} \Phi(\alpha)$. Let $X = \{[r]_{E_\alpha} : \alpha < \kappa\}$. $\Phi^{-1}[X]$ is an uncountable sequence of distinct reals in $\mathbb{R}$. Contradiction. $\square$

[17] Theorem 2 shows that under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R}$, the only uncountable cardinals below $\mathbb{R} \times \omega_1$ are ω_1, $\mathbb{R}$, $\mathbb{R} \sqcup \omega_1$, and $\mathbb{R} \times \omega_1$. Thus under these assumptions, if $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ is not in bijection with $\mathbb{R} \times \omega_1$, then $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ cannot inject into $\mathbb{R} \times \omega_1$. Moreover, [17] Theorem 2 shows that under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R}$, the only uncountable cardinals below $[\omega_1]^\omega$ are ω_1, $\mathbb{R}$, $\mathbb{R} \sqcup \omega_1$, $\mathbb{R} \times \omega_1$, and $[\omega_1]^\omega$. An interesting question would be to compare the cardinality of $[\omega_1]^\omega$ and $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ when each E_α is an equivalence relation with all classes countable.

Fact 4.18. ($\mathsf{ZF} + \mathsf{AD}$) *Let $\langle E_\alpha : \alpha < \omega_1\rangle$ be a sequence of equivalence relations on $\mathbb{R}$ such that each E_α has all classes $\mathbf{\Pi}^0_1$. There is no injection of $[\omega_1]^\omega$ into $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$.*

Proof. Recall that $\mathcal{U}$ is Martin's cone measure on $\mathcal{D}$, the set of Turing degrees. For each $x \in \mathcal{D}$, let $\Lambda(x)$ denote the collection of countable x-admissible ordinals. For each $x \in \mathcal{D}$, let $\Gamma(x) \in [\omega_1]^\omega$ be the increasing sequence of the first ω-many x-admissible ordinals.

Suppose $\Phi : [\omega_1]^\omega \to \bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ is an injection.

A sequence of Turing degrees $(x_n : n \in \omega)$ and a sequence $(\sigma_n : n \in \omega)$ in ${}^{<\omega}2$ will be constructed by recursion with the property that for all $n \in \omega$, $|\sigma_n| = n$, $\sigma_n \subset \sigma_{n+1}$, and whenever $f \in [\Lambda(x_n)]^\omega$, there is some $r \in \Phi(f)$ so that $\sigma_n \subset r$.

Let $x_0 = [\bar{0}]_T$, where $\bar{0}$ is the constant 0 function. Let $\sigma_0 = \emptyset$.

Suppose x_n and σ_n have been defined with the desired properties. Let $E_0^{n+1} = \{x \in \mathcal{D} : (\exists r \in \Phi(\Gamma(x)))(\sigma_n\hat{\ }0 \subseteq r)\}$ and $E_1^{n+1} = \{x \in \mathcal{D} : (\exists r \in \Phi(\Gamma(x)))(\sigma_n\hat{\ }1 \subseteq r)\}$. Note that the cone above x_n is contained in $E_0^{n+1} \cup E_1^{n+1}$. Since $\mathcal{U}$ is an ultrafilter, there is some $i \in 2$ and $x_{n+1} \geq_T x_n$ so that E_i^{n+1} contains the cone above x_{n+1}. Let $\sigma_{n+1} = \sigma_n\hat{\ }i$, for this $i \in 2$.

Let $f \in [\Lambda(x_{n+1})]^\omega$. A result of Jensen ([9]) shows that for every increasing ω-sequence of x_{n+1}-admissible ordinals f, there is some $y \geq_T x_{n+1}$ so that $\Gamma(y) = f$. Then $y \in E_i^{n+1}$. Hence there is some $r \in \Phi(\Gamma(y)) = \Phi(f)$ so that $\sigma_{n+1} \subseteq r$.

Let $r = \bigcup_{n\in\omega} \sigma_n$. Let z be the join $\bigoplus_{n\in\omega} x_n$. Suppose $f \in [\Lambda(z)]^\omega$. For all $n \in \omega$, there is some $r_n^f \in \Phi(f)$ so that $\sigma_n \subseteq r_n^f$. $\Phi(f)$ is an E_α class for some $\alpha < \omega$ so $\Phi(f)$ is $\mathbf{\Pi}^0_1$. Since r is the limit of $\{r_n^f : n \in \omega\} \subseteq \Phi(f)$, $r \in \Phi(f)$. It has been shown that for all $f \in [\Lambda(z)]^\omega$, $\Phi(f) \in \{[r]_{E_\alpha} : \alpha < \omega_1\} \approx \omega_1$. Then Φ induces an injection of $[\Lambda(z)]^\omega$ into ω_1. This is impossible since such an injection would yield a wellordering of $\mathbb{R}$ since $\mathbb{R}$ injects into $[\Lambda(z)]^\omega$. $\square$

The above argument incorporates Martin's proof of the partition relation $\omega_1 \to (\omega_1)^\omega_2$. The following result captures the essential idea of the above argument.

Fact 4.19. ($\mathsf{ZF} + \mathsf{AD}$) *Let $\kappa \in \mathrm{ON}$. Let $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of equivalence relations on $\mathbb{R}$. Let $\Phi : [\omega_1]^\omega \to \bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$. Let $R \subseteq [\omega_1]^\omega \times \mathbb{R}$ be defined by $R(f, x) \Leftrightarrow x \in \Phi(f)$. If R has a uniformizing function then Φ is not an injection.*

Proof. Let Ψ be a uniformizing function for R.

For each $n \in \omega$, let $E_i^n = \{x \in \mathcal{D} : \Psi(\Gamma(x))(n) = i\}$. Since $\mathcal{U}$ is an ultrafilter, there is some $a_n \in 2$ such that $E_{a_n}^n \in \mathcal{U}$. Let $x_n \in \mathcal{D}$ be such that the cone above x_n lies inside of $E_{a_n}^n$. Now suppose that $f \in [\Lambda(x_n)]^\omega$. A result of Jensen ([9]) states that for any such f, there is some $y \geq_T x_n$ so that $\Gamma(y) = f$. As $y \in E_{a_n}^n$, $\Psi(\Gamma(y))(n) = \Psi(f)(n) = a_n$.

Let $r \in \mathbb{R}$ be such that for all n, $r(n) = a_n$. Let $x = \bigoplus x_n$. If $f \in [\Lambda(x)]^n$, then $\Psi(f) = r$.

It has been shown that there is an uncountable set $X \subseteq \omega_1$ and some real r so that $\Psi[[X]^\omega] = \{r\}$. By definition of R, $\Phi[[X]^\omega] \subseteq \{[r]_{E_\alpha} : \alpha < \kappa\}$. The latter set is in bijection with κ. $[X]^\omega \approx [\omega_1]^\omega$. Therefore, Φ induces an injection of $[\omega_1]^\omega$ into the ordinal κ. As $\mathbb{R}$ injects into $[\omega_1]^\omega$, this would imply that one could wellorder $\mathbb{R}$. $\square$

Note that in Fact 4.19, R only needs to be uniformized on a set of cardinality $[\omega_1]^\omega$. To see this, suppose R is uniformized on $Z \subseteq [\omega_1]^\omega$ of cardinality $[\omega_1]^\omega$. Let $L : [\omega_1]^\omega \to Z$ be a bijection. Let $\Phi' = \Phi \circ L$. The relation R' associated to Φ' can be uniformized. Hence Φ' is not injective by Fact 4.19. This implies Φ is not injective.

The class of equivalence relations with $\boldsymbol{\Pi}^0_1$ classes is very restrictive. However, it does include equivalence relations with all finite classes. However, in such cases, there is a more natural argument: Fix some linear ordering $<$ of $\mathbb{R}$. For $f \in [\omega_1]^\omega$, let $L(x)$ denote the $<$-least element of $\Phi(x)$ (which exists since $\Phi(x)$ is finite). Now apply Fact 4.19.

Fact 4.20. *(With Jackson.) Assume* $\mathsf{ZF} + \mathsf{AD}^+$. *Let* $\kappa \in \mathrm{ON}$ *and* $\langle E_\alpha : \alpha < \kappa \rangle$ *be a sequence of equivalence relations on* $\mathbb{R}$ *such that each* E_α *has all classes countable. Then there is no injection* $\Phi : [\omega_1]^\omega \to \bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$.

Proof. This is proved by verifying the uniformization condition of Fact 4.19. Note that if $\langle E_\alpha : \alpha < \kappa \rangle$ is a sequence so that each E_α is an equivalence relation with all classes countable, then for any Φ, the associated relation has all countable sections.

Woodin's countable section uniformization states that every relation on $\mathbb{R} \times \mathbb{R}$ with countable section can be uniformized under AD^+. In the present situation, the relations are on $[\omega_1]^\omega \times \mathbb{R}$. Some modification of Woodin's ideas can be used to show countable section uniformization holds for such relations under AD^+. The main ideas of Woodin's countable section uniformization on $\mathbb{R}$ can be found in [1] and [14]. □

Originally, Theorem 4.13 was proved under AD^+ using Woodin's countable section uniformization. However, it was observed that for the purpose of the Jónsson property, one did not need total uniformization provided by Woodin's countable section uniformization but rather partial uniformization on a set of cardinality $\mathbb{R}$ (as provided by comeager uniformization) was adequate. As mentioned above, partial uniformization on a set of cardinality $[\omega_1]^\omega$ is adequate for the conclusion of Fact 4.19. This suggests the following:

Question 4.21. Using just AD, is it provable that for all relations $R \subseteq [\omega_1]^\omega \times \mathbb{R}$ with countable sections, there is some $Z \subseteq [\omega_1]^\omega$ and $\Phi : Z \to \mathbb{R}$ such that $|Z| = |[\omega_1]^\omega|$ and for all $z \in Z$, $R(z, \Phi(z))$?

The rest of this section will show the failure of the Jónsson property for $(\mathbb{R}/E_0) \times \kappa$ where E_0 is the equivalence relation from Definition 2.15 and $\kappa < \Theta$.

Fact 4.22. ($\mathsf{ZF} + \mathsf{AD}$) *Suppose* $A \subseteq (\mathbb{R}/E_0) \times \kappa$ *and* $A \approx \mathbb{R}/E_0$, *where* κ *is an ordinal. Let* $\pi_1 : (\mathbb{R}/E_0) \times \kappa \to \mathbb{R}/E_0$ *be the projection onto the first coordinate. Then* $\pi_1[A] \approx \mathbb{R}/E_0$.

Proof. Note that A injects into $\pi_1[A] \times \kappa$. Hence $\mathbb{R}/E_0$ injects into $\pi_1[A] \times \kappa$. Let $f : \mathbb{R}/E_0 \to \pi_1[A] \times \kappa$ denote this injection. For each $\alpha < \kappa$, let $A_\alpha = \{x \in \mathbb{R} : \pi_2(f([x]_{E_0})) = \alpha\}$, where $\pi_2 : (\mathbb{R}/E_0) \times \kappa \to \kappa$ is the projection onto the second coordinate. Then $\bigcup_{\alpha<\kappa} A_\alpha = \mathbb{R}$. By Fact 4.6, there must be some $\alpha < \kappa$ so that A_α is nonmeager. Using the Baire property, A_α is comeager in some basic open set O. (Actually since A_α is E_0-invariant, it can be shown that A_α is comeager.) Hence $A_\alpha \supseteq \bigcap_{n\in\omega} D_n$, where $\langle D_n : n \in \omega \rangle$ is a sequence of topologically dense open sets relative to O. One can build an E_0-tree inside of A_α. (See [2] Definition 5.2.) This implies that there is a continuous reduction of E_0 into $E_0 \restriction A_\alpha$. Hence $\mathbb{R}/E_0$ injects into A_α/E_0. Using f, A_α/E_0 injects into $\pi_1[A] \times \{\alpha\} \approx \pi_1[A]$. It has been shown that $\mathbb{R}/E_0$ injects into $\pi_1[A]$. Thus $\pi_1[A] \approx \mathbb{R}/E_0$. □

Fact 4.23. *Let* $\kappa < \Theta$. *There is a 6-Jónsson function for* $(\mathbb{R}/E_0) \times \kappa$.
$(\mathbb{R}/E_0) \times \kappa$ *is not Jónsson.*

Proof. By Fact 4.2, let $\Phi : [\mathbb{R}/E_0]^3_= \to \mathbb{R}/E_0$ be a 3-Jónsson map for $\mathbb{R}/E_0$. Let $\Psi : \mathbb{R} \to \kappa$ be a surjection. Since $=$ reduces into E_0, there is an injection $\Gamma : \mathbb{R} \to \mathbb{R}/E_0$. Let $\Lambda : [\mathbb{R}/E_0]^3_= \to \kappa$ be defined by

$$\Lambda(x) = \begin{cases} 0 & (\forall r \in \mathbb{R})(\Phi(x) \neq \Gamma(r)) \\ \Psi(r) & \Phi(x) = \Gamma(r) \end{cases}$$

Finally, let $\Upsilon : [(\mathbb{R}/E_0) \times \kappa]^6_= \to (\mathbb{R}/E_0) \times \kappa$ be defined by

$$((x_1,\alpha_1),(x_2,\alpha_2),(x_3,\alpha_3),(x_4,\alpha_4),(x_5,\alpha_5),(x_6,\alpha_6)) \mapsto (\Phi(x_1,x_2,x_3),\Lambda(x_4,x_5,x_6))$$

Suppose $B \subseteq (\mathbb{R}/E_0) \times \kappa$ is in bijection with $(\mathbb{R}/E_0) \times \kappa$. Let $f : (\mathbb{R}/E_0) \times \kappa \to B$ be a bijection. Let $A = f[(\mathbb{R}/E_0) \times \{0\}]$. Then $A \approx \mathbb{R}/E_0$. By Fact 4.22, $\pi_1[A] \approx \mathbb{R}/E_0$.

Supppose that $(y, \beta) \in (\mathbb{R}/E_0) \times \kappa$. Suppose $\Psi(r) = \beta$. Since Φ is a 3-Jónsson map and $\pi_1[A] \approx \mathbb{R}/E_0$, one can find $((x_1, \alpha_1), (x_2, \alpha_2), (x_3, \alpha_3), (x_4, \alpha_4), (x_5, \alpha_5), (x_6, \alpha_6)) \in [A]^6_= \subseteq [B]^6_=$ so that $\Phi(x_1, x_2, x_3) = y$ and $\Phi(x_4, x_5, x_6) = \Gamma(r)$. Then $\Upsilon((x_1, \alpha_1), (x_2, \alpha_2), (x_3, \alpha_3), (x_4, \alpha_4), (x_5, \alpha_5), (x_6, \alpha_6)) = (y, \beta)$. Υ is a 6-Jónsson function for $(\mathbb{R}/E_0) \times \kappa$. ☐

Question 4.24. [2] showed that $\mathbb{R}/E_0$ has no 2-Jónsson map but has a 3-Jónsson map. What is the least n so that $(\mathbb{R}/E_0) \times \kappa$ has a n-Jónsson map, where $\kappa < \Theta$?

If κ is any ordinal, is $(\mathbb{R}/E_0) \times \kappa$ also not Jónsson?

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