

WHEN A RELATION WITH ALL BOREL SECTIONS WILL BE BOREL SOMEWHERE?

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ABSTRACT. In ZFC, if there is a measurable cardinal with infinitely many Woodin cardinals below it, then for every binary relation $R \in L(\mathbb{R})$ on \mathbb{R} with all sections Δ_1^1 (Σ_1^1 or Π_1^1) and every σ -ideal I on \mathbb{R} so that the associated forcing \mathbb{P}_I of I^+ Δ_1^1 subsets is proper, there exists some I^+ Δ_1^1 set C so that $R \cap (C \times \mathbb{R})$ is Δ_1^1 (Σ_1^1 or Π_1^1 , respectively).

1. INTRODUCTION

The basic question of interest is:

Question 1.1. If E is an equivalence relation on ${}^\omega\omega$, is E a simpler equivalence relation when restricted to some subset?

This question can also be asked for graphs and more generally for binary relations. (This paper will state all results for binary relations. This introduction will briefly focus on equivalence relations which is the original form of this question from [6] and [1].)

The measure of complexity will be definability. There are various useful notions of definability given by ideas from topology, recursion theory, logical complexity, and set theory. The class of Borel sets (denoted Δ_1^1) is chosen to be the base of complexity since it is a simple class characterized by all the notions of definability mentioned above. Moreover Δ_1^1 objects seem to be well behaved and relatively well understood. Now the question can be more precisely formulated:

Question 1.2. If E is an equivalence relation on ${}^\omega\omega$, is there a Δ_1^1 set $C \subseteq {}^\omega\omega$ so that $E \upharpoonright C$ is a Δ_1^1 equivalence relation?

Here, $E \upharpoonright C = E \cap (C \times C)$. However, there is one obvious triviality. If C is countable, then any equivalence relation restricted to C is Δ_1^1 . This egregious triviality disappears if one asks that, in the above question, C be Δ_1^1 and non-trivial according to a σ -ideal on ${}^\omega\omega$. Subsets of ${}^\omega\omega$ that are not in the ideal I are called I^+ sets. In this paper, σ -ideals will always contain all the singletons.

However, it is unclear how to approach this question for arbitrary σ -ideals. The collection of available techniques is greatly enriched by considering σ -ideals on ${}^\omega\omega$ so that the associated forcing \mathbb{P}_I of Δ_1^1 I^+ sets is a proper forcing. Considering such σ -ideals makes available powerful tools from models of set theory and absoluteness. (In fact, the questions below all have negative answers for arbitrary σ -ideals. See Section 2.) Analytic (denoted Σ_1^1) sets are continuous images of Δ_1^1 or even closed sets.

Question 1.3. Let E be a Σ_1^1 equivalence relation on ${}^\omega\omega$. Let I be a σ -ideal on ${}^\omega\omega$ so that \mathbb{P}_I is a proper forcing. Is there an I^+ Δ_1^1 set C so that $E \upharpoonright C$ is a Δ_1^1 equivalence relation?

Note that questions like the above are very familiar. For example, the ideal of Lebesgue null set and the ideal of meager sets have the property that their associated forcings are proper forcings. It is very common in mathematics to ask questions about properties that hold on positive measure sets or on non-meager sets. Unfortunately, Question 1.3 has a negative answer:

Proposition 1.4. ([6], Example 4.25) *There is a Σ_1^1 equivalence relation E and a σ -ideal I with \mathbb{P}_I proper so that for all Δ_1^1 I^+ set C , $E \upharpoonright C$ is not Δ_1^1 .*

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A positive answer to any variation of the basic question will likely only be feasible if the equivalence relations bear at least some resemblance to Δ_1^1 equivalence relations. A positive answer does hold for many important examples:

Example 1.5. ([6]) Let I be a σ -ideal on a Polish space X with \mathbb{P}_I proper.

If E is a Σ_1^1 equivalence relation with all classes countable or E is Δ_1^1 reducible to an orbit equivalence relation of a Polish group action, then there is some I^+ Δ_1^1 set C so that $E \upharpoonright C$ is Δ_1^1 .

In both these examples, the equivalence relations have all Δ_1^1 classes. Of course, Δ_1^1 equivalence relations have all Δ_1^1 classes. Perhaps those two examples give evidence that a sufficient resemblance for a positive answer is the property of having all Δ_1^1 classes. [6] asked the following question:

Question 1.6. ([6] Question 4.28) Let E be a Σ_1^1 equivalence relation on ${}^\omega\omega$ with all Δ_1^1 classes. Let I be a σ -ideal on ${}^\omega\omega$ so that \mathbb{P}_I is a proper forcing. Let B be an I^+ Δ_1^1 set. Is there some $C \subseteq B$ which is I^+ Δ_1^1 so that $E \upharpoonright C$ is a Δ_1^1 equivalence relation?

Further examples and partial results suggest that a positive answer is consistent: Let I_{meager} be the ideal of meager sets. Let I_{null} be the ideal of Lebesgue measure zero sets. The associated forcings are Cohen forcing and Random real forcing which both satisfy the countable chain condition and are hence proper.

Example 1.7. ([1]) Assume ZFC + MA + \neg CH. Let I be I_{meager} (or I_{null}). Let E be a Σ_1^1 equivalence relation with all classes Δ_1^1 . Then there exists comeager (or measure 1) Δ_1^1 set C such that $E \upharpoonright C$ is Δ_1^1 .

Example 1.8. ([1]) Let κ be a remarkable cardinal in L . Let G be $\text{Coll}(\omega, < \kappa)$ -generic over L . In $L[G]$, if I is a σ -ideal with \mathbb{P}_I proper and E is a Π_1^1 equivalence relation with all classes countable, then there is a I^+ Δ_1^1 set C such that $E \upharpoonright C$ is Δ_1^1 .

It was then shown that, under large cardinal assumptions, this question has a positive answer:

Theorem 1.9. ([1]. Also see [2].) Suppose for all $X \in H_{(2^{\aleph_0})^+}$, $X^\#$ exists. Then for all Σ_1^1 and Π_1^1 equivalence relations with all Δ_1^1 classes, any σ -ideal I on ${}^\omega\omega$ with \mathbb{P}_I proper, and B an I^+ Δ_1^1 set, there exists some I^+ Δ_1^1 set $C \subseteq B$ so that $E \upharpoonright C$ is Δ_1^1 .

It should be noted that the proofs of Theorem 1.9 in both [1] and [2], an approximation result of Burgess is used: for every Σ_1^1 equivalence relation E there is (in a uniform way) an ω_1 -length decreasing sequence $(E_\alpha : \alpha < \omega_1)$ of Δ_1^1 equivalence relations so that $E = \bigcap_{\alpha < \omega_1} E_\alpha$.

Neeman asked the following generalization of Question 1.6: Projective sets are those obtainable by applying finitely many applications of complements and continuous images starting with the Δ_1^1 sets.

Question 1.10. Assume some large cardinal hypotheses. Let E be a projective equivalence relation with all Δ_1^1 classes. Let I be a σ -ideal on ${}^\omega\omega$ with \mathbb{P}_I proper. Let $B \subseteq {}^\omega\omega$ be an I^+ Δ_1^1 subset. Does there exist some I^+ Δ_1^1 $C \subseteq B$ so that $E \upharpoonright C$ is Δ_1^1 ?

It is unclear if the proofs of Theorem 1.9 can be generalized to give an answer to this question since there does not appear to be any form of Δ_1^1 approximation to arbitrary projective equivalence relations. Moreover, it is known to be consistent that there is a negative answer to Question 1.10 even when restricted to the next level of the projective hierarchy above Σ_1^1 and Π_1^1 .

Example 1.11. In the constructible universe L , there is a Δ_2^1 equivalence relation E_L with all classes countable so that for every σ -ideal I and every I^+ Δ_1^1 set B , $E_L \upharpoonright B$ is not Δ_1^1 .

Proof. The equivalence E_L on ${}^\omega\omega$ is roughly defined by $x E_L y$ if and only if the least admissible level of L for which x and y appear is the same. See [1] or [2] for more details and the complete proof. \square

It is not known what is the status of Question 1.6 or its Π_1^1 analog in L . An interesting open question is whether it is consistent that Question 1.6 or its Π_1^1 analog has a negative answer.

This paper will be concerned with extending a positive answer to these types of questions to larger classes of equivalence relations on ${}^\omega\omega$ with all Δ_1^1 classes. A certain game idea will need to be developed to take the role of Burgess's approximation in Theorem 1.9. Question 1.10 will be answered by an even more general result which will be proved in ZFC augmented by some large cardinal axioms, which are well accepted and

have proven to be very useful in descriptive set theory. The model $L(\mathbb{R})$ is the smallest inner model of ZF containing all the reals of the original universe. It contains all the sets which are “constructible” (in the sense of Gödel) from the reals of the original universe. All projective subsets of ${}^\omega\omega$ belong to $L(\mathbb{R})$. The following is a main result of the paper answering Question 1.10.

Theorem 4.3. Suppose there is a measurable cardinal with infinitely many Woodin cardinals below it. Let I be a σ -ideal on ${}^\omega\omega$ so that \mathbb{P}_I is proper. Let $R \in L(\mathbb{R})$ be a binary relation on ${}^\omega\omega$. If R has all Σ_1^1 (Π_1^1 , Δ_1^1) sections, then for every I^+ Δ_1^1 set B , there is an I^+ Δ_1^1 $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is Σ_1^1 (Π_1^1 , Δ_1^1 , respectively).

Example 1.12. The following is an application: Assume there is a measurable cardinal with infinitely many Woodin cardinals below it. Let I be a σ -ideal on ${}^\omega\omega$ with \mathbb{P}_I proper. Define an equivalence relation E on ${}^\omega\omega$ by $x E y$ if and only if $L(\mathbb{R})^V \models x \in \text{OD}_y \wedge y \in \text{OD}_x$. Note that $E \in L(\mathbb{R})$ and, in fact, is $(\Sigma_1^2)^{L(\mathbb{R})}$. E has all classes countable (and hence Δ_1^1). Theorem 4.3 implies that there is an I^+ Δ_1^1 set C so that $E \upharpoonright C$ is a Δ_1^1 equivalence relation.

Having answered Question 1.10 positively and even given a positive answer for the larger class of $L(\mathbb{R})$ equivalence relation with all Δ_1^1 classes, the ultimate natural question is the following:

Question 1.13. Is it consistent relative to some large cardinals, that (the axiom of choice fails and) for every equivalence relation E with all Δ_1^1 classes and every σ -ideal I on ${}^\omega\omega$ such that \mathbb{P}_I is a proper forcing, there is an I^+ Δ_1^1 set C so that $E \upharpoonright C$ is a Δ_1^1 equivalence relation?

The notion of an absolutely proper forcing is defined in [10]. [10] established an absoluteness result given by an embedding theorem for absolutely proper forcing under determinacy assumptions which is analogous to the proper forcing embedding theorem shown in [11] which holds under AC with large cardinals. [10] used this embedding theorem for absolutely proper forcings to establish a positive answer under AD^+ to a more general form of Question 1.10 for σ -ideals with associated forcing absolutely proper.

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2. BASICS

Definition 2.1. Let I be a σ -ideal on ${}^\omega\omega$. Let $\mathbb{P}_I = (\Delta_1^1 \setminus I, \subseteq, {}^\omega\omega)$ be the forcing of I^+ Δ_1^1 subsets of ${}^\omega\omega$ ordered by $\leq_{\mathbb{P}_I} = \subseteq$ and has largest element $1_{\mathbb{P}_I} = {}^\omega\omega$. Often \mathbb{P}_I is identified with $\Delta_1^1 \setminus I$.

Fact 2.2. ([14], Proposition 2.1.2) Let I be a σ -ideal on ${}^\omega\omega$. There is a name $\dot{x}_{\text{gen}} \in V^{\mathbb{P}_I}$ so that for all \mathbb{P}_I -generic filters G over V and all Δ_1^1 sets B coded in V , $V[G] \models B \in G \Leftrightarrow \dot{x}_{\text{gen}}[G] \in B$.

Definition 2.3. Let I be a σ -ideal on ${}^\omega\omega$. Let $M \prec H_\Xi$ be a countable elementary substructure for some sufficiently large cardinal Ξ . $x \in {}^\omega\omega$ is \mathbb{P}_I -generic over M if and only if the collection $\{B \in \mathbb{P}_I \cap M : x \in B\}$ is a \mathbb{P}_I -generic filter over M .

Proposition 2.4. ([14], Proposition 2.2.2) Let I be a σ -ideal on ${}^\omega\omega$. The following are equivalent:

- (i) \mathbb{P}_I is a proper forcing.
- (ii) For any sufficiently large cardinal Ξ , every $B \in \mathbb{P}_I$, and every countable $M \prec H_\Xi$ with $\mathbb{P}_I \in M$ and $B \in M$, the set $C = \{x \in B : x \text{ is } \mathbb{P}_I\text{-generic over } M\}$ is an I^+ Δ_1^1 set.

This proposition shows that σ -ideals whose associated forcings are proper may be useful for answering Question 1.6 since it indicates how to produce I^+ Δ_1^1 sets. The following example shows that some restrictions on the type of σ -ideals considered in Question 1.6 are necessary:

Let F_{ω_1} denote the countable admissible ordinal equivalence relation defined by $x F_{\omega_1} y$ if and only if $\omega_1^x = \omega_1^y$. F_{ω_1} is a thin Σ_1^1 equivalence relation with all Δ_1^1 classes. Thin means that F_{ω_1} does not have a perfect set of pairwise F_{ω_1} -inequivalent elements. Let I be the σ -ideal which is σ -generated by the F_{ω_1} -classes. Suppose there is an I^+ Δ_1^1 set C so that $F_{\omega_1} \upharpoonright C$ is Δ_1^1 . By definition of I , each F_{ω_1} -class is in I . So since C is I^+ , C must intersect nontrivially uncountably many classes of F_{ω_1} . So $F_{\omega_1} \upharpoonright C$ has uncountable many classes. Since F_{ω_1} is thin, there is also no perfect set of $F_{\omega_1} \upharpoonright C$ inequivalent elements. This contradicts

Silver's dichotomy [12] which states that every Δ_1^1 (even Π_1^1) equivalence relation E on ${}^\omega\omega$ has countably many classes or there exists a perfect set of pairwise E -inequivalent elements.

In this example, I is not proper or even ω_1 -preserving: Let $G \subseteq \mathbb{P}_I$ be a \mathbb{P}_I -generic filter over V . Fact 2.2 implies that $\dot{x}_{\text{gen}}[G]$ is not in any ground model coded Δ_1^1 set in I . $\omega_1^{\dot{x}_{\text{gen}}[G]}$ can not be a countable admissible ordinal of V since if it was countable then a theorem of Sacks shows that there is a $z \in ({}^\omega\omega)^V$ so that $\omega_1^z = \omega_1^{\dot{x}_{\text{gen}}[G]}$. Then $x \in [z]_{F_{\omega_1}}$. By definition of I , $[z]_{F_{\omega_1}}$ is a Δ_1^1 set coded in V that belongs to I . Hence $\omega_1^{\dot{x}_{\text{gen}}[G]}$ must be an uncountable admissible ordinal of V , but in $V[G]$, $\omega_1^{\dot{x}_{\text{gen}}[G]}$ is a countable admissible ordinal. Hence \mathbb{P}_I collapses ω_1 .

Definition 2.5. A measure μ on a set X is a nonprincipal ultrafilter on X . If κ is a cardinal, then μ is κ -complete if and only if for all $\beta < \kappa$ and sequences $(A_\alpha : \alpha < \beta)$ with each $A_\alpha \in \mu$, $\bigcap_{\alpha < \beta} A_\alpha \in \mu$. \aleph_1 -completeness is often called countable completeness. Let $\text{meas}_\kappa(X)$ be the set of all κ -complete ultrafilters on X .

Suppose $\mu \in \text{meas}_{\aleph_1}(<^\omega X)$. By countable completeness, there is a unique m so that ${}^m X \in \mu$. In this case, m is called the dimension of μ and this is denoted by $\dim(\mu) = m$.

Definition 2.6. Let X be a set. For $m \leq n < \omega$, let $\pi_{n,m} : {}^n X \rightarrow {}^m X$ be defined by $\pi_{n,m}(f) = f \upharpoonright m$.

Let $m \leq n < \omega$. Let ν be a measure of dimension m and μ be a measure of dimension n . μ is an extension of ν (or ν is a projection of μ) if and only if for all $A \in \nu$ with $A \subseteq {}^m X$, $\pi_{n,m}^{-1}[A] \in \mu$.

A tower of measures over X is a sequence $(\mu_n : n \in \omega)$ so that

- (i) For all n , $\mu_n \in \text{meas}_{\aleph_1}(<^\omega X)$ and $\dim(\mu_n) = n$.
- (ii) For all $m \leq n < \omega$, μ_n is an extension of μ_m .

A tower of measures over X , $(\mu_n : n \in \omega)$, is countably complete if and only if for all sequence $(A_n : n \in \omega)$ with the property that for all $n \in \omega$, $A_n \in \mu_n$, there exists a $f : \omega \rightarrow X$ so that for all $n \in \omega$, $f \upharpoonright n \in A_n$.

Definition 2.7. A tree T on X is a subset of $<^\omega X$ so that if $s \subseteq t$ and $t \in T$, then $s \in T$.

If $s \in {}^n(X \times Y)$ where $n \in \omega$, then in a natural way, s be may be considered as a pair (s_0, s_1) with $s_0 \in {}^n X$ and $s_1 \in {}^n Y$.

Let T be a tree on X . The body of T , denoted $[T]$, is the set of infinite paths through T , that is $[T] = \{f \in {}^\omega X : (\forall n \in \omega)(f \upharpoonright n \in T)\}$.

Suppose T is a tree on $X \times Y$. For each $s \in <^\omega X$, define $T^s = \{t \in {}^{|\omega|} Y : (s, t) \in T\}$. If $f \in {}^\omega X$, then define $T^f = \bigcup_{n \in \omega} T^{f \upharpoonright n}$.

Let T be a tree on $X \times Y$, then $p[T] = \{f \in {}^\omega X : T^f \text{ is ill-founded}\} = \{f \in {}^\omega X : [T^f] \neq \emptyset\}$.

Definition 2.8. For any $k \in \omega$, $A \subseteq {}^k({}^\omega\omega)$ is Σ_1^1 if and only if there exists a tree on ${}^k\omega \times \omega$ so that $A = p[T]$. $A \subseteq {}^k({}^\omega\omega)$ is Π_1^1 if and only if $A = {}^k({}^\omega\omega) \setminus B$ for some Σ_1^1 set $B \subseteq {}^k({}^\omega\omega)$. $A \subseteq {}^k({}^\omega\omega)$ is Δ_1^1 if and only if A is both Σ_1^1 and Π_1^1 .

Definition 2.9. Let γ be an ordinal and $k \in \omega$. A tree T on ${}^k\omega \times \gamma$ is homogeneous if and only if there is a collection $(\mu_s : s \in <^\omega({}^k\omega))$ so that

- (i) For each $s \in <^\omega({}^k\omega)$, $\mu_s \in \text{meas}_{\aleph_1}(<^\omega\gamma)$ and concentrates on T^s (that is, $T^s \in \mu_s$).
- (ii) For all $s, t \in <^\omega({}^k\omega)$, if $s \subseteq t$, then μ_t is an extension of μ_s .
- (iii) For all $f \in p[T]$, $(u_{f \upharpoonright n} : n \in \omega)$ is a countably complete tower of measures on γ .

A collection $(u_s : s \in <^\omega({}^k\omega))$ which witnesses the homogeneity of T is called a homogeneity system for T . Let κ be a cardinal. The homogeneous tree T is κ -homogeneous if and only if each μ_s is κ -complete.

Definition 2.10. For any $k \in \omega$, $A \subseteq {}^k({}^\omega\omega)$ is homogeneously Suslin if and only if there exists an ordinal γ and a homogeneous tree on ${}^k\omega \times \gamma$ so that $A = p[T]$. If the tree T is κ -homogeneous, then A is said to be κ -homogeneously Suslin.

Later, homogeneity systems will be used to show a certain player has a winning strategy in a particular game using techniques that are very similar to Martin's proof of Σ_1^1 determinacy from a measurable cardinal.

Definition 2.11. Let X be some set. Let $A \subseteq {}^\omega X$. The game associated to A , denoted G_A , is the following: The game has two players, Player 1 and Player 2, who alternately take turns playing elements of X with Player 1 playing first. The picture below denotes a partial play where Player 1 plays the sequence $(a_i : i \in \omega)$

and Player 2 plays the sequence $(b_i : i \in \omega)$.

$$\frac{a_0 \quad a_1 \quad \dots \quad a_{k-1}}{b_0 \quad b_1 \quad \dots \quad b_{k-1}}$$

Player 2 is said to win this play of G_A if and only if the infinite sequence $(a_0 b_0 a_1 b_1 \dots) \in A$. Otherwise Player 1 wins.

A function $\tau : {}^{<\omega}X \rightarrow X$ is a winning strategy for Player 1 if and only if for all sequence $(b_i : i \in \omega)$ played by Player 2, Player 1 wins by playing $(a_i : i \in \omega)$ where this sequence is defined recursively by $a_0 = \tau(\emptyset)$ and $a_{k+1} = \tau(a_0 b_0 \dots a_k b_k)$. A winning strategy $\tau : {}^{<\omega}X \rightarrow X$ for Player 2 is defined similarly. The game G_A is determined if Player 1 or Player 2 has a winning strategy.

Let X be a set. ${}^\omega X$ is given the topology with basis $\{U_s : s \in {}^{<\omega}X\}$, where $U_s = \{f \in {}^\omega X : s \subseteq f\}$.

Fact 2.12. ([4], Gale-Stewart) *If X is wellorderable and $A \subseteq {}^\omega X$ is open, then G_A is determined. Hence if A is closed, then G_A is also determined.*

3. THE GAME

Assume ZFC which includes the axiom of choice.

Definition 3.1. If R be a relation on $({}^\omega \omega)^2$, then let $R_x = \{y : (x, y) \in R\}$.

Let S be a homogeneous tree on $\omega \times \omega \times \gamma$, where γ is some ordinal. Denote $p[S]$ by R_S . Denote $({}^\omega \omega \times {}^\omega \omega) \setminus p[S] = R^S$.

Definition 3.2. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$ for some ordinal γ . Let I be a σ -ideal on ${}^\omega \omega$ so that \mathbb{P}_I is proper.

Let **A** assert that $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} \check{S}$ is a homogeneous tree.

Statement **A** just asserts that the tree S remains homogeneous in \mathbb{P}_I -generic extensions. Note that **A** is true since the completeness of countably complete measures is a measurable cardinal and $|\mathbb{P}_I|$ is always less than a measurable cardinal under AC.

Definition 3.3. Let S be a homogeneously Suslin tree on $\omega \times \omega \times \gamma$ for some ordinal γ . Let I be a σ -ideal on ${}^\omega \omega$ such that \mathbb{P}_I is a proper forcing.

Let D_Σ be the formula on ${}^\omega \omega \times {}^\omega \omega$ asserting:

$$D_\Sigma(x, T) \Leftrightarrow (T \text{ is tree on } \omega \times \omega) \wedge (\forall y)(R_S(x, y) \Leftrightarrow T^y \text{ is ill-founded})$$

Let D_Π be the formula on ${}^\omega \omega \times {}^\omega \omega$ asserting:

$$D_\Pi(x, T) \Leftrightarrow (T \text{ is a tree on } \omega \times \omega) \wedge (\forall y)(\neg(R^S(x, y)) \Leftrightarrow T^y \text{ is ill-founded})$$

If $D_\Sigma(x, T)$ holds, then T is a tree which witnesses $(R_S)_x$ is Σ_1^1 . Similarly, if $D_\Pi(x, T)$ holds, then T is a tree which witnesses ${}^\omega \omega \setminus (R^S)_x$ is Σ_1^1 , i.e. $(R^S)_x$ is Π_1^1 .

Definition 3.4. Let S be a homogeneously Suslin tree on $\omega \times \omega \times \gamma$ for some ordinal γ . Let I be a σ -ideal on ${}^\omega \omega$ such that \mathbb{P}_I is a proper forcing.

Let **B $_\Sigma$** say: $(\forall x)(\exists T)D_\Sigma(x, T)$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} (\forall x)(\exists T)D_\Sigma(x, T)$.

Let **B $_\Pi$** say: $(\forall x)(\exists T)D_\Pi(x, T)$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} (\forall x)(\exists T)D_\Pi(x, T)$.

B $_\Sigma$ states that all R_S sections are Σ_1^1 and all R_S sections remain Σ_1^1 in \mathbb{P}_I -generic extensions. Similarly, **B $_\Pi$** states that all R^S sections are Π_1^1 and all R^S sections remain Π_1^1 in \mathbb{P}_I -generic extensions.

Definition 3.5. Let S be a homogeneously Suslin tree on $\omega \times \omega \times \gamma$ for some ordinal γ . Let I be a σ -ideal on ${}^\omega \omega$ such that \mathbb{P}_I is a proper forcing.

Let **C $_\Sigma$** state: There is an ordinal ϵ and a tree U on $\omega \times \omega \times \epsilon$ so that $p[U] = \{(x, T) : D_\Sigma(x, T)\}$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x, T) : D_\Sigma(x, T)\}$.

Let **C $_\Pi$** state: There is an ordinal ϵ and a tree U on $\omega \times \omega \times \epsilon$ so that $p[U] = \{(x, T) : D_\Pi(x, T)\}$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x, T) : D_\Pi(x, T)\}$.

C $_\Sigma$ states that the set defined by D_Σ has a tree representation that continues to represent the formula D_Σ in \mathbb{P}_I -generic extensions. **C $_\Pi$** is similar. The following game plays an important role the next theorem.

Definition 3.6. Let S be a tree on $\omega \times \omega \times \gamma$. Let T be a tree on $\omega \times \omega$. Let $g \in {}^\omega\omega$. Consider the following game $G^{g,T}$:

$$\frac{m_0, n_0 \quad m_1, n_1 \quad \dots \quad m_{k-1}, n_{k-1}}{\alpha_0 \quad \alpha_1 \quad \dots \quad \alpha_{k-1}}$$

The rules are:

- (1) Player 1 plays $m_i, n_i \in \omega$. Player 2 plays $\alpha_i < \gamma$.
- (2) $(m_0 \dots m_{k-1}, n_0 \dots n_{k-1}) \in T$
- (3) $(g \upharpoonright k, m_0 \dots m_{k-1}, \alpha_0 \dots \alpha_{k-1}) \in S$.

The first player to violate these rules loses. If the game continues forever, then Player 2 wins. This game is open for Player 1 and hence closed for Player 2.

The following shows under certain assumptions a more general canonicalization property holds for relations. [2] defines this phenomenon as the rectangular canonization property.

Theorem 3.7. Let γ be an ordinal. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$. Let I be a σ -ideal on ${}^\omega\omega$ so that \mathbb{P}_I is proper. Assume \mathbf{A} , \mathbf{B}_Σ , and \mathbf{C}_Σ hold for S and I . Then for any I^+ Δ_1^1 set $B \subseteq {}^\omega\omega$, there exists an I^+ Δ_1^1 set $C \subseteq B$ so that $R_S \cap (C \times {}^\omega\omega)$ is a Σ_1^1 relation.

Proof. Let U be the tree on $\omega \times \omega \times \epsilon$ witnessing \mathbf{C}_Σ for S and I . Let $M \prec H_\Xi$ be a countable elementary substructure with Ξ sufficiently large and $B, I, \mathbb{P}_I, S, U \in M$.

Claim 1 : Let g be \mathbb{P}_I -generic over M . If $x, T \in M[g]$ and $M[g] \models D_\Sigma(x, T)$, then $V \models D_\Sigma(x, T)$.

Proof of Claim 1: By \mathbf{C}_Σ for S and I and the fact that $M \prec H_\Xi$, $M[g] \models D_\Sigma(x, T)$ implies $M[g] \models (x, T) \in p[U]$. There exists some $f \in M[g]$ with $f : \omega \rightarrow \epsilon$ so that $M[g] \models (x, T, f) \in [U]$. Hence for each $n \in \omega$, $M[g] \models (x \upharpoonright n, T \upharpoonright n, f \upharpoonright n) \in U$. For each $n \in \omega$, $(x \upharpoonright n, T \upharpoonright n, f \upharpoonright n) \in M$. So by absoluteness, $M \models (x \upharpoonright n, T \upharpoonright n, f \upharpoonright n) \in U$. For all $n \in \omega$, $V \models (x \upharpoonright n, T \upharpoonright n, f \upharpoonright n) \in U$. $V \models (x, T) \in p[U]$. $V \models D_\Sigma(x, T)$.

Now fix a $g \in {}^\omega\omega$ so that g is \mathbb{P}_I -generic over M . As $M \prec H_\Xi$, $M \models (\forall x)(\exists T)D_\Sigma(x, T)$. $M[g] \models (\forall x)(\exists T)D_\Sigma(x, T)$ by \mathbf{B}_Σ and the fact that $M \prec H_\Xi$. So fix a tree T on $\omega \times \omega$ so that $M[g] \models D_\Sigma(g, T)$.

Claim 2 : In $M[g]$, Player 2 has a winning strategy in the game $G^{g,T}$.

Proof of Claim 2: Work in $M[g]$: By an appropriate coding, $G^{g,T}$ is equivalent to a game G_A , where $A \subseteq {}^\omega\gamma$ is an open subset. Suppose Player 2 does not have a winning strategy. By Fact 2.12, Player 1 must have a winning strategy τ^* . By \mathbf{A} , S is a homogeneous tree in $M[g]$. Let $(\mu_t : t \in {}^{<\omega}(\omega \times \omega))$ be a homogeneity system witnessing the homogeneity of S .

Now two sequences of natural numbers, $(a_i : i \in \omega)$ and $(b_i : i \in \omega)$, and a sequence $(A_n : n \in \omega)$ so that $A_n \subseteq {}^n\gamma$ will be constructed (in $M[g]$) by recursion:

Let $a_0, b_0 \in \omega$ so that $(a_0, b_0) = \tau^*(\emptyset)$. Let $A_0 = \{\emptyset\}$.

Suppose $a_0, \dots, a_{k-1}, b_0, \dots, b_{k-1}$, and A_0, \dots, A_{k-1} has been constructed. Let $h_k : S^{(g \upharpoonright k, a_0 \dots a_{k-1})} \rightarrow \omega \times \omega$ be defined by

$$h_k(\beta_0 \dots \beta_{k-1}) = \tau^*(a_0, b_0, \beta_0, \dots, a_{k-1}, b_{k-1}, \beta_{k-1})$$

$\mu_{(g \upharpoonright k, a_0 \dots a_{k-1})}$ concentrates on $S^{(g \upharpoonright k, a_0 \dots a_{k-1})}$ and is countably complete; therefore, there is a unique (a_k, b_k) so that $h_k^{-1}[\{(a_k, b_k)\}] \in \mu_{(g \upharpoonright k, a_0 \dots a_{k-1})}$. Let $A_k = h_k^{-1}[\{(a_k, b_k)\}]$.

This completes the construction of $(a_i : i \in \omega)$, $(b_i : i \in \omega)$, and $(A_i : i \in \omega)$.

Let $L \in {}^\omega(\omega \times \omega)$ be such that for all $i \in \omega$, $L(i) = (a_i, b_i)$. Note that $L \in [T]$. To see this, suppose not. Then there is some least $k \in \omega$ so that $L \upharpoonright (k+1) = (a_0 \dots a_k, b_0 \dots b_k) \notin T$. For $i \leq k$, define $\mu_i = \mu_{g \upharpoonright i, a_0 \dots a_{i-1}}$. For $0 \leq i \leq j \leq k$, let $\pi_{j,i} : {}^j\gamma \rightarrow {}^i\gamma$ be defined by $\pi_{j,i}(s) = s \upharpoonright i$. By definition of the homogeneity system for S , for $0 \leq i \leq j \leq k$, μ_j is an extension of μ_i . Hence for all $0 \leq i \leq k$, $\pi_{k,i}^{-1}[A_i] \in \mu_k$. By countable completeness of μ_k , $\bigcap_{0 \leq i \leq k} \pi_{k,i}^{-1}[A_i] \in \mu_k$. Let $(\beta_0 \dots \beta_{k-1}) \in \bigcap_{0 \leq i \leq k} \pi_{k,i}^{-1}[A_i]$. Consider the following play of $G^{g,T}$ where player 1 uses the strategy τ^* and Player 2 plays $(\beta_0 \dots \beta_{k-1})$:

$$\frac{a_0, b_0 \quad a_1, b_1 \quad \dots \quad a_{k-1}, b_{k-1} \quad a_k, b_k}{\beta_0 \quad \beta_1 \quad \dots \quad \beta_{k-1}}$$

Note that for all $0 \leq i \leq k$, $(\beta_0 \dots \beta_{i-1}) \in A_i = h_i^{-1}[\{(a_i, b_i)\}] \subseteq S^{(g \upharpoonright i, a_0 \dots a_{i-1})}$. So rule (3) of the game $G^{g,T}$ is not violated by Player 2. However, $(a_0 \dots a_k, b_0 \dots b_k) = L \upharpoonright (k+1) \notin T$. Player 1 violates rule (2) and is the first player to violate any rules. Player 1 loses this game. This contradicts the assumption that τ^* is a winning strategy for Player 1. So this completes the proof that $L \in [T]$.

Let $\mathbf{a} = (a_i : i \in \omega)$. Since $L \in [T]$ and $D_\Sigma(g, T)$, this implies that $R_S(g, \mathbf{a})$. Now let $J \in {}^\omega(\omega \times \omega)$ be such that for all $k \in \omega$, $J \upharpoonright k = (g \upharpoonright k, a_0 \dots a_{k-1})$. Then by definition of S , $J \in p[S]$. Since S is a homogeneous tree via $(u_t : t \in {}^{<\omega}(\omega \times \omega))$, $(\mu_{J \upharpoonright k} : k \in \omega)$ is a countably complete tower of measures.

Each $A_k \in \mu_{g \upharpoonright k, a_0 \dots a_{k-1}} = \mu_{J \upharpoonright k}$. So by the countable completeness of the tower, there exists some $\Phi : \omega \rightarrow \gamma$ so that for all $k \in \omega$, $\Phi \upharpoonright k \in A_k$. Now consider the play of $G^{g,T}$ where Player 1 uses its winning strategy τ^* and Player 2 plays Φ . By construction of the sequences $(a_i : i \in \omega)$, $(b_i, i \in \omega)$, and $(A_i : i \in \omega)$, a finite partial play of the game looks as follows:

$$\frac{a_0, b_0 \quad a_0, b_1 \quad \dots \quad a_{k-1}, b_{k-1}}{\Phi(0) \quad \Phi(1) \quad \dots \quad \Phi(k-1)}$$

Neither players violate any rules in this play. Hence the game continues forever, and so Player 2 wins this play of $G^{g,T}$. This contradicts the fact that τ^* was a winning strategy for Player 1.

So Player 1 could not have had a winning strategy. Player 2 must have a winning strategy in $G^{g,T}$. This completes the proof of Claim 2.

By Claim 2, fix a Player 2 winning strategy $\tau \in M[g]$.

Claim 3 : τ is a winning strategy for $G^{g,T}$ in V .

Proof of Claim 3: Suppose the following is a play of $G^{g,T}$ (in V) in which Player 2 uses τ and loses

$$\frac{m_0, n_0 \quad m_1, n_1 \quad \dots \quad m_{k-1}, n_{k-1}}{\alpha_0 \quad \alpha_1 \quad \dots \quad \alpha_{k-1}}$$

Since $\tau \in M[g]$ and ${}^{<\omega}\omega \subseteq M[g]$, this entire finite play belongs to $M[g]$. So, Player 2 loses this game in $M[g]$, as well. This contradicts τ being a winning strategy in $M[g]$. This completes the proof of Claim 3.

Claim 4 : For all $y \in {}^\omega\omega$, $R_S(g, y)$ if and only if $(S \cap M)^{(g,y)}$ is ill-founded.

Proof of Claim 4: By Claim 1, $M[g] \models D_\Sigma(g, T)$ implies $V \models D_\Sigma(g, T)$. Hence in V , T gives the Σ_1^1 definition of $(R_S)_g$.

Suppose $R_S(g, y)$. Then T^y is ill-founded. Let $f \in [T^y]$. Consider the following play of the game $G^{g,T}$ where Player 1 plays y and f , and Player 2 responds using its winning strategy τ .

$$\frac{y(0), f(0) \quad y(1), f(1) \quad \dots \quad y(k-1), f(k-1)}{\alpha_0 \quad \alpha_1 \quad \dots \quad \alpha_{k-1}}$$

Since $f \in [T^y]$, Player 1 can not lose. Since τ is a winning strategy for Player 2, Player 2 also does not lose at a finite stage. Hence Player 2 wins by having the game continue forever. Let $\Phi : \omega \rightarrow \gamma$ be the sequence coming from Player 2's response, i.e. for all k , $\Phi(k) = \alpha_k$.

Since $\tau \in M[g]$ and ${}^{<\omega}\omega \subseteq M[g]$, each finite partial play of $G^{g,T}$ above belongs to $M[g]$. Hence $\Phi \upharpoonright k \in M[g]$ for all $k \in \omega$. As $\text{On}^M = \text{On}^{M[g]}$, $(g \upharpoonright k, y \upharpoonright k, \Phi \upharpoonright k) \in (S \cap M)$ for all $k \in \omega$.

It has been shown that $R_S(g, y)$ implies $(S \cap M)^{(g,y)}$ is ill-founded. Of course, if $(S \cap M)^{(g,y)}$ is ill-founded, then $S^{(g,y)}$ is ill-founded. By definition, $R_S(g, y)$. This completes the proof of Claim 4.

Let $b : \omega \rightarrow \text{On}^M$ be a bijection. Define a new tree S' on $\omega \times \omega \times \omega$ by $(s_1, s_2, s_3) \in S' \Leftrightarrow (s_1, s_2, b \circ s_3) \in S$. By Fact 2.4, let $C \subseteq B$ be the I^+ Δ_1^1 set of \mathbb{P}_I -generic reals over M inside B . By Claim 4, for all $y \in {}^\omega\omega$, $R_S(g, y) \Leftrightarrow (S')^{(g,y)}$ is ill-founded. $R_S \cap (C \times {}^\omega\omega)$ is Σ_1^1 . The proof of the theorem is complete. \square

Theorem 3.8. *Let γ be an ordinal. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$. Let I be a σ -ideal on ${}^\omega\omega$ so that \mathbb{P}_I is proper. Assume \mathbf{A} , \mathbf{B}_\square , and \mathbf{C}_\square hold for S and I . Then for any I^+ Δ_1^1 set $B \subseteq {}^\omega\omega$, there exists an I^+ Δ_1^1 set $C \subseteq B$ so that $R^S \cap (C \times {}^\omega\omega)$ is a Π_1^1 relation.*

Proof. The proof of this is very similar to the proof of Theorem 3.7. \square

Theorem 3.9. *Let γ and ν be ordinals. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$. Let U be a homogeneous tree on $\omega \times \omega \times \nu$. Suppose $p[S] = ({}^\omega\omega \times {}^\omega\omega) \setminus p[U]$. Let $R = R_S = R^U$. Let I be a σ -ideal on ${}^\omega\omega$ such that \mathbb{P}_I is a proper forcing. Suppose \mathbf{A} , \mathbf{B}_Σ , and \mathbf{C}_Σ holds for S and I . Suppose \mathbf{A} , \mathbf{B}_Π , and \mathbf{C}_Π holds for U and I .*

Then for any I^+ Δ_1^1 set $B \subseteq {}^\omega\omega$, there exists an I^+ Δ_1^1 set $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is a Δ_1^1 relation.

Proof. By Theorem 3.7, there is some I^+ Δ_1^1 set $C' \subseteq B$ so that $R \cap (C' \times {}^\omega\omega)$ is Σ_1^1 . By Theorem 3.8, there is some I^+ Δ_1^1 set $C \subseteq C'$ so that $R \cap (C \times {}^\omega\omega)$ is Π_1^1 . Therefore, $R \cap (C \times {}^\omega\omega)$ is Δ_1^1 . \square

If the above assumptions holds and $R_S = E$ defines an equivalence relation with all Σ_1^1 classes, then there is some I^+ Δ_1^1 set so that $E \upharpoonright C$ is an Σ_1^1 equivalence relation. Similarly, suppose $R_S = G$ is a graph on ${}^\omega\omega$. Then $G_x = \{y : x G y\}$ is the set of neighbors of x . Suppose G_x is Σ_1^1 for all x . Then there is an I^+ Δ_1^1 set C so that the induced subgraph $G \upharpoonright C$ is an Σ_1^1 graph.

4. CANONICALIZATION FOR RELATIONS IN $L(\mathbb{R})$

The main results of this section are the following. The definition of some terms are stated further below.

Theorem 4.1. *Let λ be a limit of Woodin cardinals. Let I be a σ -ideal on ${}^\omega\omega$ so that \mathbb{P}_I is proper. Let $R \in \text{Hom}_{<\lambda}$ be a binary relation on ${}^\omega\omega$. If R has all Σ_1^1 (Π_1^1 or Δ_1^1) sections, then for every I^+ Δ_1^1 set B , there is an I^+ Δ_1^1 $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is Σ_1^1 (Π_1^1 or Δ_1^1 , respectively).*

Theorem 4.2. *Suppose there are infinitely many Woodin cardinals. Let I be a σ -ideal on ${}^\omega\omega$ so that \mathbb{P}_I is proper. Let R be a projective binary relation on ${}^\omega\omega$. If R has all Σ_1^1 (Π_1^1 , Δ_1^1) sections, then for every I^+ Δ_1^1 set B , there is an I^+ Δ_1^1 $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is Σ_1^1 (Π_1^1 , Δ_1^1 , respectively).*

Theorem 4.3. *Suppose there is a measurable cardinal with infinitely many Woodin cardinals below it. Let I be a σ -ideal on ${}^\omega\omega$ so that \mathbb{P}_I is proper. Let $R \in L(\mathbb{R})$ be a binary relation on ${}^\omega\omega$. If R has all Σ_1^1 (Π_1^1 , Δ_1^1) sections, then for every I^+ Δ_1^1 set B , there is an I^+ Δ_1^1 $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is Σ_1^1 (Π_1^1 , Δ_1^1 , respectively).*

This section will provide a brief description of the theory of tree representations of subsets of ${}^\omega\omega$ and absoluteness which will be used to indicate some circumstances in which the \mathbf{A} , \mathbf{B}_Σ , \mathbf{C}_Σ , \mathbf{A} , \mathbf{B}_Π , and \mathbf{C}_Π hold.

Definition 4.4. Let κ be a cardinal. A κ -weak homogeneity system with support some ordinal γ is a sequence of κ -complete measures on ${}^{<\omega}\gamma$, $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$, so that

- (i) If $s \neq t$, then $\mu_s \neq \mu_t$.
- (ii) $\dim(\mu_s) \leq |s|$.
- (iii) If μ_s is an extension of some measure ν , then there exists some $k < |s|$ so that $\mu_{s \upharpoonright k} = \nu$.

Define $W_{\bar{\mu}}$ by

$$W_{\bar{\mu}} = \{x \in {}^\omega\omega : (\exists f \in {}^\omega\omega)(f \text{ is an increasing sequence } \wedge (\mu_{x \upharpoonright f(k)} : k \in \omega) \text{ is a countably complete tower})\}$$

A set $A \subseteq {}^\omega\omega$ is κ -weakly homogeneous if and only there is a κ -weak homogeneity system $\bar{\mu}$ so that $A = W_{\bar{\mu}}$.

Definition 4.5. Let γ be an ordinal. A tree on $\omega \times \gamma$ is κ -weakly homogeneous if and only there is some κ -weak homogeneity system $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ so that $p[T] = W_{\bar{\mu}}$ and for all $s \in {}^{<\omega}\omega$, there is some $k \leq |s|$ so that μ_s concentrates on $T^{s \upharpoonright k}$. $A \subseteq {}^\omega\omega$ is κ -weakly homogeneously Suslin if and only if $A = p[T]$ for some tree T which is κ -weakly homogeneous.

Fact 4.6. ([13], Proposition 1.12.) *If $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ is a κ -weak homogeneity system with support γ , then there is a tree T on $\omega \times \gamma$ so that $\bar{\mu}$ witnesses T is κ -weakly homogeneously Suslin. Hence a set is κ -weakly homogeneous if and only if it is κ -weakly homogeneously Suslin.*

Definition 4.7. Let μ be a countably complete measure on ${}^{<\omega}X$. Let M_μ be the Mostowski collapse of the ultrapower $\text{Ult}(V, \mu)$. Let $j_\mu : V \rightarrow M_\mu$ be the composition of the ultrapower map and the Mostowski collapse map.

Suppose ν and μ are countably complete measures on ${}^{<\omega}X$. Suppose for some $m \leq n$, $\dim(\mu) = m$ and $\dim(\nu) = n$, and ν is an extension of μ . Define $\Lambda_{m,n} : {}^m X V \rightarrow {}^n X V$ by $\Lambda_{m,n}(f)(s) = f(s \upharpoonright m)$ for each

$s \in {}^n X$. Define an elementary embedding $\text{Ult}(V, \mu) \rightarrow \text{Ult}(V, \nu)$ by $[f]_\mu \mapsto [\Lambda_{m,n}(f)]_\nu$. This induces an elementary embedding $j_{\mu,\nu} : M_\mu \rightarrow M_\nu$.

Definition 4.8. Let γ and θ be ordinals. Let $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ be a weak homogeneity system with support γ . The Martin-Solovay tree with respect to $\bar{\mu}$ below θ , denoted $\text{MS}_\theta(\bar{\mu})$, is a tree on $\omega \times \theta$ defined by: for all $s \in {}^{<\omega}\omega$ and $h \in {}^{|\mathit{s}|\theta}$

$$(s, h) \in \text{MS}_\theta(\bar{\mu}) \Leftrightarrow (\forall i < j < |\mathit{s}|)(\mu_{s \upharpoonright j} \text{ is an extension of } \mu_{s \upharpoonright i} \Rightarrow j_{\mu_{s \upharpoonright i}, \mu_{s \upharpoonright j}}(h(i)) > h(j))$$

If $(u_n : n \in \omega)$ is a tower of measure, then the tower is countably complete if and only if the directed limit of the directed system $(M_{\mu_i} : j_{\mu_i, \mu_j} : i < j < \omega)$ is well-founded. Suppose $x \in p[\text{MS}_\theta(\bar{\mu})]$. If $(x, \Phi) \in [\text{MS}_\theta(\bar{\mu})]$, then Φ witnesses in a continuous way that the directed limit model is ill-founded. This shows that $x \in p[\text{MS}_\theta(\bar{\mu})]$ implies that $x \notin W_{\bar{\mu}}$. In fact, the converse is also true giving the following result:

Fact 4.9. *Let κ be a cardinal. Suppose $\bar{\mu}$ is a κ -weak homogeneity system with support γ . Then if $\theta > |\gamma|^+$, then $p[\text{MS}_\theta(\bar{\mu})] = {}^\omega\omega \setminus W_{\bar{\mu}}$.*

Proof. See [13] Lemma 1.19, [7] Fact 1.3.12, or [5] Theorem 4.10. □

Let μ be a κ -complete ultrafilter on some set X . Let \mathbb{P} be a forcing with $|\mathbb{P}| < \kappa$. Let $G \subseteq \mathbb{P}$ be \mathbb{P} -generic over V . In $V[G]$, define $\mu^* \subseteq \mathcal{P}(X)$ by $A \in \mu^*$ if and only there exists a $B \in \mu$ so that $B \subseteq A$. In $V[G]$, μ^* is a κ -complete ultrafilter on X . Suppose $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ is a κ -weak homogeneity system. Denote $\bar{\mu}^* = (\mu_s^* : s \in {}^{<\omega}\omega)$. $\bar{\mu}^*$ is a κ -weak homogeneity system. It can be shown that $\text{MS}_\theta(\bar{\mu})^V = \text{MS}_\theta(\bar{\mu}^*)^V[G]$. Hence Fact 4.9 implies that $V[G] \models p[\text{MS}_\theta(\bar{\mu})] = {}^\omega\omega \setminus W_{\bar{\mu}^*}$. (Also if one has a κ -homogeneous tree S with κ -homogeneity system $\bar{\mu}$ and $|\mathbb{P}| < \kappa$, then $\bar{\mu}^*$ is a κ -homogeneity system for S in $V[G]$. This shows A.)

Now suppose that T is a κ -weakly homogeneous tree on $\omega \times \alpha$ witnessed by the κ -weak homogeneity system $\bar{\mu}$. This gives that $p[T] = W_{\bar{\mu}}$. One also has that $p[\text{MS}_\theta(\bar{\mu})^V]$ continues to represent ${}^\omega\omega \setminus p[T]$ in $V[G]$. So in summary:

Fact 4.10. *(ZF + DC) Let κ be a cardinal. Let T be a κ -weakly homogeneous tree on $\omega \times \gamma$, for some ordinal γ , with κ -weak homogeneity system $\bar{\mu}$. Let $\theta > |\gamma|^+$. Let \mathbb{P} be a forcing with $|\mathbb{P}| < \kappa$ and $G \subseteq \mathbb{P}$ be \mathbb{P} -generic over V . $V[G] \models \text{MS}_\theta(\bar{\mu}^*) = \text{MS}_\theta(\bar{\mu})^V$. $V[G] \models p[\text{MS}_\theta(\bar{\mu})^V] = {}^\omega\omega \setminus p[T]$.*

Proof. See [13], Section 1 and especially Lemma 1.19. Also see [7], Section 1.3. □

So if T is κ -weakly homogeneous, an appropriate Martin-Solovay tree will continue to represent the complement of $p[T]$ in generic extensions by forcings of cardinality less than κ . The Martin-Solovay trees give the generically-correct tree representations for complements of κ -weakly homogeneously Suslin sets. However, the formulas D_Σ and D_Π involve more negations and quantifications over ${}^\omega\omega$. Multiple iterations of the Martin-Solovay construction will be needed. The following results are useful for continuing the Martin-Solovay construction of generically-correct tree representation for more complex sets. In addition, these results will also imply that these representations are also homogeneously Suslin.

Definition 4.11. If $B \subseteq {}^k({}^\omega\omega) \times {}^\omega\omega$, denote

$$\exists^{\mathbb{R}} B = \{x : (\exists y)((x, y) \in B)\} \text{ and } \forall^{\mathbb{R}} B = \{x : (\forall y)((x, y) \in B)\}$$

If $A \subseteq {}^k({}^\omega\omega)$, then denote

$$\neg A = {}^k({}^\omega\omega) \setminus A$$

Fact 4.12. ([13], Proposition 1.10) *Let $A \subseteq {}^\omega\omega$. A is κ -weakly homogeneously Suslin if and only if there is a κ -homogeneously Suslin set $B \subseteq {}^\omega\omega \times {}^\omega\omega$ so that $A = \exists^{\mathbb{R}} B$.*

A Woodin cardinal is a technical large cardinal which has been very useful in descriptive set theory. (See [7], Section 1.5 for more information about Woodin cardinals.)

Fact 4.13. ([8]) *Let δ be a Woodin cardinal. Let $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ be a δ^+ -weak homogeneity system with support $\gamma \in \text{ON}$. Then for sufficiently large θ , $\text{MS}_\theta(\bar{\mu})$ is κ -homogeneous for all $\kappa < \delta$.*

Definition 4.14. If κ is a cardinal, then let Hom_κ be the collection of κ -homogeneously Suslin subsets of ${}^\omega\omega$. Let $\text{Hom}_{<\kappa} = \bigcap_{\gamma < \kappa} \text{Hom}_\gamma$.

Fact 4.15. (Martin-Steel; [13] Section 2) Let λ be a limit of Woodin cardinals. Then $\text{Hom}_{<\lambda}$ is closed under complements and $\forall^{\mathbb{R}}$.

Fact 4.16. (Martin; [9], Theorem 4.15) If κ is a measurable cardinal, then every $\mathbf{\Pi}_1^1$ set is κ -homogeneously Suslin.

Fact 4.17. (Martin-Steel) Let λ be a limit of Woodin cardinals, then all projective sets are in $\text{Hom}_{<\lambda}$.

Proof. Every Woodin cardinal has a stationary set of measurable cardinals below it. Hence every $\mathbf{\Pi}_1^1$ set is κ -homogeneously Suslin for all $\kappa < \lambda$. That is, all $\mathbf{\Pi}_1^1$ sets are in $\text{Hom}_{<\lambda}$. Then by the closure properties given by Fact 4.15, all projective sets are in $\text{Hom}_{<\lambda}$. \square

Fact 4.18. (Woodin) Suppose λ is a limit of Woodin cardinals and there is a measurable cardinal greater than λ . Then every subset of ${}^\omega\omega$ in $L(\mathbb{R})$ is in $\text{Hom}_{<\lambda}$.

Homogeneously Suslin sets were defined to be those sets that can be presented as projections of some trees satisfying certain properties. In the ground model, there could be many homogeneous trees representing the same homogeneously Suslin set A . For instance, suppose $\kappa_1 < \kappa_2$. In the ground model, suppose $A = p[T_1]$ where T_1 is a κ_1 -homogeneous tree and $A = p[T_2]$ where T_2 is a κ_2 -homogeneous tree. Suppose \mathbb{P}_1 and \mathbb{P}_2 are two different forcings. Which tree should represent A in each forcing extension? What are the relations between $p[T_1]$ and $p[T_2]$ in various forcing extensions? Absolutely complemented trees and universal Baireness provide a way to interpret homogeneously Suslin sets in a way which is independent of the homogeneous tree representation in some sense:

Definition 4.19. (See [3]) Let κ be an ordinal. Let T be a tree on $\omega \times X$ and let U be a tree on $\omega \times Y$, for some sets X and Y . T and U are κ -absolute complements if and only if for all forcings $\mathbb{P} \in V_\kappa$ and all $G \subseteq \mathbb{P}$ which are \mathbb{P} -generic over V , $V[G] \models p[T] = {}^\omega\omega \setminus p[U]$.

A tree T on $\omega \times X$ is κ -absolutely complemented if and only if there exists some tree U on $\omega \times Y$ (for some set Y) so that T and U are κ -absolute complements.

A set $A \subseteq {}^\omega\omega$ is κ -universally Baire if and only if $A = p[T]$ for some tree T which is κ -absolutely complemented.

Fact 4.20. Let T_1 and T_2 be trees on $\omega \times \gamma_1$ and $\omega \times \gamma_2$ which are κ -absolutely complemented and $p[T_1] = p[T_2]$. If $\mathbb{P} \in V_\kappa$ and $G \subseteq \mathbb{P}$ is \mathbb{P} -generic over V , then $V[G] \models p[T_1] = p[T_2]$.

So if A is a κ -universally Baire set and if T_1 and T_2 are two κ -absolutely complemented trees so that $V \models A = p[T_1] = p[T_2]$, then either tree can be used to represent A in extensions by forcings in V_κ . As a matter of convention, if A is κ -universally Baire and $\mathbb{P} \in V_\kappa$, the set A will always refer to $p[T]$ for some and any κ -absolutely complemented tree $T \in V$ so that $V \models p[T] = A$.

Fact 4.21. ([13] Corollary 1.21) Let κ be a cardinal. κ -weakly homogeneously Suslin sets are κ -universally Baire.

In particular, κ -homogeneously Suslin sets can be interpreted unambiguously in \mathbb{P} -extensions whenever $\mathbb{P} \in V_\kappa$.

Let λ be a limit of Woodin cardinals. Let \dot{A} be a new unary relation symbol. Let $A \subseteq ({}^\omega\omega)^n$ be such that $A \in \text{Hom}_{<\lambda}$. Let (H_{\aleph_1}, \in, A) be the $\{\dot{A}, \dot{A}\}$ -structure with domain H_{\aleph_1} (the hereditarily countable sets) and with \dot{A} interpreted as A . Now let $\mathbb{P} \in V_\lambda$ be some forcing and $G \subseteq \mathbb{P}$ be a \mathbb{P} -generic filter over V . $\mathbb{P} \in V_\kappa$ for some $\kappa < \lambda$. The structure $(H_{\aleph_1}, \in, A^{V[G]})$ is understood in the following way: It is a structure with domain $H_{\aleph_1}^{V[G]}$ (the hereditarily countable subsets of $V[G]$) and $A^{V[G]}$ is $p[T]^{V[G]}$ for any γ -homogeneous tree T so that $V \models A = p[T]$ and $\gamma \geq \kappa$. By the above discussion, this is independent of which tree T is chosen. Actually, in the proof of the fact below, depending on the quantifier complexity of a particular formula φ involving \dot{A} , A will be considered as $p[T]$ for a sufficiently homogeneous tree T so that after the appropriate number of applications of the Martin-Solovay tree construction, the resulting tree representation of φ will be at least κ -homogeneous.

Using ideas very similar to the proof of Fact 4.15, one has the following absoluteness result:

Fact 4.22. (Woodin; [13], Theorem 2.6) Let λ be a limit of Woodin cardinals. Let $A \in \text{Hom}_{<\lambda}$. Let $\mathbb{P} \in V_\lambda$ and $G \subseteq \mathbb{P}$ be \mathbb{P} -generic over V . Then $(H_{\aleph_1}^V, \in, A)$ and $(H_{\aleph_1}^{V[G]}, \in, A^{V[G]})$ are elementarily equivalent.

In this setting, V and $V[G]$ satisfy the same formulas involving \dot{A} and quantifications over the reals with the above intended interpretation. In particular, V and $V[G]$ satisfy the same projective formulas.

Now, the above discussion will be applied to indicate when A , B_Σ , C_Σ , B_Π , and C_Π hold. Fix a σ -ideal I on ${}^\omega\omega$ so that \mathbb{P}_I is proper.

Let λ be a limit of Woodin cardinals. By the above discussion about universal Baireness, one may speak about a binary relation $R \in \text{Hom}_{<\lambda}$ without explicit reference to a fix tree defining R . By Fact 4.15, if $R \in \text{Hom}_{<\lambda}$, then ${}^\omega\omega \setminus R \in \text{Hom}_{<\lambda}$. Given an κ -weakly homogeneous tree representation of R for sufficiently large γ , the associated Martin-Solovay tree will be a sufficiently homogeneous tree representation of ${}^\omega\omega \setminus R$ by Fact 4.13. Hence in this setting, $R_S = R^T$, where T is the appropriate Martin-Solovay tree using the homogeneity system on S . (So if R_S has all Π_1^1 classes, then the results of Section 3 should be applied to R^T using A , B_Π , C_Π for T and I .)

The formula D_Σ and D_Π both involve complements and real quantification over the homogeneously Suslin set R . By Fact 4.15, $D_\Sigma, D_\Pi \in \text{Hom}_{<\lambda}$. Starting with an appropriate weakly homogeneous tree representation of R , the process described in the proof of Fact 4.15 produces a tree U representing D_Σ or D_Π that is generically correct for \mathbb{P}_I , in the sense that $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x, T) : D_\Sigma(x, T)\}$. So C_Σ holds for R and I . (A similar argument holds for C_Π .)

R having all Σ_1^1 sections can be expressed as a formula using some real quantifiers over the relation $R \in \text{Hom}_{<\lambda}$. Fact 4.22 implies that these statements are absolute to the \mathbb{P}_I -extension. The tree S remains homogeneous in the \mathbb{P}_I -extension by the remark mentioned before Fact 4.10. This shows that A and B_Σ holds for R and I .

Finally, using the above discussion and results of the previous section, the three main theorems of this section are obtained.

With the appropriate assumptions, even more sets of reals are homogeneously Suslin and these canonicalization results would hold for relations in those classes. For example, Chang's model $L({}^\omega\text{ON}) = \bigcup_{\alpha \in \text{ON}} L({}^\omega\alpha)$ is the smallest inner model of ZF containing all the countable sequences of ordinals of V . Woodin has shown that with a proper class of Woodin cardinals every set of reals in $L({}^\omega\text{ON})$ is ∞ -homogeneously Suslin. Hence under this assumption, the above result would hold for binary relations in $L({}^\omega\text{ON})$ with all Σ_1^1 , Π_1^1 , or Δ_1^1 sections.

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