THE SIZE OF THE CLASS OF COUNTABLE SEQUENCES OF ORDINALS

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Abstract. Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. There is no injection of ${}^{<\omega_1}\omega_1$ (the set of countable length sequences of countable ordinals) into ${}^{\omega}\mathrm{ON}$ (the class of ω length sequences of ordinals). There is no injection of $[\omega_1]^{\omega_1}$ (the powerset of ω_1) into ${}^{<\omega_1}\mathrm{ON}$ (the class of countable length sequences of ordinals).

1. Introduction

Mathematical size between two sets is compared through injections and bijections. If A and B are two sets, then $|A| \leq |B|$ indicates that there is an injection from A to B. One writes $|A| < |B|$ if and only if $|A| \leq |B|$ and not $|B| \leq |A|$. One writes $|A| = |B|$ if there is a bijection between A and B. In ZF, the Cantor-Schröder-Bernstein theorem asserts that $|A| \leq |B|$ and $|B| \leq |A|$ imply that $|A| = |B|$. The axiom of choice, AC, implies every set can be wellordered and is in bijection with an ordinal and the least such one is called its cardinality. Thus under AC, the class of cardinals is wellordered under the injection relation. A frequent phenomenon is that the relation between the size of two sets with explicit definitions are often independent of $\mathsf{ZF} + \mathsf{AC}$. The continuum hypothesis is a notable example.

The axiom of determinacy, AD, asserts that all integer games between two players of a certain form have a winning strategy for one of the two players. Cardinalities of sets are no longer wellorderable under injections. However, under AD and its extensions, mathematical size of sets become more natural in that size corresponds more closely to the identity of the object or its fundamental combinatorial properties. The computation of size under determinacy involves techniques that are closely connected to definability. For instance, under AD, $|\mathbb{R}|$ and ω_1 are incomparable cardinalities. Even under AC, it seems that one cannot explicitly specify a wellordering of the reals without imposing conditions on the structure of the universe, such as the reals all belong to the constructible universe L or some other canonical inner model. Moreover, various large cardinal principles imply that wellorderings of the reals must necessarily be quite complicated. Under AD, the incomparability of $|\mathbb{R}|$ and ω_1 follows from the measurability of ω_1. Measurability of ω_1 can be proved using the Martin measure on the Turing degrees. Alternatively, Solovay showed the club measure on ω_1 is a normal measure using the $\mathbf{\Sigma}^1_1$ boundedness principle.

Let ON denote the class of ordinals. Let $\epsilon \in \mathrm{ON}$ and $X \subseteq \mathrm{ON}$ be a set or class. Let ${}^{\epsilon}X$ be the set of functions $f : \epsilon \to X$. Let ${}^{<\epsilon}X = \bigcup_{\delta<\epsilon} {}^{\delta}X$. Let $[X]^{\epsilon}$ be the set of functions $f : \epsilon \to X$ which are increasing. Let $[X]^{<\epsilon} = \bigcup_{\delta<\epsilon}[X]^{\delta}$.

Suppose κ and δ are two ordinals greater than 1. The main question in this paper is whether ${}^{<\omega_1}\kappa$ can inject into ${}^{\omega}\delta$. These two sets seem to have a fundamental difference. ${}^{<\omega_1}\kappa$ consists of sequences of arbitrary countable length and ${}^{\omega}\delta$ consists entirely of sequences of one fixed length ω. Assuming the axiom of choice, these sets may not be distinguishable through size since, for example, $|[\omega_1]^{\omega}| = |[\omega_1]^{<\omega_1}|$ under ZFC.

Observe that $|{}^{<\omega_1}2| = |[\omega_1]^{<\omega_1}| = |{}^{<\omega_1}\omega_1|$. Thus a negative answer to the above question follows from a negative answer to the following question.

Question 1.1. Is there an injection from $[\omega_1]^{<\omega_1}$ into ${}^{\omega}\mathrm{ON}$?

Another related question is that if κ and δ are two ordinals greater than 1, then does $|{}^{\omega_1}\kappa| \leq |{}^{<\omega_1}\delta|$ hold? Again these two sets have a fundamental difference: ${}^{\omega_1}\kappa$ consists of sequences of length ω_1 and ${}^{<\omega_1}\delta$ consists of countable length sequences. Observe that $|{}^{\omega_1}2| = |\mathscr{P}(\omega_1)| = |[\omega_1]^{\omega_1}|$. Thus a negative answer to the above question follows from a negative answer to the following question.

Question 1.2. Is there an injection from $[\omega_1]^{\omega_1}$ into ${}^{<\omega_1}\mathrm{ON}$?

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Woodin [17] was aware of a negative answer to a particular instance of Question 1.1 under $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_{\mathbb{R}}$, namely that $[\omega_1]^{<\omega_1}$ does not inject into $[\omega_1]^{\omega}$. Result from [3] show that Question 1.1 has a negative answer under a fragment of AD^+. These earlier results however pass through an analysis of a set S_1 which uses AD^+ techniques. This article will answer these questions using classical determinacy techniques which hold under $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$.

The theory AD^+ isolated by Woodin ([18] Section 9.1) consists of the following statements.

- $\mathsf{DC}_{\mathbb{R}}$.
- Every set of reals has an ∞-Borel code. (An ∞-Borel code is a pair (S,φ) where S is a set of ordinals and φ is a formula of set theory. Let $\mathfrak{B}_{(S,\varphi)} = \{r \in \mathbb{R} : L[S,r] \models \varphi(S,r)\}$. (S,φ) is an ∞-Borel code for a set $A \subseteq \mathbb{R}$ if and only if $A = \mathfrak{B}_{(S,r)}$.)
- Ordinal Determinacy, which is the statements that for every $\lambda < \Theta$, $X \subseteq \mathbb{R}$, and continuous function $\pi : {}^{\omega}\lambda \to \mathbb{R}$, the two player game on λ with payoff set $\pi^{-1}(X)$ is determined.

Results of Kechris and Woodin showed that if AD holds, then $L(\mathbb{R}) \models \mathsf{AD}^+$. The relation between AD and AD^+ as well as the relations between the three statements in AD^+ are not known.

Let $S_1 = \{\sigma \in [\omega_1]^{<\omega_1} : \omega_1^{L[\sigma]} = \sup(\sigma)\}$. Woodin [17] was aware that under $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}+\mathsf{DC}$, $|S_1| \leq |[\omega_1]^{<\omega_1}|$ and S_1 does not inject into $[\omega_1]^{\omega}$. Moreover, since $S_1 \subseteq [\omega_1]^{<\omega_1}$, a negative answer to Question 1.1 follows from AD^+ by the following result.

Fact 1.3. *([3]) Assume $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ and all sets of reals have an ∞-Borel code. Then there is no injection of S_1 into ${}^{\omega}\mathrm{ON}$. As a consequence, there is no injection of $[\omega_1]^{<\omega_1}$ into ${}^{\omega}\mathrm{ON}$.*

This result uses ∞-Borel codes to absorb fragments of functions into suitable ZFC models. To the authors' knowledge, the most interesting properties about S_1 (and even to distinguish $|S_1|$ from $|\mathbb{R}|$) require arguments using ∞-Borel codes. Unlike S_1, $[\omega_1]^{<\omega_1}$ is a more combinatorial object and [5] distinguished $[\omega_1]^{\omega}$ and $[\omega_1]^{<\omega_1}$ in AD alone using the almost everywhere continuity property (on sequences of a fixed countable length).

First, this paper will show that under just $\mathsf{ZF}+\mathsf{AD}$, one can prove the following.

Theorem 2.9. Assuming $\mathsf{ZF}+\mathsf{AD}$, $\neg(|[\omega_1]^{<\omega_1}| \leq |{}^{\omega}(\omega_\omega)|)$.

Then one will obtain the conclusion of Fact 1.3 under just $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$.

Theorem 4.4. Assume $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$. There is no injection of $[\omega_1]^{<\omega_1}$ into ${}^{\omega}\mathrm{ON}$.

These two theorems are then used to prove the following theorem.

Theorem 4.7. Assume $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$. There is no injection of $[\omega_1]^{\omega_1}$ into ${}^{<\omega_1}\mathrm{ON}$.
Assuming just $\mathsf{ZF}+\mathsf{AD}$, $\neg(|[\omega_1]^{\omega_1}| \leq |{}^{<\omega_1}(\omega_\omega)|)$.

Fact 1.3 is proved using techniques that clearly have an AD^+ flavor. In contrast, the results of this paper are proved using an eclectic combination of classical determinacy arguments and more recent results of classical flavor. Almost everywhere continuity results for functions $\Phi : [\omega_1]^{\epsilon} \to \omega_1$ from [5] use the Kunen tree which is an important tool for analyzing the ultrapower of ω_1 by the partition measures. Various almost everywhere club uniformization results will be employed. One consequence of these club uniformization results is the almost everywhere continuity property for functions of the form $\Phi : [\omega_1]^{\omega_1} \to \omega_1$ from [4]. Generic coding arguments, category notions, and the Banach-Mazur games will be used to make uniform selection of cofinal sets and uniform selection of $\mathbf{\Sigma}^1_2$ bounding prewellorderings. Martin's good coding system and the Martin style games, which are used to prove the partition relations, will indirectly appear in the almost everywhere good code uniformization.

Ideas involving S_1 require forcing techniques over ZFC-models that do not seem to generalize to cardinals higher than ω_1. [5] used classical determinacy arguments to prove $\neg(|[\omega_1]^{<\omega_1}| \leq |[\omega_1]^{\omega}|)$ under just AD, but the techniques used could be generalized to prove $\neg(|[\omega_2]^{<\omega_2}| \leq |[\omega_2]^{\omega_1}|)$ under AD which has no known AD^+ style proof to the authors' knowledge. The methods used here seem to be more suitable to generalizations of the main questions to higher strong partition cardinals such as δ^1_3.

2. Below ω_ω

This section will give a negative answer to Question 1.1 below ω_ω under just $\mathsf{ZF} + \mathsf{AD}$. A brief review of Kunen functions and the ultrapower representations of ω_n for $n < \omega$ will be given. Fact 2.6 will show under AD that $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$. Using the ultrapower representations, Fact 2.6 will be used to show that $[\omega_1]^{<\omega_1}$ does not inject into ${}^\omega\omega_n$ (Theorem 2.8) or even ${}^\omega(\omega_\omega)$ (Theorem 2.9). Later, the main result (Theorem 4.4) will be derived by reducing back to Fact 2.6.

Let $\epsilon \in \mathrm{ON}$ and $f : \epsilon \to \mathrm{ON}$. The function f is discontinuous everywhere if and only if for all $\alpha < \epsilon$, $\sup(f \restriction \alpha) < f(\alpha)$. The function f has uniform cofinality ω if and only if there is a function $F : \epsilon \times \omega \to \mathrm{ON}$ so that for all $\alpha < \epsilon$ and $n \in \omega$, $F(\alpha, n) < F(\alpha, n+1)$ and $F(\alpha) = \sup\{F(\alpha, k) : k \in \omega\}$. The function f has the correct type if and only if it is discontinuous everywhere and has uniform cofinality ω. If X is a set or class of ordinals, then $[X]^\epsilon_*$ and $[X]^{<\epsilon}_*$ are the subsets of $[X]^\epsilon$ and $[X]^{<\epsilon}$ (respectively) consisting of those functions of the correct type. Using the ideas from [4] Fact 2.2, one can show that if $\epsilon \leq \kappa$ and κ is a cardinal, then $|[\kappa]^\epsilon| = |[\kappa]^\epsilon_*|$. The notion of having correct type is used to formulate a correct type partition property which provides club homogeneous sets.

Definition 2.1. Let $\epsilon \leq \kappa$ be ordinals. Let $\kappa \to_* (\kappa)^\epsilon_2$ indicate that for all $P : [\kappa]^\epsilon_* \to 2$, there is a club $C \subseteq \kappa$ and an $i \in 2$ so that for all $f \in [C]^\epsilon_*$, $P(f) = i$. If $\kappa \to_* (\kappa)^\epsilon_2$ for all $\epsilon < \kappa$, then κ is said to be a weak partition cardinal. If $\kappa \to_* (\kappa)^\kappa_2$, then κ is said to be a strong partition cardinal.

Fact 2.2. *(Martin; [7] Theorem 12.2, [2] Fact 4.9, [2] Corollary 4.27) Assume $\mathsf{ZF} + \mathsf{AD}$. For all $\epsilon \leq \omega_1$, $\omega_1 \to_* (\omega_1)^\epsilon_2$. (Martin and Paris; [2] Theorem 5.19) For all $\epsilon < \omega_2$, $\omega_2 \to_* (\omega_2)^\epsilon_2$.*

For $1 \leq \epsilon \leq \omega_1$, define the filter W^ϵ_1 on $[\omega_1]^\epsilon_*$ by $A \in W^\epsilon_1$ if and only if there is a club $C \subseteq \omega_1$ so that $[C]^\epsilon_* \subseteq A$. The partition relations imply that W^ϵ_1 is a countably complete measure. In particular, ω_1 is a measurable cardinal. Using the Kunen tree analysis, it can be shown in $\mathsf{ZF} + \mathsf{AD}$ (without $\mathsf{DC}_\mathbb{R}$) that for each $1 \leq n < \omega$, $\prod_{[\omega_1]^n} \omega_1 / W^n_1$ is a wellordering under the usual ultrapower ordering and in fact $\omega_{n+1} = \prod_{[\omega_1]^n} \omega_1 / W^n_1$. The Martin and Paris weak partition property for ω_2 from Fact 2.2 follows from the ultrapower representation of ω_2 (see [2] Theorem 5.19 for the details). These partition properties on ω_2 imply that ω_2 is measurable (for instance using the ω-club filter or the ω_1-club filter on ω_2) and hence regular. The ultrapower representation also shows that for all $n \geq 3$, $\mathrm{cof}(\omega_n) = \omega_2$.

One can explicitly give an ω_2 cofinal sequence through ω_n when $n \geq 2$. Let $\mathfrak{V}$ be the subset of the ultrapower $\prod_{\omega_1} \omega_1 / W^1_1$ consisting of those elements which have a representative $f : \omega_1 \to \omega_1$ which is an increasing function of the correct type. Since $|\prod_{\omega_1} \omega_1 / W^1_1| = \omega_2$, one also has that $|\mathfrak{V}| = \omega_2$. For $n \geq 2$ and $f : \omega_1 \to \omega_1$, define $K_n(f) : [\omega_1]^{n-1} \to \omega_1$ by $K_n(f)(\alpha_1, ..., \alpha_{n-1}) = f(\alpha_{n-1})$. Define $\rho_n : \mathfrak{V} \to \prod_{[\omega_1]^{n-1}} \omega_1 / W^{n-1}_1 = \omega_n$ by $\rho_n([f]_{W^1_1}) = [K_n(f)]_{W^{n-1}_1}$. (Observe that ρ_n is well defined and independent of the choice of representative f.) One can check that $\rho_n : \mathfrak{V} \to \omega_n$ is cofinal. (See [6] Section 4 and [12] Theorem 5.2.)

Fact 2.3. *(Kunen; [2] Theorem 5.10; [6] Lemma 4.1) Assume $\mathsf{ZF} + \mathsf{AD}$. Let $f : \omega_1 \to \omega_1$ be a function. There is a function $\mathcal{K} : \omega_1 \times \omega_1 \to \omega_1$ so that for each $\omega \leq \alpha < \omega_1$, $f(\alpha) < \sup\{\mathcal{K}(\alpha, \beta) : \beta < \alpha\} = \{\mathcal{K}(\alpha, \beta) : \beta < \alpha\}$. (The latter means that the set $\{\mathcal{K}(\alpha, \beta) : \beta < \alpha\}$ is the ordinal $\sup\{\mathcal{K}(\alpha, \beta) : \beta < \alpha\}$.)*

Fact 2.4. *Assume $\mathsf{ZF} + \mathsf{AD}$. Let $\delta < \epsilon \leq \omega_1$ and $\Phi : [\omega_1]^\epsilon_* \to \omega_1$ have the property that there is a club $C \subseteq \omega_1$ so that for all $f \in [C]^\epsilon_*$, $\Phi(f) < f(\delta)$. Then there is a club $F \subseteq \omega_1$ so that for all $f, g \in [F]^\epsilon_*$, if $f \restriction \delta = g \restriction \delta$, then $\Phi(f) = \Phi(g)$.*

Proof. Let δ, ϵ, Φ, and C be as in the statement. Let $\epsilon' = \delta + 1 + (\epsilon - \delta)$. Let $h : \epsilon' \to \omega_1$. Define $\mathsf{main}(h) : \epsilon \to \omega_1$ by

$$\mathsf{main}(h)(\alpha) = \begin{cases} h(\alpha) & \alpha < \delta \\ h(\delta + 1 + (\alpha - \delta)) & \alpha \geq \delta \end{cases}.$$

Let $\mathsf{extra}(h) \in \omega_1$ be $\mathsf{extra}(h) = h(\delta)$. Define a partition $\Phi : [C]^{\epsilon'}_* \to 2$ by $P(h) = 0$ if and only if $\Phi(\mathsf{main}(h)) < \mathsf{extra}(h)$. By $\omega_1 \to_* (\omega_1)^{\epsilon'}_2$, let $D_0 \subseteq C$ be a club homogeneous for P. Next, it will be shown that D_0 is homogeneous for P taking value 0. Let $D_1 \subseteq D_0$ be the set of limit points of D_0. Pick $f \in [D_1]^\epsilon_*$. Since $f \in [C]^\epsilon_*$, $\Phi(f) < f(\delta) \in D_1$. Let $\gamma \in D_0$ be so that $\sup(f \restriction \delta) < \gamma < f(\delta)$. Let $h : \epsilon' \to D_0$ be such that $\mathsf{main}(h) = f$ and $\mathsf{extra}(h) = \gamma$. Since $P(h) = 0$ and $h \in [D_0]^{\epsilon'}_*$, this shows D_0 is homogeneous for P taking value 0.

Let $\mathsf{next}_{D_0} : \omega_1 \to D_0$ be defined by $\mathsf{next}_{D_0}(\alpha)$ is the least element of D_0 larger than α. For any $f \in [D_1]^\epsilon_*$, $\Phi(f) < \mathsf{next}_{D_0}(\sup(f \restriction \delta))$. This follows from $P(h) = 0$ where $h : \epsilon' \to D_0$ is defined so that $\mathsf{main}(h) = f$ and $\mathsf{extra}(h) = \mathsf{next}_{D_0}(\sup(f \restriction \delta))$. For each $\sigma \in [D_1]^\delta_*$, let $V_\sigma : [D_1 \setminus (\sup(\sigma)+1)]^{\epsilon-\delta}_* \to \mathsf{next}_{D_0}(\sup(\sigma))$ be defined by $V_\sigma(k) = \Phi(\sigma\hat{\ }k)$. By the countable completeness of $W_1^{\epsilon-\delta}$, there is a club E and a $\gamma_\sigma < \mathsf{next}_{D_0}(\sup(\sigma))$ so that for all $k \in [E]^{\epsilon-\delta}_*$, $V_\sigma(k) = \gamma_\sigma$.

Define $Q : [D_1]^\epsilon_* \to 2$ by $Q(f) = 0$ if and only if $\Phi(f) = \gamma_{f \restriction \delta}$. By $\omega_1 \to_* (\omega_1)^\epsilon_2$, there is a club $F \subseteq D_1$ which is homogeneous for Q. Pick any $\sigma \in [F]^\delta_*$. There is a club $E \subseteq F$ so that for all $k \in [E]^{\epsilon-\delta}_*$, $V_\sigma(k) = \gamma_\sigma$. Pick a $k \in [E]^{\epsilon-\delta}_*$ and let $f = \sigma\hat{\ }k$. Then $\Phi(f) = \Phi(\sigma\hat{\ }k) = V_\sigma(k) = \gamma_\sigma = \gamma_{f\restriction\delta}$. So $Q(f) = 0$ and hence F must be homogeneous for Q taking value 0. Thus F is the desired club. ◻

The following is the almost everywhere continuity property for functions $\Phi : [\omega_1]^\epsilon \to \omega_1$ when $\epsilon < \omega_1$.

Fact 2.5. *([5] Theorem 2.15) Assume $\mathsf{ZF} + \mathsf{AD}$. Let $\epsilon < \omega_1$ and $\Phi : [\omega_1]^\epsilon_* \to \omega_1$. There is an $\delta < \epsilon$ and a club $C \subseteq \omega_1$ so that for all $f, g \in [C]^\epsilon_*$, if $\sup(f) = \sup(g)$ and $f \restriction \delta = g \restriction \delta$, then $\Phi(f) = \Phi(g)$.*

Proof. In [5], this fact is derived from a finer and complete analysis of continuity given by [5] Theorem 2.14. The following is a simpler proof of just the coarser continuity stated above.

Define a partition $P : [\omega_1]^{\epsilon+1}_* \to 2$ by $P(h)$ if and only if $\Phi(h \restriction \epsilon) < h(\epsilon)$. By $\omega_1 \to_* (\omega_1)^{\epsilon+1}_2$, there is a club $C_0 \subseteq \omega_1$ which is homogeneous for P. Pick an $f \in [C_0]^\epsilon_*$ and let $\gamma \in C_0$ be such that $\Phi(f) < \gamma$. Define $h = f\hat{\ }\gamma$. Since $h \in [C_0]^{\epsilon+1}_*$ and $P(h) = 0$, this shows that C_0 is homogeneous for P taking value 0. For all $f \in [C_0]^\epsilon_*$, $\Phi(f) < \mathsf{next}_{C_0}(\sup(f))$ by using the fact that $P(h) = 0$ where $h \in [C_0]^{\epsilon+1}_*$ is defined so that $h \restriction \epsilon = f$ and $h(\epsilon) = \mathsf{next}_{C_0}(\sup(f))$. By Fact 2.3, there is a function $\mathcal{K} : \omega_1 \times \omega_1 \to \omega_1$ with the property that for all $\omega \leq \alpha < \omega_1$, $\mathsf{next}_{C_0}(\alpha) < \sup\{\mathcal{K}(\alpha, \beta) : \beta < \alpha\} = \{\mathcal{K}(\alpha, \beta) : \beta < \alpha\}$. Since for all $f \in [C_0]^\epsilon_*$, $\Phi(f) < \mathsf{next}_{C_0}(\sup(f))$, define $\Psi : [C_0]^\epsilon_* \to \omega_1$ by $\Psi(f)$ is the least $\beta < \sup(f)$ so that $\Phi(f) = \mathcal{K}(\sup(f), \beta)$. For each $f \in [C_0]^\epsilon_*$, let δ_f be the least δ so that $\Psi(f) < f(\delta)$. Since W_1^ϵ is countably additive, there is a $\delta < \epsilon$ and a club $C_1 \subseteq C_0$ so that for all $f \in [C_1]^\epsilon_*$, $\delta_f = \delta$. Fact 2.4 implies there is a club $C \subseteq C_1$ so that for all $f, g \in [C]^\epsilon_*$, if $f \restriction \delta = g \restriction \delta$, then $\Psi(f) = \Psi(g)$. For any $f, g \in [C]^\epsilon_*$, if $f \restriction \delta = g \restriction \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \mathcal{K}(\sup(f), \Psi(f)) = \mathcal{K}(\sup(g), \Psi(g)) = \Phi(g)$. C is the desired club. ◻

[5] uses Fact 2.5 to give an argument in $\mathsf{ZF} + \mathsf{AD}$ that $[\omega_1]^{<\omega_1}$ does not inject into ${}^\omega\omega_1$. Since this result is the backbone of all other results in the paper, the proof will be given for completeness.

Fact 2.6. *([5] Theorem 2.16) Assuming $\mathsf{ZF} + \mathsf{AD}$, $\neg(|[\omega_1]^{<\omega_1}| \leq |{}^\omega\omega_1|)$. In particular, $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$.*

Proof. Suppose there is an injection $\Phi : [\omega_1]^{<\omega_1} \to {}^\omega\omega_1$. For each $\epsilon < \omega_1$ and $n \in \omega$, define $\Phi^\epsilon_n : [\omega_1]^\epsilon \to \omega_1$ by $\Phi^\epsilon_n(f) = \Phi(f)(n)$. By Fact 2.5, there is a $\delta < \epsilon$ and a club $C \subseteq \omega_1$ so that for all $f, g \in [C]^\epsilon_*$, if $f \restriction \delta = g \restriction \delta$ and $\sup(f) = \sup(g)$, then $\Phi^\epsilon_n(f) = \Phi^\epsilon_n(g)$. Let δ^ϵ_n be the least such δ. For each $n \in \omega$, let $\Lambda_n : \omega_1 \to \omega_1$ be defined by $\Lambda_n(\epsilon) = \delta^\epsilon_n$. Since Λ_n is regressive and the club measure W^1_1 is normal, there is a $\delta_n < \omega_1$ so that there exists a club $C \subseteq \omega_1$ with the property that $\Lambda_n(\epsilon) = \delta_n$ for all $\epsilon \in C$. By the Moschovakis coding lemma, there is a surjection $\pi : \mathbb{R} \to \mathscr{P}(\omega_1)$. Define $R \subseteq \omega \times \mathbb{R}$ by $R(n, x)$ if and only if $\pi(x)$ is a club with the property that for all $\epsilon \in \pi(x)$, $\Lambda_n(\epsilon) = \delta_n$. By $\mathsf{AC}^\mathbb{R}_\omega$, there is a sequence $\langle x_n : n \in \omega\rangle$ so that for all $n \in \omega$, $R(n, x_n)$. Let $C_n = \pi(x_n)$ and $C = \bigcap_{n\in\omega} C_n$. Let $\delta = \sup\{\delta_n : n \in \omega\}$ and note that $\delta < \omega_1$ since ω_1 is regular. Fix an ordinal $\epsilon > \delta + 1$ with $\epsilon \in C$. Using the Moschovakis coding lemma and $\mathsf{AC}^\mathbb{R}_\omega$ again, there is a sequence $\langle D_n : n \in \omega\rangle$ of club subsets of ω_1 with the property that for all $n \in \omega$, for all $f, g \in [D_n]^\epsilon_*$, if $f \restriction \delta^\epsilon_n = g \restriction \delta^\epsilon_n$ and $\sup(f) = \sup(g)$, then $\Phi^\epsilon_n(f) = \Phi^\epsilon_n(g)$. Let $D = \bigcap_{n\in\omega} D_n$. Pick $f, g \in [D]^\epsilon_*$ so that $f \restriction \delta = g \restriction \delta$, $\sup(f) = \sup(g)$, and $f \neq g$. Since $\delta \geq \delta_n = \delta^\epsilon_n$ (since $\epsilon \in C$) for all $n \in \omega$, one has that $\Phi(f) = \Phi(g)$. This contradicts Φ being an injection. ◻

Let club_{ω_1} denote the collection of club subsets of ω_1. The following is the everywhere ω_1 club uniformization.

Fact 2.7. *([2] Fact 4.8) Assume $\mathsf{ZF} + \mathsf{AD}$. Suppose $R \subseteq \omega_1 \times \mathsf{club}_{\omega_1}$ is a relation which is $\subseteq$-downward closed in the club coordinate. (This means for all $\alpha \in \omega_1$, if $C \subseteq D$ are club subsets of ω_1 and $R(\alpha, D)$, then $R(\alpha, C)$.) Then there is a function $\Phi : \mathrm{dom}(R) \to \mathsf{club}_{\omega_1}$ so that for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, \Phi(\alpha))$.*

The following is useful notation. If $f : \omega_1 \to \omega_1$ and $\epsilon < \omega_1$, then let $\mathsf{drop}(f, \epsilon) : \omega_1 \to \omega_1$ be defined by $\mathsf{drop}(f, \epsilon)(\alpha) = f(\epsilon + \alpha)$. The following argument appears in [5] for just ω_2. The next result adapts the arguments for all ω_n such that $2 \leq n < \omega$.

Theorem 2.8. *Assume* $\mathsf{ZF} + \mathsf{AD}$, $\neg(|[\omega_1]^{<\omega_1}| \leq {}^\omega\omega_n)$ *for all* $n \in \omega$.

Proof. Suppose $\Phi : [\omega_1]^{<\omega_1} \to {}^\omega\omega_n$ is an injection. Fact 2.6 implies this is impossible if $n \leq 1$. So suppose $n \geq 2$ and inductively this result has been shown for all $k < n$.

For each $\epsilon < \omega_1$, define a partition $P_\epsilon : [\omega_1]^{\omega_1}_* \to 2$ by $P_\epsilon(f) = 0$ if and only if $\sup(\Phi(f \restriction \epsilon)) < \rho_n([\mathsf{drop}(f, \epsilon)]_{W^1_1})$, where $\rho_n : \mathfrak{V} \to \omega_n$ is cofinal. (Note that $\sup(\Phi(f \restriction \epsilon)) < \omega_n$ since $\Phi(f \restriction \epsilon) \in {}^\omega\omega_n$ and $\mathrm{cof}(\omega_n) = \omega_2$.) By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club $C \subseteq \omega_1$ which is homogeneous for P_ϵ. Next, one will show C is homogeneous for P_ϵ taking value 0. Pick any $\sigma \in [C]^\epsilon_*$. Since ρ_n is cofinal, there is an $h \in [C]^{\omega_1}_*$ with $\sup(\sigma) < h(0)$ and $\sup(\Phi(\sigma)) < \rho_n([h]_{W^1_1})$. Let $f = \sigma\hat{\ }h$. Note that $f \in [C]^{\omega_1}_*$ and $P_\epsilon(f) = 0$ since $\sup(\Phi(f \restriction \epsilon)) = \sup(\Phi(\sigma)) < \rho_n([h]_{W^1_1}) = \rho_n([\mathsf{drop}(f, \epsilon)]_{W^1_1})$. Thus C is homogeneous for P_ϵ taking value 0. Fix a $g \in [C]^{\omega_1}_*$. Let $\beta = \rho_n([g]_{W^1_1})$. For any $\sigma \in [C]^\epsilon_*$, let γ_σ be the least $\gamma < \omega_1$ so that $\sup(\sigma) < g(\gamma)$. Let $f_\sigma = \sigma\hat{\ }\mathsf{drop}(g, \gamma_\sigma)$ and note that $f_\sigma \in [C]^{\omega_1}_*$. $P(f_\sigma) = 0$ implies that $\sup(\Phi(\sigma)) < \rho_n([\mathsf{drop}(f_\sigma, \epsilon)]_{W^1_1}) = \rho_n([g]_{W^1_1}) = \beta$ since $[\mathsf{drop}(f_\sigma, \epsilon)]_{W^1_1} = [g]_{W^1_1}$. Let β_ϵ be the least ordinal γ for which there exists a club $D \subseteq \omega_1$ so that for all $\sigma \in [D]^\epsilon_*$, $\sup(\Phi(\sigma)) < \gamma$. (The previous argument shows such objects exists as witnessed by β and C above.)

Define a relation $R \subseteq \omega_1 \times \mathsf{club}_{\omega_1}$ by $R(\epsilon, D)$ if and only if for all $\sigma \in [D]^\epsilon_*$, $\sup(\Phi(\sigma)) < \beta_\epsilon$. Note that R is $\subseteq$-downward in the club_{ω_1} coordinate and $\mathrm{dom}(R) = \omega_1$. By Fact 2.7, there is a sequence $\langle C_\epsilon : \epsilon < \omega_1\rangle$ so that for all $\epsilon < \omega_1$, $R(\epsilon, C_\epsilon)$. Let $\delta = \sup\{\beta_\epsilon : \epsilon < \omega_1\}$ and note that $\delta < \omega_n$ since $\mathrm{cof}(\omega_n) = \omega_2$. Let $B : \delta \to \omega_{n-1}$ be an injection. Define an injection $\Psi : \bigcup_{\epsilon<\omega_1}[C_\epsilon]^\epsilon_* \to {}^\omega\omega_{n-1}$ by $\Psi(\sigma)(n) = B(\Phi(\sigma)(n))$. This is well defined since for all $\sigma \in [C_\epsilon]^\epsilon_*$, $\Phi(\sigma)(n) < \beta_\epsilon < \delta$. Since $|\bigcup_{\epsilon<\omega_1}[C_\epsilon]^\epsilon_*| = |[\omega_1]^{<\omega_1}|$, Ψ induces an injection of $[\omega_1]^{<\omega_1}$ into ${}^\omega\omega_{n-1}$, which is impossible by the induction hypothesis. ☐

Note that the next result implies Theorem 2.8; however, the proof is more tedious.

Theorem 2.9. *Assume* $\mathsf{ZF} + \mathsf{AD}$, $\neg(|[\omega_1]^{<\omega_1}| \leq |{}^\omega(\omega_\omega)|)$.

Proof. Suppose there is an injection $\Phi : [\omega_1]^{<\omega_1} \to {}^\omega\omega_\omega$. By the countable additivity of W^1_1, for each $n \in \omega$ and $\epsilon < \omega_1$, there is a club C and an integer b so that $\Phi(\sigma)(n) < \omega_b$ for all $\sigma \in [C]^\epsilon_*$. Let b^ϵ_n be the least such integer. Again by the countable additivity of W^1_1, for each $n \in \omega$, there is a club D and an integer b_n so that for all $\epsilon \in D$, $b^\epsilon_n = b_n$. By the Moschovakis coding lemma and $\mathsf{AC}^{\mathbb{R}}_\omega$, there is a sequence $\langle D_n : n \in \omega\rangle$ of club subsets of ω_1 so that for all $\epsilon \in D_n$, $b^\epsilon_n = b_n$. Let $D^* = \bigcap_{n\in\omega} D_n$.

Claim 1: For each $n \in \omega$, there is a sequence $\langle E_\epsilon : \epsilon \in D^*\rangle$ of club subsets of ω_1 and an injection $I : A \to \omega_1$, where $A = \{\Phi(\sigma)(n) : \sigma \in \bigcup_{\epsilon\in D^*}[E_\epsilon]^\epsilon_*\}$.

To see Claim 1: Fix $n \in \omega$. Recall for each $i \geq 2$, there are cofinal maps $\rho_i : \mathfrak{V} \to \omega_i$ of $\mathfrak{V}$ (which has cardinality ω_2) into ω_i. Define $R \subseteq D^* \times \mathsf{club}_{\omega_1}$ by $R(\epsilon, C)$ if and only if for all $\sigma \in [C]^\epsilon_*$, $\Phi(\sigma)(n) < \omega_{b_n}$. Since R is $\subseteq$-downward closed in the club_{ω_1}-coordinate and $\mathrm{dom}(R) = D^*$, Fact 2.7 implies there is a sequence $\langle C'_\epsilon : \epsilon \in D^*\rangle$ so that for all $\epsilon \in D^*$, $R(\epsilon, C'_\epsilon)$. Let $C^{b_n}_\epsilon = C'_\epsilon$, $a_{b_n} = b_n$, and $\Psi_{b_n} : \bigcup_{\epsilon\in D^*}[C^{b_n}_\epsilon]^\epsilon_* \to \omega_{a_{b_n}}$ be defined by $\Psi_{b_n}(\sigma) = \Phi(\sigma)(n)$. (In the following construction, the indices of the objects created will be decreasing.)

Suppose for $0 < k \leq b_n$, the following objects have been defined.

- For all $k \leq j \leq b_n$, $a_j \leq b_n$ and for all $k < j \leq b_n$, if $a_j > 1$, then $a_{j-1} < a_j$.
- For all $k \leq j \leq b_n$, $\langle C^j_\epsilon : \epsilon \in D^*\rangle$ is a sequence of clubs and for all $k \leq j_0 \leq j_1 \leq b_n$, $C^{j_0}_\epsilon \subseteq C^{j_1}_\epsilon$.
- For all $k \leq j \leq b_n$, $\Psi_j : \bigcup_{\epsilon\in D^*}[C^j_\epsilon]^\epsilon_* \to \omega_{a_j}$.
- For $k < j \leq b_n$, $I_j : T_j \to \omega_{a_{j-1}}$ is an injection where $T_j = \{\Psi_j(\sigma) : \sigma \in \bigcup_{\epsilon\in D^*}[C^j_\epsilon]^\epsilon_*\}$.

Next, one will define a_{k-1}, C^{k-1}_ϵ for each $\epsilon \in D^*$, Ψ_{k-1}, I_k, and T_k with the above property at $k - 1$.

(Case I: $a_k > 1$.) Fix $\epsilon \in D^*$. Define $P^k_\epsilon : [C^k_\epsilon]^{\omega_1}_* \to 2$ by $P^k_\epsilon(f) = 0$ if and only if $\Psi_k(f \restriction \epsilon) < \rho_{a_k}([\mathsf{drop}(f, \epsilon)]_{W^1_1})$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club $K \subseteq C^k_\epsilon$ which is homogeneous for P^k_ϵ. Fix a $\sigma \in [K]^\epsilon_*$. Pick any $\ell \in [K]^{\omega_1}_*$ so that $\ell(0) > \sup(\sigma)$ and $\Psi_k(\sigma) < \rho_{a_k}([\ell]_{W^1_1})$ which is possible since $\rho_{a_k} : \mathfrak{V} \to \omega_{a_k}$ is cofinal. Let $f = \sigma\hat{\ }\ell$. Note that $P^k_\epsilon(f) = 0$ since $\Psi_k(f \restriction \epsilon) = \Psi_k(\sigma) < \rho_{a_k}([\ell]_{W^1_1}) = \rho_{a_k}([\mathsf{drop}(f, \epsilon)]_{W^1_1})$. Since $f \in [K]^{\omega_1}_*$, K is homogeneous for P^k_ϵ taking value 0.

Now fix an $\ell \in [K]^{\omega_1}_*$ and let $\beta = \rho_{a_k}([\ell]_{W^1_1})$. For any $\sigma \in [K]^\epsilon_*$, let γ_σ be the least γ so that $\sup(\sigma) < \ell(\gamma)$. Let $f_\sigma = \sigma\hat{}\mathsf{drop}(\ell, \gamma_\sigma)$. $P^k_\epsilon(f_\sigma) = 0$ implies that $\Psi_k(\sigma) = \Psi_k(f_\sigma \upharpoonright \epsilon) < \rho_{a_k}([\mathsf{drop}(f_\sigma, \epsilon)]_{W^1_1}) = \rho_{a_k}([\ell]_{W^1_1}) = \beta$ since $[\mathsf{drop}(f_\sigma, \epsilon)]_{W^1_1} = [\ell]_{W^1_1}$. It has been shown that there is a $\bar{\beta}$ so that there exists a club $\bar{K} \subseteq C^k_\epsilon$ with the property that for all $\sigma \in [\bar{K}]^\epsilon_*$, $\Psi_k(\sigma) < \beta$. Let β_ϵ be the least such $\bar{\beta} \geq \omega$ with this property. Let $\delta = \sup\{\beta_\epsilon : \epsilon \in D^*\}$. Since $a_k > 1$, $\mathrm{cof}(\omega_{a_k}) = \omega_2$ so $\delta < \omega_{a_k}$.

Now define $S \subseteq D^* \times \mathsf{club}_{\omega_1}$ by $S(\epsilon, K)$ if and only if $K \subseteq C^k_\epsilon$ and for all $\sigma \in [K]^\epsilon_*$, $\Psi_k(\sigma) < \delta$. Note that $\mathrm{dom}(S) = D^*$ by the previous discussion and S is $\subseteq$-downward closed in the club_{ω_1}-coordinate. By Fact 2.7, there is a sequence of clubs $\langle C^{k-1}_\epsilon : \epsilon \in D^*\rangle$ with the property that for all $\epsilon \in D^*$, $S(\epsilon, C^{k-1}_\epsilon)$. For all $\sigma \in \bigcup_{\epsilon \in D^*}[C^{k-1}_\epsilon]^\epsilon_*$, $\Psi_k(\sigma) < \beta_{|\sigma|} < \delta < \omega_{a_k}$. Since $\omega \leq \delta < \omega_{a_k}$, there is an $a_{k-1} < a_k$ so that $\omega_{a_{k-1}} \leq \delta < \omega_{a_{k-1}+1}$. Let $I_k : \delta \to \omega_{a_{k-1}}$ be an injection. Let $T_k = \{\Psi_k(\sigma) : \sigma \in \bigcup_{\epsilon \in D^*}[C^{k-1}_\epsilon]^\epsilon_*\}$ and note that the restriction $I_k : T_k \to \omega_{a_{k-1}}$ is an injection. Let $\Psi_{k-1} : \bigcup_{\epsilon \in D^*}[C^{k-1}_\epsilon]^\epsilon_* \to \omega_{a_{k-1}}$ be defined by $I_k \circ \Psi_k$.

(Case II: $a_k \leq 1$) Let $a_{k-1} = a_k$, $C^{k-1}_\epsilon = C^k_\epsilon$ for each $\epsilon \in D^*$, $T_k = \{\Psi_k(\sigma) : \sigma \in \bigcup_{\epsilon \in D^*}[C^{k-1}_\epsilon]^\epsilon_*\}$, $I_k : T_k \to \omega_{a_{k-1}}$ be the inclusion map, and $\Psi_{k-1} = \Psi_k$.

By recursion, one has constructed $\langle a_k : 0 \leq k \leq b_n\rangle$, $\langle\langle C^k_\epsilon : \epsilon \in D^*\rangle : 0 \leq k \leq b_n\rangle$, $\langle T_k : 0 < k \leq b_n\rangle$, and $\langle I_k : 0 < k \leq b_n\rangle$. Since $a_{k-1} < a_k$ for all $0 < k \leq b_n$ such that $a_k > 1$ and $a_{b_n} = b_n$, one must have that $a_0 \leq 1$. For each $\epsilon \in D^*$, let $E_\epsilon = C^0_\epsilon$ and $I : A \to \omega_1$ be defined by $I = I_1 \circ ... \circ I_{b_n}$, where recall that $A = \{\Phi(\sigma)(n) : \sigma \in \bigcup_{\epsilon \in D^*}[E_\epsilon]^\epsilon_*\}$. This completes the proof of Claim 1.

Using Claim 1, the Moschovakis coding lemma, and $\mathsf{AC}^{\mathbb{R}}_\omega$, there exist sequences $\langle\langle E^n_\epsilon : \epsilon \in D^*\rangle : n \in \omega\rangle$ and $\langle I_n : n \in \omega\rangle$ so that for all $n \in \omega$, $I_n : A_n \to \omega_1$ is an injection where $A_n = \{\Phi(\sigma)(n) : \sigma \in \bigcup_{\epsilon \in D^*}[E^n_\epsilon]^\epsilon_*\}$. For each $\epsilon \in D^*$, let $E_\epsilon = \bigcap_{n \in \omega} E^n_\epsilon$. Let $\Sigma : \bigcup_{\epsilon \in D^*}[E_\epsilon]^\epsilon_* \to {}^\omega\omega_1$ be defined by $\Sigma(\sigma)(n) = I_n(\Phi(\sigma)(n))$. Now suppose $\sigma_0, \sigma_1 \in \bigcup_{\epsilon \in D^*}[E_\epsilon]^\epsilon_*$ and $\sigma_0 \neq \sigma_1$. Since Φ is an injection, $\Phi(\sigma_0) \neq \Phi(\sigma_1)$. Thus there is some $n \in \omega$ so that $\Phi(\sigma_0)(n) \neq \Phi(\sigma_1)(n)$. Since $\Phi(\sigma_0)(n), \Phi(\sigma_1)(n) \in A_n$ and $I_n : A_n \to \omega_1$ is an injection, one has that $\Sigma(\sigma_0) \neq \Sigma(\sigma_1)$ because $\Sigma(\sigma_0)(n) = I_n(\Phi(\sigma_0)(n)) \neq I_n(\Phi(\sigma_1)(n)) = \Sigma(\sigma_1)(n)$. It has been shown that Σ is an injection. Since $|\bigcup_{\epsilon \in D^*}[E_\epsilon]^\epsilon_*| = |[\omega_1]^{<\omega_1}|$, Σ induces an injection from $[\omega_1]^{<\omega_1}$ to ${}^\omega\omega_1$. This is impossible by Fact 2.6. ☐

3. Uniform Choice of Unbounded Subsets and Bounding Prewellorderings

Since AD has a limited amount of choice, this section will establish two results concerning the ability to make uniform choices. Fact 3.3 will show that if $\langle \nu_\alpha : \alpha < \omega_1\rangle$ is a sequence of limit ordinals bounded below Θ of cofinality less than or equal to ω_1, then there is a sequence $\langle K_\alpha : \alpha < \omega_1\rangle$ so that each K_α is an unbounded subset of ν_α of size less than or equal to ω_1. Fact 3.10 will show that if $\langle \delta_\alpha : \alpha < \omega_1\rangle$ is a sequence of limit ordinals bounded below Θ with cofinality greater than ω_1, then there is a sequence $\langle (P_\alpha, \preceq_\alpha) : \alpha < \omega_1\rangle$ so that each $\preceq_\alpha$ is a $\boldsymbol{\Sigma}^1_2$ bounded prewellordering on P_α of length δ_α. These two results will be used to establish Theorem 4.4.

Definition 3.1. Let $2 \leq \alpha < \omega_1$. For $s \in {}^{<\omega}\alpha$, let $N^\alpha_s = \{f \in {}^\omega\alpha : s \subseteq f\}$. Give ${}^\omega\alpha$ the topology generated by $\{N^\alpha_s : s \in {}^{<\omega_1}\alpha\}$ as a basis. Using this topology, one can define the usual category notions. Note that since α is countable, ${}^\omega\alpha$ is homeomorphic to the usual topology on ${}^\omega\omega$. Let surj_α be the set of functions $f : \omega \to \alpha$ which are surjections, i.e. $f[\omega] = \alpha$. Observe that surj_α is a comeager subset of ${}^\omega\alpha$.

Under AD, the meager ideal on ${}^\omega\omega$ has full wellordered additivity. That is, if $\lambda \in \mathrm{ON}$ and $\langle A_\alpha : \alpha < \lambda\rangle$ is a sequence of meager subsets of ${}^\omega\omega$, then $\bigcup_{\alpha<\lambda} A_\alpha$ is a meager subset of ${}^\omega\omega$. Since, ${}^\omega\omega$ and ${}^\omega\alpha$ are homeomorphic for each $\alpha < \omega_1$, the meager ideal on ${}^\omega\alpha$ also has full wellordered additivity.

The following is a simple form of the Kechris-Woodin generic coding function [11] for ω_1.

Fact 3.2. *There is a function $\mathfrak{G} : {}^\omega\omega_1 \to \mathrm{WO}$ so that for $\alpha < \omega_1$, if $f \in \mathsf{surj}_\alpha$, then $\mathrm{ot}(\mathfrak{G}(f)) = \alpha$.*

Proof. Let $f \in {}^\omega\alpha$. Let $A_f = \{n \in \omega : (\forall m)(m < n \Rightarrow f(m) \neq f(n))\}$. Define $\mathfrak{G}(f)$ to be the element of WO with domain A_f so that for all $m, n \in A_f$, $m <_{\mathfrak{G}(f)} n$ if and only if $f(m) < f(n)$. Note that if $f \in \mathsf{surj}_\alpha$, then $(A_f, <_{\mathfrak{G}(f)})$ is order isomorphic to α. ☐

Fact 3.3. *Assume $\mathsf{ZF} + \mathsf{AD}$. Let $\langle \nu_\alpha : \alpha < \omega_1\rangle$ be a sequence of limit ordinals so that for all $\alpha < \omega_1$, $\mathrm{cof}(\nu_\alpha) \leq \omega_1$ and $\sup\{\nu_\alpha : \alpha < \omega_1\} < \Theta$. Then there is a sequence $\langle K_\alpha : \alpha < \omega_1\rangle$ so that for all $\alpha < \omega_1$, $K_\alpha \subseteq \nu_\alpha$, $|K_\alpha| \leq \omega_1$, and $\sup K_\alpha = \nu_\alpha$.*

Proof. Let $\delta = \sup\{\nu_\alpha : \alpha < \omega_1\} < \Theta$. By the Moschovakis coding lemma, there is a surjection $\pi : \mathbb{R} \to \mathscr{P}(\delta)$. Define a relation $S \subseteq \mathrm{WO} \times \mathbb{R}$ by $S(w, r)$ if and only if $\pi(r)$ codes a function $\rho : \omega_1 \to \nu_{\mathrm{ot}(w)}$ such that $\sup \rho[\omega_1] = \nu_{\mathrm{ot}(w)}$. By the finer form of the Moschovakis coding lemma ([14] 7D.5 or [6] Theorem 2.12) applied to the pointclass $\mathbf{\Sigma}^1_2$ and the $\mathbf{\Sigma}^1_2$ (and even $\mathbf{\Pi}^1_1$) prewellordering on WO given by the ordertype function, there is relation R with the following properties.

- $R \subseteq S$ and R is $\mathbf{\Sigma}^1_2$.
- For all $\alpha < \omega_1$, $R \cap (\mathrm{WO}_\alpha \times \mathbb{R}) \neq \emptyset$.

Let $T \subseteq \mathrm{WO} \times \mathbb{R}$ be defined by

$$T(w, r) \Leftrightarrow (\exists v)(v \in \mathrm{WO} \wedge \mathrm{ot}(v) = \mathrm{ot}(w) \wedge R(v, r)).$$

T is $\mathbf{\Sigma}^1_2$, $\mathrm{dom}(T) = \mathrm{WO}$, and for all $w \in \mathrm{WO}$, $T(w, r)$ if and only if $\pi(r)$ codes a function $\rho : \omega_1 \to \nu_{\mathrm{ot}(w)}$ such that $\sup \rho[\omega_1] = \nu_{\mathrm{ot}(w)}$. Since T is $\mathbf{\Sigma}^1_2$, the scale property for $\mathbf{\Sigma}^1_2$ ([9] Corollary 3.8) implies (by [14] 4E.4) that T has a uniformization function $\Psi' : \mathrm{WO} \to \mathbb{R}$, i.e. for all $w \in \mathrm{WO}$, $T(w, \Psi'(w))$. For each $w \in \mathrm{WO}$, let $\Psi(w) = \pi(\Psi'(w))$, i.e. $\Psi(w)$ is the unbounded function from ω_1 into $\nu_{\mathrm{ot}(w)}$ coded by $\Psi'(w)$.

Define a partial function H as follows. For $\alpha < \omega_1$, $\beta < \omega_1$, and $s \in {}^{<\omega}\alpha$, $(\alpha, \beta, s) \in \mathrm{dom}(H)$ if and only if there is an $\eta < \nu_\alpha$ such that $\{f \in N^\alpha_s \cap \mathsf{surj}_\alpha : \Psi(\mathfrak{G}(f))(\beta) = \eta\}$ is comeager in N^α_s. If $(\alpha, \beta, s) \in \mathrm{dom}(H)$, then let $H(\alpha, \beta, s)$ be the unique η with the above property.

Define $K_\alpha = \{H(\alpha, \beta, s) : (\alpha, \beta, s) \in \mathrm{dom}(H)\}$. Note $|K_\alpha| \leq \omega_1$. It remains to show that $\sup K_\alpha = \nu_\alpha$. Fix $\gamma < \nu_\alpha$. For each $f \in \mathsf{surj}_\alpha$, let β_f be the least $\beta < \omega_1$ so that $\Psi(\mathfrak{G}(f))(\beta) > \gamma$. For each $\beta < \omega_1$, let $B_\beta = \{f \in \mathsf{surj}_\alpha : \beta_f = \beta\}$. Since surj_α is comeager, $\mathsf{surj}_\alpha = \bigcup_{\beta < \omega_1} B_\beta$, and wellordered unions of meager subsets of ${}^\omega\alpha$ are meager, there is a $\beta^* < \omega_1$ so that B_{β^*} is nonmeager. For each $\zeta > \gamma$, let $C_\zeta = \{f \in B_{\beta^*} : \Psi(\mathfrak{G}(f))(\beta^*) = \zeta\}$. Since B_{β^*} is nonmeager, $B_{\beta^*} = \bigcup_{\zeta > \gamma} C_\zeta$, and wellordered unions of meager subsets of ${}^\omega\alpha$ are meager, there is a $\zeta^* > \gamma$ so that C_{ζ^*} is nonmeager. By the Baire property, there is an $s \in {}^\omega\alpha$ so that C_{ζ^*} is comeager in N^α_s. Then for comeagerly many $f \in N^\alpha_s$, $\Psi(\mathfrak{G}(f))(\beta^*) = \zeta^* > \gamma$. Hence $(\alpha, \beta^*, s) \in \mathrm{dom}(H)$ and $H(\alpha, \beta^*, s) = \zeta^* > \gamma$. Thus $\zeta^* \in K_\alpha$. Since $\gamma < \nu_\alpha$ was arbitrary, $\sup K_\alpha = \nu_\alpha$. ☐

Let Γ be a (boldface) pointclass, $\check{\Gamma}$ be the dual pointclass of Γ, and $\Delta = \Gamma \cap \check{\Gamma}$. Let $\delta(\Gamma)$ be the supremum of the prewellorderings on $\mathbb{R}$ which belong to Δ. Let $v(\Gamma)$ be the supremum of the $\check{\Gamma}$ wellfounded relations on $\mathbb{R}$. If $A \in \mathscr{P}(\mathbb{R})$, then let $\mathsf{rk}_W(A)$ denote the Wadge rank of A (which exists assuming $\mathsf{DC}_\mathbb{R}$). If Γ is a pointclass, then let $o(\Gamma) = \sup\{\mathsf{rk}_W(A) : A \in \Gamma\}$.

Fact 3.4. *([6] Lemma 2.13 and 2.16) Suppose Γ is a nonselfdual pointclass, closed under $\forall^\mathbb{R}$, $\vee$, $\wedge$, and has the prewellordering property. Then $\delta(\Gamma) = v(\Gamma)$ and is a regular cardinal.*

Fact 3.5. *([10] Lemma 2.3) Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$. Suppose Δ is a (boldface) pointclass closed under $\neg$, $\wedge$, and $\forall^\mathbb{R}$. Then $o(\Delta) = \delta(\Delta)$.*

Fact 3.6. *Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$. Suppose Γ is a nonselfdual pointclass closed under $\exists^\mathbb{R}$, $\forall^\mathbb{R}$, $\wedge$, $\vee$, and has the prewellordering property. Let $\Delta = \Gamma \cap \check{\Gamma}$. Then $\delta = \delta(\Gamma) = \delta(\Delta) = o(\Delta)$ is a regular cardinal. Let C be the set of $\eta < \delta$ so that $\Upsilon_\eta = \{A \subseteq \mathbb{R} : \mathsf{rk}_W(A) < \eta\}$ is a pointclass closed under $\exists^\mathbb{R}$. C is a club subset of δ.*

Proof. The first statement follows from Fact 3.4 and Fact 3.5. It remains to show that the set C defined above is a club subset of δ.

If $A \subseteq \mathbb{R}$, then the pointclass $\mathbf{\Sigma}^1_1(A)$ is the smallest nonselfdual pointclass containing A and closed under $\exists^\mathbb{R}$, $\wedge$, and $\vee$. Uniformly from A, one can obtain a universal set U for $\mathbf{\Sigma}^1_1(A)$. (See [13] Section 3.) Note that Δ is also closed under $\exists^\mathbb{R}$, $\forall^\mathbb{R}$, $\wedge$, and $\vee$. Thus if $A \in \Delta$, then $\mathbf{\Sigma}^1_1(A) \subseteq \Delta$.

Let $\gamma < \delta$. Since $\delta = o(\Delta)$, find some $A \in \Delta$ so that $\gamma < \mathsf{rk}_W(A)$. Let A_0 be a universal set for $\mathbf{\Sigma}^1_1(A)$. If A_n has been defined, let A_{n+1} be a universal set for $\mathbf{\Sigma}^1_1(\mathbb{R} \setminus A_n) \subseteq \Gamma$. Let $\xi_n = \mathsf{rk}_W(A_{n+1})$ and note that $\xi_n < \delta$ since $A_{n+1} \in \Delta$. Let $\eta = \sup_{n \in \omega} \xi_n$ and observe that $\eta < \delta$ since δ is regular. The claim is that Υ_η is closed under $\exists^\mathbb{R}$. Suppose $B \subseteq \mathbb{R} \times \mathbb{R}$ and $\mathsf{rk}_W(B) < \eta$. There is an $n < \omega$ so that $\mathsf{rk}_W(B) < \xi_n$. Thus $B \in \mathbf{\Sigma}^1_1(\mathbb{R} \setminus A_n)$. Since $\mathbf{\Sigma}^1_1(\mathbb{R} \setminus A_n)$ is closed under $\exists^\mathbb{R}$, $\exists^\mathbb{R} B \in \mathbf{\Sigma}^1_1(\mathbb{R} \setminus A_n)$ and thus $\mathsf{rk}_W(\exists^\mathbb{R} B) \leq \mathrm{rk}(A_{n+1}) = \xi_n < \eta$. It has been shown that $\eta \in C$ and $\gamma < \eta$. This shows that C is unbounded. Suppose ξ is a limit of elements of C. Suppose $B \subseteq \mathbb{R} \times \mathbb{R}$ and $\mathsf{rk}_W(B) < \xi$. There is a $\xi' < \xi$ with $\xi' \in C$ so that $\mathsf{rk}_W(B) < \xi' < \xi$. Since $B \in \Upsilon_{\xi'}$ and $\Upsilon_{\xi'}$ is closed under $\exists^\mathbb{R}$, $\exists^\mathbb{R} B \in \Upsilon_{\xi'}$. Hence $\mathsf{rk}_W(\exists^\mathbb{R} B) < \xi' < \xi$. Thus Υ_ξ is closed under $\exists^\mathbb{R}$. It has been shown that C is a club. ☐

For any $A \subseteq \mathbb{R}$, an example of such a pointclass closed under $\exists^{\mathbb{R}}$, $\forall^{\mathbb{R}}$, $\wedge$, $\vee$, having the prewellordering property, and containing A is $\mathbf{\Sigma}_1^{L(A,\mathbb{R})}$. Let δ_A denote the first Σ_1-stable ordinal of $L(A,\mathbb{R})$, i.e. the least δ so that $L_\delta(A,\mathbb{R}) \prec_1 L(A,\mathbb{R})$. Arguments from [13] Section 2.4 and [10] Lemma 2.3 imply that $\delta_A = o(\mathbf{\Delta}_1^{L(A,\mathbb{R})}) = \delta(\mathbf{\Sigma}_1^{L(A,\mathbb{R})})$.

Fact 3.7. *(Steel; [15] Theorem 2.1) Assume* $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$*. Let* Γ *be a nonselfdual pointclass and* $\Delta = \Gamma \cap \check{\Gamma}$*. Suppose* $\exists^{\mathbb{R}}\Delta \subseteq \Delta$*. Suppose* $\kappa < \mathrm{cof}(o(\Delta))$*. If* A *is* κ*-Suslin and* $B \in \Gamma$*, then* $A \cap B \in \Gamma$*.*

The next result is a finer version of Steel's result concerning Suslin bounded prewellorderings with a particular emphasis on a bound for the Wadge rank of the desired prewellordering.

Fact 3.8. *(Steel) Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$*. Let* δ *be such that* $\mathrm{cof}(\delta) \geq \omega_2$*. Suppose* κ *is such that* $\kappa > \delta$ *and there is a pointclass* Γ^* *which is closed under* $\forall^{\mathbb{R}}$*,* $\exists^{\mathbb{R}}$*,* $\wedge$*,* $\vee$*, has the prewellordering property, and* $o(\Gamma^*) = o(\Delta^*) = \kappa$*, where* $\Delta^* = \Gamma^* \cap \check{\Gamma}^*$*. Then there is a prewellordering* $(P, \preceq)$ *with the following properties.*

- *The length of* $(P, \preceq)$ *is* δ*. Let* $\varphi : P \to \delta$ *be the associated norm of* $\preceq$*.*
- $\varphi : P \to \delta$ *satisfies* $\mathbf{\Sigma}_2^1$ *bounding, which means that for all* $\mathbf{\Sigma}_2^1$ $S \subseteq P$*, there is a* $\zeta < \delta$ *so that* $\varphi[S] \subseteq \zeta$*.*
- $\mathsf{rk}_W(P) < \kappa$ *and* $\mathsf{rk}_W(\preceq) < \kappa$*.*

Proof. This argument follows the template from [6] Theorem 2.28 with additional complexity calculations. Let $\nu = \mathrm{cof}(\delta)$. Let η be the ν^{th}-element of the club $C \subseteq \kappa$ from Fact 3.6. Then $\Upsilon_\eta = \{A \subseteq \mathbb{R} : \mathsf{rk}_W(A) < \eta\}$ is a pointclass closed under $\exists^{\mathbb{R}}$. Let $B \subseteq \mathbb{R}$ be such that $\mathsf{rk}_W(B) = \eta$. Let $\Gamma = \{A \subseteq \mathbb{R} : A \leq_W B\}$. By a basic property of the Wadge degrees ([16] Corollary 3.5), Γ is nonselfdual because $\mathrm{cof}(\mathsf{rk}_W(B)) > \omega$. Also observe $\Upsilon_\eta = \Gamma \cap \check{\Gamma}$.

Fix some recursive coding of continuous functions $F : \mathbb{R} \to \mathbb{R}$ by $\mathbb{R}$. If $x \in \mathbb{R}$, then let $\Sigma_x : \mathbb{R} \to \mathbb{R}$ denote the continuous function coded by x. Let $\pi : \omega \times \omega \to \omega$ be a recursive bijection. If $x \in \mathbb{R} = {}^\omega\omega$ and $n \in \omega$, then let $x^{[n]}(k) = x(\pi(n,k))$.

Let $E \subseteq \mathbb{R}$ be defined by $x \in E$ if and only if $\Sigma_{x^{[0]}}^{-1}[B] = \mathbb{R} \setminus \Sigma_{x^{[1]}}^{-1}[B]$. Note that $E \in \Delta^*$ since Δ^* is closed under $\vee$, $\wedge$, $\neg$, $\forall^{\mathbb{R}}$, and $\exists^{\mathbb{R}}$. Therefore $\mathsf{rk}_W(E) < \kappa$. Note that if $x \in E$, then $\Sigma_{x^{[0]}}^{-1}[B] = \mathbb{R} \setminus \Sigma_{x^{[1]}}^{-1}[B]$ which implies that $\Sigma_{x^{[0]}}^{-1}[B] \in \Gamma \cap \check{\Gamma} = \Upsilon_\eta$; and for every $A \in \Upsilon_\eta = \Gamma \cap \check{\Gamma}$, there is some $x \in E$ so that $A = \Sigma_{x^{[0]}}^{-1}[B]$. Define a prewellordering $\varphi_0 : E \to \eta$ by $\varphi_0(x) = \mathsf{rk}_W(\Sigma_{x^{[0]}}^{-1}[B])$. For all $x, y \in E$, define $x \preceq_{\varphi_0} y$ if and only if $\varphi_0(x) \leq \varphi_0(y)$. Note that $x \preceq_{\varphi_0} y$ if and only if

$$(\exists^{\mathbb{R}} z)(\Sigma_{x^{[0]}}^{-1}[B] = \Sigma_z^{-1}[\Sigma_{y^{[0]}}^{-1}[B]]).$$

Thus the prewellordering $\preceq_{\varphi_0}$ associated to φ_0 belongs to Δ^* and hence has Wadge rank below κ.

Next, one seeks to show that $\varphi_0 : E \to \eta$ is $\mathbf{\Sigma}_2^1$ bounded. Let $S \subseteq E$ be a $\mathbf{\Sigma}_2^1$ set (so it is also an ω_1-Suslin set). Suppose for the sake of contradiction that $\sup \varphi_0[S] = \eta$. Define $F_0 \subseteq \mathbb{R} \times \mathbb{R}$ by

$$F_0(x,y) \Leftrightarrow (x \in S \wedge \Sigma_{x^{[0]}}(y) \in B) \Leftrightarrow (x \in S \wedge \Sigma_{x^{[1]}}(y) \notin B).$$

Since $\Upsilon_\eta = \Gamma \cap \check{\Gamma}$ is closed under $\exists^{\mathbb{R}}$ and $\omega_1 < \mathrm{cof}(\eta) = \mathrm{cof}(o(\Upsilon_\eta))$, one has that F_0 belongs to Υ_η by applying Fact 3.7 to the ω_1-Suslin set S, the set $B \in \Gamma$, and the set $(\mathbb{R} \setminus B) \in \check{\Gamma}$. Now suppose $A \in \Upsilon_\eta$ and thus $\mathsf{rk}_W(A) < \eta$. Since $\sup \varphi_0[S] = \eta$, there is some $x \in S$ so that $\varphi_0(x) > \mathsf{rk}_W(A)$. Thus $A \leq_W \Sigma_{x^{[0]}}^{-1}[B] = (F_0)_x \leq_W F_0$. This shows that every set in Υ_η is Wadge reducible to F_0. Thus $\mathsf{rk}_W(F_0) \geq \eta$ however $\mathsf{rk}_W(F_0) < \eta$ since $F_0 \in \Upsilon_\eta$. This contradiction shows one must have $\sup \varphi_0[S] < \eta$. It has been shown that there is $\mathbf{\Sigma}_2^1$ bounded prewellordering $\varphi_0 : E \to \eta$ where $\eta < \kappa$, $\mathrm{cof}(\eta) = \nu$, and the Wadge rank of the associated prewellordering $\preceq_{\varphi_0}$ is less than κ.

Since $\mathrm{cof}(\eta) = \nu$, let $\rho_0 : \nu \to \eta$ be an increasing cofinal map. Define $\varphi_1 : E \to \nu$ by $\varphi_1(x)$ is the least $\alpha < \nu$ so that $\rho_0(\alpha) \geq \varphi_0(x)$. Suppose $S \subseteq E$ is $\mathbf{\Sigma}_2^1$. Since φ_0 is $\mathbf{\Sigma}_2^1$ bounded, there is some $\zeta < \eta$ so that $\varphi_0[S] \subseteq \zeta$. Since ρ_0 is cofinal through η, there is some $\xi < \nu$ so that $\rho_0(\xi) > \zeta$. Thus $\varphi_1[S] \subseteq \xi$. Hence φ_1 is also $\mathbf{\Sigma}_2^1$ bounded.

Since $\nu < \kappa = o(\Delta^*) = \delta(\Delta^*)$, there is a norm $\psi_0 : \mathbb{R} \to \nu$ whose associated prewellordering $\preceq_{\psi_0}$ belongs to Δ^*. Define a relation $S \subseteq \mathbb{R} \times E$ by $S(x,y)$ if and only if $\rho_0(\psi_0(x)) = \varphi_0(y)$. By the Moschovakis coding lemma, there is an $R \subseteq S$ so that for all $\alpha < \nu$, $R \cap (\psi_0^{-1}[\{\alpha\}] \times E) \neq \emptyset$ and $R \in \Delta^*$ (in fact $R \in \mathbf{\Sigma}^1_1(\preceq_{\psi_0}) \subseteq \Delta^*$). Note that $x \preceq_{\varphi_1} y$ if and only if

$$(\forall^{\mathbb{R}} a)(\forall^{\mathbb{R}} b)[(R(a,b) \wedge y \preceq_{\varphi_0} b) \Rightarrow x \preceq_{\varphi_0} b].$$

Thus $\preceq_{\varphi_1} \in \Delta^*$ and $\mathsf{rk}_W(\preceq_{\varphi_1}) < \kappa$. It has been shown that there is a prewellordering $\varphi_1 : E \to \nu$ which is $\mathbf{\Sigma}^1_2$ bounded and $\mathsf{rk}_W(\preceq_{\varphi_1}) < \kappa$.

Since $\mathrm{cof}(\delta) = \nu$, let $\rho_1 : \nu \to \delta$ be an increasing cofinal map. Since $o(\Delta^*) = \delta(\Delta^*) = \kappa$ and $\delta < \kappa$, let $\psi_1 : \mathbb{R} \to \delta$ be a surjective map so that $\mathsf{rk}_W(\preceq_{\psi_1}) < \kappa$. Define $S_1 \subseteq E \times \mathbb{R}$ by $S_1(x,y)$ if and only if $\rho_1(\varphi_1(x)) = \psi_1(y)$. By the Moschovakis coding lemma, there is a relation $R_1 \subseteq S_1$ such that $R_1 \in \mathbf{\Sigma}^1_1(\preceq_{\varphi_1}) \subseteq \Delta^*$ and for all $\alpha < \nu$, $R_1 \cap (\varphi_1^{-1}[\{\alpha\}] \times \mathbb{R}) \neq \emptyset$.

For each $\beta < \delta$, define $P_\beta \subseteq \mathbb{R}$ by $x \in P_\beta$ if and only if

$$\psi_1(x^{[0]}) = \beta \wedge x^{[1]} \in E \wedge \beta < \rho_1(\varphi_1(x^{[1]})).$$

Fix some $w \in \mathbb{R}$ so that $\psi_1(w) = \beta$. Then $x \in P_\beta$ if and only if

$$\psi_1(x^{[0]}) = \psi_1(w) \wedge (\exists^{\mathbb{R}} z)(\exists^{\mathbb{R}} v)\Big(x^{[1]} \in E \wedge z \in E \wedge \varphi_1(x^{[1]}) = \varphi_1(z) \wedge R_1(z,v) \wedge \psi_1(w) < \psi_1(v)\Big).$$

Using $\preceq_{\psi_1} \in \Delta^*$, $R_1 \in \Delta^*$, and the closure properties of Δ^*, one has that $P_\beta \in \Delta^*$ for any $\beta < \delta$ and thus $\mathsf{rk}_W(P_\beta) < \kappa$. Let $P = \bigcup_{\beta < \delta} P_\beta$. Since κ is regular, $\sup\{\mathsf{rk}_W(P_\beta) : \beta < \delta\} < \kappa$. Pick a set $D \in \Delta^*$ such that for all $\beta < \delta$, $\mathsf{rk}_W(P_\beta) < \mathsf{rk}_W(D)$. Define $S_2 \subseteq \mathbb{R} \times \mathbb{R}$ by $S_2(x,y)$ if and only if $P_{\psi_1(x)} = \Sigma_y^{-1}[D]$. By the Moschovakis coding lemma, there is a $R_2 \subseteq S_2$ with $R_2 \in \Delta^*$ so that for all $\alpha < \delta$, $R_2 \cap (\psi_1^{-1}[\{\alpha\}] \times \mathbb{R}) \neq \emptyset$. Then $x \in P$ if and only if

$$(\exists^{\mathbb{R}} w)(\exists^{\mathbb{R}} y)(R_2(w,y) \wedge \Sigma_y(x) \in D).$$

Thus $P \in \Delta^*$ and hence $\mathsf{rk}_W(P) < \kappa$.

Define a norm $\varphi : P \to \delta$ by $\varphi(x) = \psi_1(x^{[0]})$. Since $\mathsf{rk}_W(\preceq_{\psi_1}) < \kappa$ and $\mathsf{rk}_W(P) < \kappa$, one has that $\mathsf{rk}_W(\preceq_\varphi) < \kappa$. Let $T \subseteq P$ be $\mathbf{\Sigma}^1_2$. Let $T' = \{x : (\exists y)(y \in T \wedge x = y^{[1]})\}$. Since $T \subseteq P$, one has that T' is a $\mathbf{\Sigma}^1_2$ subset of E. Since φ_1 is $\mathbf{\Sigma}^1_2$ bounded, there is a $\zeta < \nu$ so that $\varphi_1[T'] \subseteq \zeta$. By definition of $P = \bigcup_{\beta < \delta} P_\beta$, one has that $\varphi[T] \subseteq \rho_1(\zeta) < \delta$. It has been shown that $\varphi : P \to \delta$ is a $\mathbf{\Sigma}^1_2$ bounded prewellordering of length δ so that $\mathsf{rk}_W(\leq_\varphi) < \kappa$. This completes the proof. $\square$

Fix a coding of strategies by reals. If $x \in \mathbb{R}$, let $\rho_x : {}^{<\omega}\omega \to \omega$ denote the strategy on ω coded by x. If $w \in \mathrm{WO}_{\geq\omega}$, then $\rho_x^w : {}^{<\omega}\mathrm{ot}(w) \to \mathrm{ot}(w)$ denote the strategy on $\mathrm{ot}(w)$ which results from transferring ρ_x via the bijection $B_w : \omega \to \mathrm{ot}(w)$ naturally induced from w. In this way, one says that (w,x) with $w \in \mathrm{WO}$ and $x \in \mathbb{R}$ codes the strategy ρ_x^w.

Let $\alpha < \omega_1$. A $*$-game on α takes the following form.

1	s_0		s_2		s_4		s_6		$\cdots$	f
2		s_1		s_3		s_5		s_7	$\cdots$	
		z_0		z_1		z_3		z_4		z

For all $i \in \omega$, $s_i \in {}^{<\omega}\alpha$ and $z_i \in \omega$. Player 1 plays s_{2i} for all $i \in \omega$. Player 2 plays s_{2i+1} and z_i for all $i \in \omega$. Let $f = s_0{}^\frown s_1{}^\frown s_2 ...$ and $z \in {}^\omega\omega$ be defined by $z(i) = z_i$. The concept of a $*$-game on α does not include any payoff set or winning conditions. A strategy for Player 1 or Player 2 in a $*$-game on α is merely a strategy satisfying the above conditions for the respective player.

A Banach-Mazur game on $\alpha < \omega_1$ is a $*$-game on α with a certain payoff set: Let $A \subseteq {}^\omega\alpha$ and $B \subseteq {}^\omega\alpha \times {}^\omega\omega$. $G^*_{A,B}$ is a $*$-game on α so that Player 2 wins $G^*_{A,B}$ if and only if $f \in A$ and $B(f,z)$.

The following (in the ω case) is a well-known result concerning the unfolded Banach-Mazur game. If $2 \leq \alpha < \omega_1$, then one has that ${}^\omega\alpha$ is homeomorphic to ${}^\omega\omega$ so the result transfers to the countable ordinal α.

Fact 3.9. ([8] *Theorem 21.8) Let $\alpha < \omega_1$. Let $A \subseteq {}^\omega\alpha$ and $B \subseteq {}^\omega\alpha \times {}^\omega\omega$. Assume $G^*_{A,B}$ is determined (for instance, under* AD*). If A is comeager in ${}^\omega\alpha$ and $A \subseteq \mathrm{dom}(B)$, then Player 2 has a winning strategy in $G^*_{A,B}$. Thus for every $w \in \mathrm{WO}$ with $\mathrm{ot}(w) = \alpha$, there is an $x \in \mathbb{R}$ so that ρ_x^w is a Player 2 winning strategy for $G^*_{A,B}$.*

Fact 3.10. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. *Let* $\langle \delta_\alpha : \alpha < \omega_1 \rangle$ *be a sequence of limit ordinals such that* $\sup\{\delta_\alpha : \alpha < \omega_1\} < \Theta$ *and for all* $\alpha < \omega_1$, $\mathrm{cof}(\delta_\alpha) \geq \omega_2$. *Then there is a sequence* $\langle (P_\alpha, \preceq_\alpha) : \alpha < \omega_1 \rangle$ *so that for all* $\alpha < \omega_1$, $\preceq_\alpha$ *is a prewellordering on* P_α *of length* δ_α *so that the associated surjective norm* $\varphi_\alpha : P_\alpha \to \delta_\alpha$ *is* $\mathbf{\Sigma}^1_2$ *bounded.*

Proof. Let $\delta = \sup\{\delta_\alpha : \alpha < \omega_1\}$ which is less than Θ by assumption. Let $\preceq_\delta$ denote a prewellordering on $\mathbb{R}$ of length δ. Let Γ be a nonselfdual pointclass closed under $\forall^{\mathbb{R}}$, $\exists^{\mathbb{R}}$, $\wedge$, $\vee$, having the prewellordering property, and containing $\preceq_\delta$ (for example, $\mathbf{\Sigma}_1^{L(\preceq_\delta, \mathbb{R})}$). Then $o(\Gamma) > \delta$. Let A^* be a universal set in Γ and note that $\mathsf{rk}_W(A^*) = o(\Gamma)$. Define a relation $S \subseteq \mathrm{WO} \times \mathbb{R}$ by $S(w, x)$ if and only if $\Sigma_x^{-1}[A^*]$ is a $\mathbf{\Sigma}^1_2$ bounded prewellordering of length $\delta_{\mathrm{ot}(w)}$, where recall that $\Sigma_x : \mathbb{R} \to \mathbb{R}$ is the continuous function coded by the real x. Note that $\mathrm{dom}(S) = \mathrm{WO}$ since by Fact 3.8, for each $\alpha < \omega_1$, there is a $\mathbf{\Sigma}^1_2$ bounded prewellordering $(P, \preceq)$ of length δ_α so that $(P, \preceq) \leq_W A^*$. By the Moschovakis coding lemma, there is a $\mathbf{\Sigma}^1_2$ relation $R' \subseteq S$ so that for all $\alpha < \omega_1$, $R' \cap (\mathrm{WO}_\alpha \times \mathbb{R}) \neq \emptyset$. Define $R \subseteq \mathrm{WO} \times \mathbb{R}$ by $R(w, x)$ if and only if $(\exists v)(v \in \mathrm{WO} \wedge \mathrm{ot}(v) = \mathrm{ot}(w) \wedge R'(v, x))$. Note that R is $\mathbf{\Sigma}^1_2$ and $\mathrm{dom}(R) = \mathrm{WO}$. By the scale property on $\mathbf{\Sigma}^1_2$ and its associated uniformization properties, let $\Phi : \mathrm{WO} \to \mathbb{R}$ be a uniformization for R, which means for all $w \in \mathrm{WO}$, $R(w, \Phi(w))$. For each $w \in \mathrm{WO}$, let $(Q_w, \preceq_w)$ denote the $\mathbf{\Sigma}^1_2$ bounded prewellordering of length $\delta_{\mathrm{ot}(w)}$ coded by $\Sigma^{-1}_{\Phi(w)}[A^*]$. Let $\varphi_w : Q_w \to \delta_{\mathrm{ot}(w)}$ be the associated surjective norm of $(Q_w, \preceq_w)$.

For each $\alpha < \omega_1$, let P_α consists of the collection of (w, x) such that $w \in \mathrm{WO}_\alpha$ and $x \in \mathbb{R}$ with the following two properties.

(1) ρ^w_x is a Player 2 strategy for $*$-games on α.
For each $p \in {}^\omega({}^{<\omega}\alpha)$, let $s^p_{2n} = p(n)$. Let $\langle s^p_{2n+1} : n \in \omega \rangle$ and $\langle z^p_i : i \in \omega \rangle$ be the response of Player 2 using ρ^w_x. Let $\mathfrak{f}(w, x, p) = s^p_0 {}^\frown s^p_1 {}^\frown s^p_2 ...$ and let $\mathfrak{z}(w, x, p) \in \mathbb{R}$ be defined by $\mathfrak{z}(w, x, p)(i) = z^p_i$.

(2) There exists a $\beta_{w,x} < \delta_\alpha$ so that for all $p \in {}^\omega({}^{<\omega}\alpha)$,

$$\mathfrak{z}(w, x, p) \in Q_{\mathfrak{G}(\mathfrak{f}(w,x,p))} \text{ and } \varphi_{\mathfrak{G}(\mathfrak{f}(w,x,p))}(\mathfrak{z}(w, x, p)) = \beta_{w,x}$$

where $\mathfrak{G}$ is the generic coding function from Fact 3.2.

Define $\psi_\alpha : P_\alpha \to \delta_\alpha$ by $\psi_\alpha((w, x)) = \beta_{w,x}$. Let $\preceq_\alpha$ be the prewellordering on P_α induced from ψ_α.

Claim 1: ψ_α is surjective.

To see Claim 1: Fix a $\beta < \delta_\alpha$. Let $A = \mathsf{surj}_\alpha$ which is a comeager subset of ${}^\omega\alpha$. Let $B \subseteq \mathsf{surj}_\alpha \times \mathbb{R}$ be defined by $(f, z) \in B$ if and only if $z \in Q_{\mathfrak{G}(f)}$ and $\varphi_{\mathfrak{G}(f)}(z) = \beta$. Since $f \in \mathsf{surj}_\alpha$ implies $\mathfrak{G}(f) = \alpha$, one has that $\mathrm{dom}(B) = \mathsf{surj}_\alpha = A$. Fact 3.9 implies that there is a (w, x) with $w \in \mathrm{WO}_\alpha$ and $x \in \mathbb{R}$ so that ρ^w_x is a Player 2 winning strategy in $G^*_{A,B}$. From the definitions, one has that $(w, x) \in P_\alpha$ and $\psi_\alpha((w, x)) = \beta$.

Claim 2: ψ_α is $\mathbf{\Sigma}^1_2$ bounded.

To see Claim 2: Recall the following basic fact about Banach-Mazur type games. For any $(w, x) \in P_\alpha$, there is a comeager set $K_{w,x}$ of $f \in {}^\omega\alpha$ so that there exists a $p : \omega \to {}^{<\omega}\alpha$ such that $\mathfrak{f}(w, x, p) = f$. For each $f \in K_{w,x}$, there is a canonical $p^{f,w,x} : \omega \to {}^{<\omega}\alpha$ so that $\mathfrak{f}(w, x, p^{f,w,x}) = f$. For a fixed f, there is a Borel function with the property that given a (w, x) such that $f \in K_{w,x}$, the function will output $p^{f,w,x}$.

Let $S \subseteq P_\alpha$ be $\mathbf{\Sigma}^1_2$. Fix an $f \in \mathsf{surj}_\alpha$. Let $S_f = \{(w, x) \in S : (\exists p)(\mathfrak{f}(w, x, p) = f)\}$ which is a $\mathbf{\Sigma}^1_2$ set. For any $(w, x) \in S_f \subseteq S \subseteq P_\alpha$, one has that $\mathfrak{z}(w, x, p^{f,w,x}) \in Q_{\mathfrak{G}(f)}$. Thus $T_f = \{\mathfrak{z}(w, x, p^{f,w,x}) : (w, x) \in S_f\}$ is a subset of $Q_{\mathfrak{G}(f)}$ and is $\mathbf{\Sigma}^1_2$. Since $\varphi_{\mathfrak{G}(f)}$ is $\mathbf{\Sigma}^1_2$ bounded, let γ_f be the least ordinal below δ_α so that $\varphi_{\mathfrak{G}(f)}[T_f] \subseteq \gamma_f$.

For each $\gamma < \delta_\alpha$, let $B_\gamma = \{f \in \mathsf{surj}_\alpha : \gamma_f = \gamma\}$ and $B_{<\gamma} = \{f \in \mathsf{surj}_\alpha : \gamma_f < \gamma\}$. The claim is that there is a γ^* so that $B_{<\gamma^*}$ is comeager. Suppose not, then for all γ, $B_{<\gamma}$ is not comeager. Since $\mathsf{surj}_\alpha \setminus B_{<\gamma} = \bigcup_{\gamma' \geq \gamma} B_\gamma$ and wellordered union of meager sets are meager, one must have some $\gamma' \geq \gamma$ such that $B_{\gamma'}$ is nonmeager. Thus it has been shown that for all $\gamma < \delta_\alpha$, there exists a γ' so that $\gamma \leq \gamma' < \delta_\alpha$ and $B_{\gamma'}$ is nonmeager. Let ϵ_0 be the least ordinal $\epsilon > 0$ so that B_ϵ is nonmeager. Suppose $\beta < \omega_1$ and for all $\iota < \beta$, ϵ_ι has been defined. Let $\zeta_\beta = \sup\{\epsilon_\iota : \iota < \beta\}$ and observe that $\zeta_\beta < \delta_\alpha$ since $\mathrm{cof}(\delta_\alpha) = \omega_2$ and $\beta < \omega_1$. Let ϵ_β be the least ordinal ϵ greater than or equal to ζ_β so that B_ϵ is nonmeager. This defines a sequence $\langle B_{\epsilon_\iota} : \iota < \omega_1 \rangle$ of disjoint nonmeager subsets of ${}^\omega\alpha$. Since AD implies all sets of reals have the Baire property, the existence of this sequence contradicts the countable chain condition of the topology on ${}^\omega\alpha$. Thus it has been shown that there is a γ^* so that $B_{<\gamma^*}$ is comeager.

Now fix a $(w,x) \in S$. Since $K_{w,x}$ and $B_{<\gamma^*}$ are comeager, $K_{w,x} \cap B_{<\gamma^*} \neq \emptyset$. Let $f \in K_{w,x} \cap B_{<\gamma^*}$. Observe that

$$\psi_\alpha((w,x)) = \beta_{w,x} = \varphi_{\mathfrak{G}(\mathfrak{f}(w,x,p^{f,w,x})}(\mathfrak{z}(w,x,p^{f,w,x})) = \varphi_{\mathfrak{G}(f)}(\mathfrak{z}(w,x,p^{f,w,x})) < \gamma_f < \gamma^*.$$

The second equation follows from the definition of $(w,x) \in P_\alpha$. The third equation comes from the fact that $f \in K_{w,x}$ and the definition of $p^{f,w,x}$. The first inequality follows from the fact that $\mathfrak{z}(w,x,p^{f,w,x}) \in T_f$. The second inequality is obtained from $f \in B_{<\gamma^*}$. Thus it has been shown that $\psi_\alpha[S] \subseteq \gamma^*$, and so ψ_α satisfies $\mathbf{\Sigma}^1_2$ bounding. □

4. General Result for All Ordinals

This section will establish the two main results. Theorem 4.4 will show that there is no injection of $[\omega_1]^{<\omega_1}$ into ${}^\omega\mathrm{ON}$. Theorem 4.7 will show that there is no injection of $[\omega_1]^{\omega_1}$ into ${}^{<\omega_1}\mathrm{ON}$.

The following is the good coding system for functions from $\epsilon < \omega_1$ into ω_1. Such coding systems are used to prove partition properties.

Definition 4.1. (Martin, [2] Fact 4.9) Let $\epsilon < \omega_1$. Fix $w^* \in \mathrm{WO}$ so that w^* codes a wellordering with domain ω of ordertype ϵ (assuming without loss of generality that $\omega \leq \epsilon$). For $n \in \omega$, let $\mathrm{ot}(w^*, n) \in \epsilon$ denote the rank of n in the wellordering w^*. For each $\alpha < \epsilon$, let $\mathsf{num}(w^*, \alpha) \in \omega$ denote the integer n so that $\mathrm{ot}(w^*, n) = \alpha$. Define $\mathsf{decode} : \mathbb{R} \to \mathscr{P}(\epsilon \times \omega_1)$ by $\mathsf{decode}(x)(\beta, \gamma)$ if and only if $x^{[\mathsf{num}(w^*,\beta)]} \in \mathrm{WO}_\gamma$. Define $\mathsf{GC}_{\beta,\gamma} \subseteq \mathbb{R}$ by $x \in \mathsf{GC}_{\beta,\gamma}$ if and only if $\mathsf{decode}(x)(\beta,\gamma)$.

Observe that if $x \in \mathsf{GC}_{\beta,\gamma}$, then for any $\xi < \omega_1$, if $\mathsf{decode}(x)(\beta,\xi)$ holds, then one must have that $\gamma = \xi$. By $\mathsf{AC}^{\mathbb{R}}_\omega$, for any $f : \epsilon \to \omega_1$, there is some $x \in \mathbb{R}$ so that $\mathsf{decode}(x)$ is the graph of f.

This defines a good coding system $(\mathbf{\Sigma}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \omega_1)$ for ${}^\epsilon\omega_1$. Let $\mathsf{GC} = \bigcap_{\beta<\epsilon} \bigcup_{\gamma<\omega_1} \mathsf{GC}_{\beta,\gamma}$. Thus for all $x \in \mathsf{GC}$, $\mathsf{decode}(x)$ is the graph of a function $f : \epsilon \to \omega_1$.

Fact 4.2. *([1]) Fix $\epsilon < \omega_1$. Let $(\mathbf{\Sigma}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \omega_1)$ be a fixed good coding system for ${}^\epsilon\omega_1$. For any club $D \subseteq \omega_1$, there is a club $C \subseteq D$ so that $\mathsf{INC}^\epsilon(C)$, which is the set of $x \in \mathsf{GC}$ so that $\mathsf{decode}(x) \in [C]^\epsilon$, is $\mathbf{\Pi}^1_1$.*

If $\epsilon < \omega_1$ and $f : \omega \cdot \epsilon \to \omega_1$, let $\mathsf{block}(f) : \epsilon \to \omega_1$ be defined by $\mathsf{block}(f)(\alpha) = \sup\{f(\omega \cdot \alpha + n) : n \in \omega\}$.

Fact 4.3. *([2] Theorem 3.8; [1]) (Almost Everywhere Good Code Uniformization) Assume $\mathsf{ZF} + \mathsf{AD}$. Let $\epsilon < \omega_1$ and $(\mathbf{\Sigma}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \omega \cdot \epsilon, \gamma < \omega_1)$ be a good coding system for ${}^{\omega\cdot\epsilon}\omega_1$. Suppose $R \subseteq [\omega_1]^\epsilon_* \times \mathbb{R}$. Then there is a club $C \subseteq \omega_1$ and a Lipschitz continuous function $\Xi : \mathbb{R} \to \mathbb{R}$ so that for all $x \in \mathsf{INC}^{\omega\cdot\epsilon}(C)$, $R(\mathsf{block}(\mathsf{decode}(x)), \Xi(x))$.*

Theorem 4.4. *Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. There is no injection of $[\omega_1]^{<\omega_1}$ into ${}^\omega\mathrm{ON}$.*

Proof. Suppose there is an injection $\Phi : [\omega_1]^{<\omega_1} \to {}^\omega\mathrm{ON}$. By the Moschovakis coding lemma, there is a surjection $\pi : \mathbb{R} \to [\omega_1]^{<\omega_1}$. Define $\Psi : \mathbb{R} \times \omega \to \mathrm{ON}$ by $\Psi(r,n) = \Phi(\pi(r))(n)$. Thus $\Psi[\mathbb{R} \times \omega]$ is a surjective image of $\mathbb{R}$. There is a $\delta < \Theta$ so that $\Psi[\mathbb{R} \times \omega]$ has ordertype δ. This implies that that there is an injection $\Phi' : [\omega_1]^{<\omega_1} \to {}^\omega\delta$ where $\delta < \Theta$. Thus it suffices to show that there is no injection $\Phi : [\omega_1]^{<\omega_1} \to {}^\omega\delta$ where $\delta < \Theta$. For the sake of contradiction, fix such an injection Φ. For each $\epsilon < \omega_1$ and $n \in \omega$, let $\Phi^\epsilon_n : [\omega_1]^\epsilon \to \delta$ be defined by $\Phi^\epsilon_n(f) = \Phi(f)(n)$.

Claim: For each $\epsilon < \omega_1$ and $n \in \omega$, there is a club $C \subseteq \omega_1$ so that $|\Phi^\epsilon_n[[C]^\epsilon_*]| \leq \omega_1$.

Given the claim, the theorem follows: If C is a club, let $A^\epsilon_C = \{\xi < \delta : (\exists f \in [C]^\epsilon_*)(\exists n \in \omega)(\Phi(f)(n) = \xi\}$. Fixing an $\epsilon < \omega_1$, one can use $\mathsf{AC}^{\mathbb{R}}_\omega$ and the claim to show that there is a sequence $\langle C_n : n \in \omega\rangle$ so that for all $n \in \omega$, $|\Phi^\epsilon_n[[C_n]^\epsilon_*]| = \omega_1$. Letting $C = \bigcap_{n\in\omega} C_n$, one has that $A^\epsilon_C \subseteq \bigcup_{n\in\omega} \Phi^\epsilon_n[[C_n]^\epsilon_*]$ and hence $|A^\epsilon_C| \leq \omega_1$.

It has been shown that for all $\epsilon < \omega_1$, there is a club C so that $|A^\epsilon_C| \leq \omega_1$. By Fact 2.7, there is a sequence $\langle C_\epsilon : \epsilon < \omega_1\rangle$ so that for all $\epsilon < \omega_1$, $|A^\epsilon_{C_\epsilon}| \leq \omega_1$. Let $T = \bigcup_{\epsilon<\omega_1} A^\epsilon_{C_\epsilon}$ and note that $|T| = \omega_1$. Observe that if $f \in \bigcup_{\epsilon<\omega_1} [C_\epsilon]^\epsilon_*$, then $\Phi(f) \in {}^\omega T$. Since Φ is an injection, $|\bigcup_{\epsilon<\omega_1} [C_\epsilon]^\epsilon_*| = |[\omega_1]^{<\omega_1}|$, and $|T| = \omega_1$, one has that Φ induces an injection of $[\omega_1]^{<\omega_1}$ into ${}^\omega\omega_1$. This is impossible by Fact 2.6.

Thus it remains to show the claim: Now fix $\epsilon < \omega_1$ and $n \in \omega$. Let $S_{-1} = \{\delta\}$. Suppose $k \in \{-1\} \cup \omega$ and $S_k \subseteq \delta + 1$ has been defined with $|S_k| \leq \omega_1$. Let $S^0_k = \{\gamma \in S_k : \gamma \text{ is a successor ordinal}\}$, $S^1_k = \{\gamma \in S_k : \gamma \text{ is a limit ordinal and } \mathrm{cof}(\gamma) \leq \omega_1\}$, and $S^2_k = \{\gamma \in S_k : \gamma \text{ is a limit ordinals and } \mathrm{cof}(\gamma) > \omega_1\}$. (If any of

S^0_k, S^1_k, or S^2_k are empty, then disregard the corresponding case in the following construction. Thus assume all three sets are nonempty.) Let $\iota_0 : \omega_1 \to S^0_k$, $\iota_1 : \omega_1 \to S^1_k$, and $\iota_2 : \omega_1 \to S^2_k$ be surjections.

Let $\varsigma_\xi = \iota_0(\xi)$ which is a successor ordinal. Let ς^*_ξ be the predecessor of ς_ξ.

Let $\nu_\xi = \iota_1(\xi)$. Applying Fact 3.3 to $\langle \nu_\xi : \xi < \omega_1\rangle$, there is a sequence $\langle K_\xi : \xi < \omega_1\rangle$ so that for all $\xi < \omega_1$, $K_\xi \subseteq \nu_\xi$, $|K_\xi| \leq \omega_1$, and $\sup K_\xi = \nu_\xi$.

Let $\delta_\xi = \iota_2(\xi)$. Applying Fact 3.10 to $\langle \delta_\xi : \xi < \omega_1\rangle$, there is a sequence $\langle (P_\xi, \preceq_\xi) : \xi < \omega_1\rangle$ of prewellorderings so that for each ξ, $(P_\xi, \preceq_\xi)$ is a $\mathbf{\Sigma}^1_2$ bounded prewellordering of length δ_ξ with $\varphi_\xi : P_\xi \to \delta_\xi$ being its associated surjective norm.

Fix an $f \in [\omega_1]^\epsilon_*$. Consider the relation $S_f \subseteq \mathrm{WO} \times \mathbb{R}$ defined by $S_f(w, x)$ if and only if $(x \in P_{\mathrm{ot}(w)}) \wedge (\Phi^\epsilon_n(f) < \varphi_{\mathrm{ot}(w)}(x))$. Note that $w \in \mathrm{dom}(S_f)$ if and only if $\Phi^\epsilon_n(f) < \delta_{\mathrm{ot}(w)}$. Fix a Σ^1_2 set $U \subseteq \mathbb{R} \times \mathbb{R}^2$ which is universal for $\mathbf{\Sigma}^1_2$ subsets of $\mathbb{R}^2$. By the Moschovakis coding lemma, there is a $z \in \mathbb{R}$ so that $U_z \subseteq S_f$ and for all $\xi < \omega_1$, $U_z \cap (\mathrm{WO}_\xi \times \mathbb{R}) \neq \emptyset$ if and only if $S_f \cap (\mathrm{WO}_\xi \times \mathbb{R}) \neq \emptyset$. Such a $z \in \mathbb{R}$ will be called an f-selector.

Define $T \subseteq [\omega_1]^\epsilon_* \times \mathbb{R}$ by $T(f, z)$ if and only if z is an f-selector. Note that $\mathrm{dom}(T) = [\omega_1]^\epsilon_*$. After fixing a good coding system $(\mathbf{\Sigma}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \omega \cdot \epsilon, \gamma < \omega_1)$ for ${}^{\omega\cdot\epsilon}\omega_1$ and using Fact 4.3, there is a club $D \subseteq \omega_1$ and a Lipschitz function $\Xi : \mathbb{R} \to \mathbb{R}$ so that for all $x \in \mathsf{Inc}^{\omega\cdot\epsilon}(D)$, $T(\mathsf{block}(\mathsf{decode}(x)), \Xi(x))$. By Fact 4.2, there is a $C' \subseteq D$ so that $\mathsf{Inc}^{\omega\cdot\epsilon}(C')$ is $\mathbf{\Pi}^1_1$. Thus $\Xi[\mathsf{Inc}^{\omega\cdot\epsilon}(C')]$ is a $\mathbf{\Sigma}^1_2$ set. Let $C \subseteq C'$ be the club subset of limit points of C'. Fix a $\xi < \omega_1$. Let

$$V_\xi = \{x : (\exists z)(\exists w \in \mathrm{WO}_\xi)(z \in \Xi[\mathsf{Inc}^{\omega\cdot\epsilon}(C')] \wedge U_z(w, x))\}.$$

V_ξ is $\mathbf{\Sigma}^1_2$ and $V_\xi \subseteq P_\xi$ since all elements of $\Xi[\mathsf{Inc}^{\omega\cdot\epsilon}(C')]$ are f-selectors for some function f (so that there exists an $u \in \mathsf{Inc}^{\omega\cdot\epsilon}(C')$ with $\mathsf{block}(\mathsf{decode}(u)) = f$). Since P_ξ is $\mathbf{\Sigma}^1_2$ bounded, there is a $\gamma < \delta_\xi$ so that $\varphi_\xi[V_\xi] \subseteq \gamma$. Note that if $f \in [C]^\epsilon_*$, then there is a $u \in \mathsf{Inc}^{\omega\cdot\epsilon}(C')$ so that $\mathsf{block}(\mathsf{decode}(u)) = f$. (To see this, because $f \in [C]^\epsilon_*$, there is a $g \in [C']^{\omega\cdot\epsilon}$ so that $\mathsf{block}(g) = f$. Choose $u \in \mathsf{GC}$ so that $\mathsf{decode}(u) = g$.) Then $\Xi(u)$ is an f-selector. If $\Phi^\epsilon_n(f) < \delta_\xi$, then there is some $w \in \mathrm{WO}_\xi$ and $x \in P_\xi$ so that $U_{\Xi(u)}(w, x)$. Hence $x \in V_\xi$ and therefore $\Phi^\epsilon_n(f) < \varphi_\xi(x) < \gamma < \delta_\xi$.

Thus it has been shown that there is a club C so that for all $\xi < \omega_1$, there is an ordinal $\gamma < \delta_\xi$ with $\Phi^\epsilon_n(f) < \gamma$ or $\Phi^\epsilon_n(f) \geq \delta_\xi$ for all $f \in [C]^\epsilon_*$. For each $\xi < \omega_1$, let δ^*_ξ be the least ordinal $\gamma < \delta_\xi$ so that for all $f \in [C]^\epsilon_*$, $\Phi^\epsilon_n(f) < \gamma$ or $\Phi^\epsilon_n(f) \geq \delta_\xi$.

Now let $S_{k+1} = \{\varsigma^*_\xi : \xi < \omega_1\} \cup (\bigcup_{\xi<\omega_1} K_\xi) \cup \{\delta^*_\xi : \xi < \omega_1\}$. Note that S_{k+1} has the following property.

(1) $S_{k+1} \subseteq \delta$ and $|S_{k+1}| = \omega_1$.
(2) If $\xi \in S_k$ is a successor ordinal, then its predecessor belongs to S_{k+1}.
(3) If $\xi \in S_k$ is a limit ordinal with $\mathrm{cof}(\xi) \leq \omega_1$, then $\sup(S_{k+1} \cap \xi) = \xi$.
(4) There is a club C so that for all $\xi \in S_k$ with $\mathrm{cof}(\xi) > \omega_1$, there is a $\xi^* < \xi$ with $\xi^* \in S_{k+1}$ so that for all $f \in [C]^\epsilon_*$, $\Phi^\epsilon_n(f) < \xi^*$ or $\Phi^\epsilon_n(f) \geq \xi$.

The construction of S_{k+1} depends on $S_k \subseteq \delta$ and the surjections ι_0, ι_1, and ι_2. Since $\delta < \Theta$, there is a surjection of $\mathbb{R}$ onto $\mathscr{P}(\delta)$. Thus $\mathsf{DC}_{\mathbb{R}}$ is sufficient to create a sequence $\langle S_k : k \in \omega\rangle$ so that the relation between S_k and S_{k+1} is as specified above. Let $S = \bigcup_{k<\omega} S_k$ and observe that $|S| = \omega_1$.

Using $\mathsf{AC}^{\mathbb{R}}_\omega$, there is a sequence $\langle C_k : k \in \omega\rangle$ which witnesses (4) for S_k (recall $S_{-1} = \{\delta\}$). Let $C = \bigcap_{k\in\omega} C_k$, and one will show that $\Phi^\epsilon_n[[C]^\epsilon_*] \subseteq S$. So suppose otherwise that there is an $f \in [C]^\epsilon_*$ so that $\Phi^\epsilon_n(f) \notin S$. Let $\nu = \Phi^\epsilon_n(f)$. Let $\nu_{-1} = \delta$. Suppose $\nu_k \in S_k$ has been defined with $\nu < \nu_k$. If ν_k is a successor ordinal, then by (2), the predecessor ς of ν_k belongs to S_{k+1}. Since $\nu \notin S$, $\varsigma \in S_{k+1}$, and $\nu < \nu_k$, one has $\nu < \varsigma < \varsigma + 1 = \nu_k$. If $\mathrm{cof}(\nu_k) \leq \omega_1$, then (3) implies there is some ordinal $\zeta \in S_{k+1}$ such that $\nu < \zeta < \nu_k$. If $\mathrm{cof}(\nu_k) > \omega_1$, then by (4) for S_k, there is a $\nu^*_k \in S_{k+1}$ so that $\Phi^\epsilon_n(f) = \nu < \nu^*_k < \nu_k$. Thus in all cases, let ν_{k+1} be the least ordinal $\zeta \in S_{k+1}$ so that $\nu < \zeta < \nu_k$. Then $\langle \nu_k : k \in \omega\rangle$ is an infinite descending sequence of ordinals which is impossible.

It has been shown that there is a club $C \subseteq \omega_1$ so that $\Phi^\epsilon_n[[C]^\epsilon_*] \subseteq S$. Since $|S| \leq \omega_1$, $|\Phi^\epsilon_n[[C]^\epsilon_*]| \leq \omega_1$. This proves the claim. □

The next result is the almost everywhere continuity property for functions of the form $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. It follows from the almost everywhere short length club uniformization for relations of the form $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ which are $\subseteq$-downward closed in the club_{ω_1} coordinate ([4] Theorem 3.10) proved under AD. Unlike

Fact 2.7, the everywhere version of this uniformization fails in $L(\mathbb{R}) \models \mathsf{AD}$ ([4] Fact 3.9); however, the everywhere version does hold under $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ ([4] Theorem 3.7).

Fact 4.5. *([4] Theorem 4.5) Assume $\mathsf{ZF} + \mathsf{AD}$. Let $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. Then there is a club $C \subseteq \omega_1$ so that $\Phi \restriction [C]^{\omega_1}_*$ is continuous. That is, for all $f \in [C]^{\omega_1}_*$, there is an $\alpha < \omega_1$ so that for all $g \in [C]^{\omega_1}_*$, if $f \restriction \alpha = g \restriction \alpha$, then $\Phi(f) = \Phi(g)$.*

Fact 4.6. *([4] Theorem 4.6) Assume $\mathsf{ZF} + \mathsf{AD}$. For all functions $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$, there is an $\alpha < \omega_1$ so that $|\Phi^{-1}[\{\alpha\}]| = |[\omega_1]^{\omega_1}_*|$.*

Proof. By Fact 4.5, there is a club so that $\Phi \restriction [C]^{\omega_1}_*$ is continuous. Pick any $f \in [C]^{\omega_1}_*$ and let $\beta = \Phi(f)$. By continuity, there is an $\alpha < \omega_1$ so that for all $g \in [C]^{\omega_1}_*$ with $f \restriction \alpha = g \restriction \alpha$, $\Phi(g) = \Phi(f) = \beta$. Let $N^C_{f\restriction\alpha} = \{g \in [C]^{\omega_1}_* : g \restriction \alpha = f \restriction \alpha\}$. Note that $|N^C_{f\restriction\alpha}| = |[\omega_1]^{\omega_1}_*|$ and $N^C_{f\restriction\alpha} \subseteq \Phi^{-1}[\{\beta\}]$. Thus $|\Phi^{-1}[\{\beta\}]| = |[\omega_1]^{\omega_1}_*|$. ☐

The above argument is inefficient since it uses the almost everywhere continuity property (Fact 4.5) which follows from a suitable almost everywhere club uniformization property. The proof of this club uniformization property at a cardinal δ requires more than δ being a strong partition cardinal and even more than the existence of a good coding system for ${}^{\delta}\delta$. [1] isolates a strengthening of the good coding system sufficient to prove the necessary club uniformization and the almost everywhere continuity property. However, Fact 4.6 can be proved by purely combinatorial arguments using only the strong partition property. In particular, [1] shows that under ZF, if δ is a cardinal so that $\delta \to_* (\delta)^\delta_2$, $\kappa \in \mathrm{ON}$, and $\Phi : [\delta]^\delta_* \to \kappa$, then there exists an $\alpha < \kappa$ so that $|\Phi^{-1}[\{\alpha\}]| = |[\delta]^\delta_*|$.

Theorem 4.7. *Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. There is no injection of $[\omega_1]^{\omega_1}$ into ${}^{<\omega_1}\mathrm{ON}$.*

Assuming just $\mathsf{ZF} + \mathsf{AD}$, $\neg(|[\omega_1]^{\omega_1}| \leq |{}^{<\omega_1}(\omega_\omega)|)$.

Proof. Since $\mathbb{R}$ surjects onto $[\omega_1]^{\omega_1}$ by the Moschovakis coding lemma, any injection $\Upsilon : [\omega_1]^{\omega_1} \to {}^{<\omega_1}\mathrm{ON}$ induces an injection $\Phi : [\omega_1]^{\omega_1} \to {}^{<\omega_1}\delta$ for some $\delta < \Theta$. If $\sigma \in {}^{<\omega_1}\delta$, then let $\mathsf{length}(\sigma) = |\sigma|$. Let $\Psi : [\omega_1]^{\omega_1} \to \omega_1$ be defined by $\Psi = \mathsf{length} \circ \Phi$. By Fact 4.6 and the fact that $|[\omega_1]^{\omega_1}_*| = |[\omega_1]^{\omega_1}|$, there is an $\epsilon < \omega_1$ so that $|\Psi^{-1}[\{\epsilon\}]| = |[\omega_1]^{\omega_1}|$. So $\Phi : \Psi^{-1}[\{\epsilon\}] \to {}^{\epsilon}\delta$ induces an injection from $[\omega_1]^{<\omega_1}$ into ${}^{\omega}\delta$ since $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$ and $|{}^{\epsilon}\delta| = |{}^{\omega}\delta|$. However this is impossible by Theorem 4.4. The second result is the same argument using Theorem 2.9. ☐

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