WHEN A RELATION WITH ALL BOREL SECTIONS WILL BE BOREL SOMEWHERE?

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Abstract. In ZFC, if there is a measurable cardinal with infinitely many Woodin cardinals below it, then for every binary relation $R \in L(\mathbb{R})$ on $\mathbb{R}$ with all sections $\mathbf{\Delta}^1_1$ ($\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$) and every σ-ideal I on $\mathbb{R}$ so that the associated forcing $\mathbb{P}_I$ of I^+ $\mathbf{\Delta}^1_1$ subsets is proper, there exists some I^+ $\mathbf{\Delta}^1_1$ set C so that $R \cap (C \times \mathbb{R})$ is $\mathbf{\Delta}^1_1$ ($\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$, respectively).

1. Introduction

The basic question of interest is:

Question 1.1. If E is an equivalence relation on ${}^\omega\omega$, is E a simpler equivalence relation when restricted to some subset?

This question can also be asked for graphs and more generally for binary relations. (This paper will state all results for binary relations. This introduction will briefly focus on equivalence relations which is the original form of this question from [6] and [1].)

The measure of complexity will be definability. There are various useful notions of definability given by ideas from topology, recursion theory, logical complexity, and set theory. The class of Borel sets (denoted $\mathbf{\Delta}^1_1$) is chosen to be the base of complexity since it is a simple class characterized by all the notions of definability mentioned above. Moreover $\mathbf{\Delta}^1_1$ objects seem to be well behaved and relatively well understood. Now the question can be more precisely formulated:

Question 1.2. If E is an equivalence relation on ${}^\omega\omega$, is there a $\mathbf{\Delta}^1_1$ set $C \subseteq {}^\omega\omega$ so that $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

Here, $E \upharpoonright C = E \cap (C \times C)$. However, there is one obvious triviality. If C is countable, then any equivalence relation restricted to C is $\mathbf{\Delta}^1_1$. This egregious triviality disappears if one asks that, in the above question, C be $\mathbf{\Delta}^1_1$ and non-trivial according to a σ-ideal on ${}^\omega\omega$. Subsets of ${}^\omega\omega$ that are not in the ideal I are called I^+ sets. In this paper, σ-ideals will always contain all the singletons.

However, it is unclear how to approach this question for arbitrary σ-ideals. The collection of available techniques is greatly enriched by considering σ-ideals on ${}^\omega\omega$ so that the associated forcing $\mathbb{P}_I$ of $\mathbf{\Delta}^1_1$ I^+ sets is a proper forcing. Considering such σ-ideals makes available powerful tools from models of set theory and absoluteness. (In fact, the questions below all have negative answers for arbitrary σ-ideals. See Section 2.) Analytic (denoted $\mathbf{\Sigma}^1_1$) sets are continuous images of $\mathbf{\Delta}^1_1$ or even closed sets.

Question 1.3. Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation on ${}^\omega\omega$. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is a proper forcing. Is there an I^+ $\mathbf{\Delta}^1_1$ set C so that $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

Note that questions like the above are very familiar. For example, let I_{meager} be the ideal of meager sets and I_{null} be the ideal of Lebesgue measure zero sets. The associated forcings are Cohen forcing and random real forcing, respectively, which both satisfy the countable chain condition and are hence proper. It is very common in mathematics to ask questions about properties that hold on positive measure sets or on non-meager sets. Unfortunately, Question 1.3 has a negative answer:

Proposition 1.4. *([6], Example 4.25) There is a $\mathbf{\Sigma}^1_1$ equivalence relation E and a σ-ideal I with $\mathbb{P}_I$ proper so that for all $\mathbf{\Delta}^1_1$ I^+ set C, $E \upharpoonright C$ is not $\mathbf{\Delta}^1_1$.*

May 10, 2021. The first author was partially supported by NSF grants DMS-1464475, EMSW21-RTG DMS-1044448, and DMS-1703708.

A positive answer to any variation of the basic question will likely only be feasible if the equivalence relations bear at least some resemblance to $\mathbf{\Delta}^1_1$ equivalence relations. A positive answer does hold for many important examples:

Example 1.5. ([6]) Let I be a σ-ideal on a Polish space X with $\mathbb{P}_I$ proper.

If E is a $\mathbf{\Sigma}^1_1$ equivalence relation with all classes countable or E is $\mathbf{\Delta}^1_1$ reducible to an orbit equivalence relation of a Polish group action, then there is some I^+ $\mathbf{\Delta}^1_1$ set C so that $E \upharpoonright C$ is $\mathbf{\Delta}^1_1$.

In both these examples, the equivalence relations have all $\mathbf{\Delta}^1_1$ classes. Of course, $\mathbf{\Delta}^1_1$ equivalence relations have all $\mathbf{\Delta}^1_1$ classes. Perhaps those two examples give evidence that a sufficient resemblance for a positive answer is the property of having all $\mathbf{\Delta}^1_1$ classes. [6] asked the following question:

Question 1.6. ([6] Question 4.28) Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation on ${}^\omega\omega$ with all $\mathbf{\Delta}^1_1$ classes. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is a proper forcing. Let B be an I^+ $\mathbf{\Delta}^1_1$ set. Is there some $C \subseteq B$ which is I^+ $\mathbf{\Delta}^1_1$ so that $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

Further examples and partial results suggest that a positive answer is consistent.

Example 1.7. ([1]) Assume $\mathsf{ZFC} + \mathsf{MA} + \neg\mathsf{CH}$. Consider I_{meager} (or I_{null}). Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation with all classes $\mathbf{\Delta}^1_1$. Then there exists comeager (or measure one) $\mathbf{\Delta}^1_1$ set C such that $E \upharpoonright C$ is $\mathbf{\Delta}^1_1$.

Example 1.8. ([1]) Let κ be a remarkable cardinal in L. Let G be $\text{Coll}(\omega, <\kappa)$-generic over L. In $L[G]$, if I is a σ-ideal with $\mathbb{P}_I$ proper and E is a $\mathbf{\Pi}^1_1$ equivalence relation with all classes countable, then there is a I^+ $\mathbf{\Delta}^1_1$ set C such that $E \upharpoonright C$ is $\mathbf{\Delta}^1_1$.

It was then shown that, under large cardinal assumptions, this question has a positive answer:

Theorem 1.9. *([1]. Also see [2].) Suppose for all $X \in H_{(2^{\aleph_0})^+}$, $X^\sharp$ exists. Then for all $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ equivalence relations with all $\mathbf{\Delta}^1_1$ classes, any σ-ideal I on ${}^\omega\omega$ with $\mathbb{P}_I$ proper, and B an I^+ $\mathbf{\Delta}^1_1$ set, there exists some I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $E \upharpoonright C$ is $\mathbf{\Delta}^1_1$.*

The proofs of Theorem 1.9 in both [1] and [2] used an approximation result of Burgess: for every $\mathbf{\Sigma}^1_1$ equivalence relation E there is (in a uniform way) an ω_1-length decreasing sequence $(E_\alpha : \alpha < \omega_1)$ of $\mathbf{\Delta}^1_1$ equivalence relations so that $E = \bigcap_{\alpha<\omega_1} E_\alpha$.

Neeman asked the following generalization of Question 1.6: Projective sets are those obtainable by applying finitely many applications of complements and continuous images starting with the $\mathbf{\Delta}^1_1$ sets.

Question 1.10. Assume some large cardinal hypotheses. Let E be a projective equivalence relation with all $\mathbf{\Delta}^1_1$ classes. Let I be a σ-ideal on ${}^\omega\omega$ with $\mathbb{P}_I$ proper. Let $B \subseteq {}^\omega\omega$ be an I^+ $\mathbf{\Delta}^1_1$ subset. Does there exist some I^+ $\mathbf{\Delta}^1_1$ $C \subseteq B$ so that $E \upharpoonright C$ is $\mathbf{\Delta}^1_1$?

It is unclear if the proofs of Theorem 1.9 can be generalized to give an answer to this question since there does not appear to be any form of $\mathbf{\Delta}^1_1$ approximation to arbitrary projective equivalence relations. Moreover, it is known to be consistent that there is a negative answer to Question 1.10 even when restricted to the next level of the projective hierarchy above $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

Example 1.11. In the constructible universe L, there is a $\mathbf{\Delta}^1_2$ equivalence relation E_L with all classes countable so that for every σ-ideal I and every I^+ $\mathbf{\Delta}^1_1$ set B, $E_L \upharpoonright B$ is not $\mathbf{\Delta}^1_1$.

Proof. The equivalence E_L on ${}^\omega\omega$ is roughly defined by $x\ E_L\ y$ if and only if the least admissible level of L for which x and y appear is the same. See [1] or [2] for more details and the complete proof. □

It is not known what is the status of Question 1.6 or its $\mathbf{\Pi}^1_1$ analog in L. An interesting open question is whether it is consistent that Question 1.6 or its $\mathbf{\Pi}^1_1$ analog has a negative answer.

This paper will be concerned with extending a positive answer to these types of questions to larger classes of equivalence relations on ${}^\omega\omega$ with all $\mathbf{\Delta}^1_1$ classes. A certain game idea will need to be developed to take the role of Burgess's approximation in Theorem 1.9. Question 1.10 will be answered by an even more general result which will be proved in ZFC augmented by some large cardinal axioms, which are well accepted and have proven to be very useful in descriptive set theory. The model $L(\mathbb{R})$ is the smallest inner model of ZF containing all the reals of the original universe. It contains all the sets which are "constructible" (in the sense of Gödel) from the reals of the original universe. All projective subsets of ${}^\omega\omega$ belong to $L(\mathbb{R})$. The following is a main result of the paper answering Question 1.10.

Theorem 4.3. Suppose there is a measurable cardinal with infinitely many Woodin cardinals below it. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Let $R \in L(\mathbb{R})$ be a binary relation on ${}^\omega\omega$. If R has all $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$) sections, then for every I^+ $\mathbf{\Delta}^1_1$ set B, there is an I^+ $\mathbf{\Delta}^1_1$ $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$, respectively).

Example 1.12. The following is an application: Assume there is a measurable cardinal with infinitely many Woodin cardinals below it. Let I be a σ-ideal on ${}^\omega\omega$ with $\mathbb{P}_I$ proper. Define an equivalence relation E on ${}^\omega\omega$ by $x \; E \; y$ if and only if $L(\mathbb{R})^V \models x \in \mathrm{OD}_y \wedge y \in \mathrm{OD}_x$. Note that $E \in L(\mathbb{R})$ and, in fact, is $(\Sigma^2_1)^{L(\mathbb{R})}$. E has all classes countable (and hence $\mathbf{\Delta}^1_1$). Theorem 4.3 implies that there is an I^+ $\mathbf{\Delta}^1_1$ set C so that $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation.

Having answered Question 1.10 positively and even given a positive answer for the larger class of $L(\mathbb{R})$ equivalence relation with all $\mathbf{\Delta}^1_1$ classes, the ultimate natural question is the following:

Question 1.13. Is it consistent relative to some large cardinals, that (the axiom of choice fails and) for every equivalence relation E with all $\mathbf{\Delta}^1_1$ classes and every σ-ideal I on ${}^\omega\omega$ such that $\mathbb{P}_I$ is a proper forcing, there is an I^+ $\mathbf{\Delta}^1_1$ set C so that $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

The notion of an absolutely proper forcing is defined in [10]. [10] established an absoluteness result given by an embedding theorem for absolutely proper forcing under determinacy assumptions which is analogous to the proper forcing embedding theorem shown in [11] which holds under AC with large cardinals. [10] used this embedding theorem for absolutely proper forcings to establish a positive answer under AD^+ to a more general form of Question 1.10 for σ-ideals with associated forcing absolutely proper.

The authors would like to thank Alexander Kechris, Itay Neeman, and Zach Norwood for many useful discussions about the contents of this paper.

2. Basics

Definition 2.1. Let I be a σ-ideal on ${}^\omega\omega$. Let $\mathbb{P}_I = (\mathbf{\Delta}^1_1 \setminus I, \subseteq, {}^\omega\omega)$ be the forcing of I^+ $\mathbf{\Delta}^1_1$ subsets of ${}^\omega\omega$ ordered by $\leq_{\mathbb{P}_I} = \subseteq$ and has largest element $1_{\mathbb{P}_I} = {}^\omega\omega$. Often $\mathbb{P}_I$ is identified with $\mathbf{\Delta}^1_1 \setminus I$.

Fact 2.2. *([14], Proposition 2.1.2) Let I be a σ-ideal on ${}^\omega\omega$. There is a name $\dot{x}_{\mathrm{gen}} \in V^{\mathbb{P}_I}$ so that for all $\mathbb{P}_I$-generic filters G over V and all $\mathbf{\Delta}^1_1$ sets B coded in V, $V[G] \models B \in G \Leftrightarrow \dot{x}_{\mathrm{gen}}[G] \in B$.*

Definition 2.3. Let I be a σ-ideal on ${}^\omega\omega$. Let $M \prec H_\Xi$ be a countable elementary substructure for some sufficiently large cardinal Ξ. $x \in {}^\omega\omega$ is $\mathbb{P}_I$-generic over M if and only if the collection $\{B \in \mathbb{P}_I \cap M : x \in B\}$ is a $\mathbb{P}_I$-generic filter over M.

Proposition 2.4. *([14], Proposition 2.2.2) Let I be a σ-ideal on ${}^\omega\omega$. The following are equivalent:*
(i) $\mathbb{P}_I$ is a proper forcing.
(ii) For any sufficiently large cardinal Ξ, every $B \in \mathbb{P}_I$, and every countable $M \prec H_\Xi$ with $\mathbb{P}_I \in M$ and $B \in M$, the set $C = \{x \in B : x \text{ is } \mathbb{P}_I\text{-generic over } M\}$ is an I^+ $\mathbf{\Delta}^1_1$ set.

This proposition shows that σ-ideals whose associated forcings are proper may be useful for answering Question 1.6 since it indicates how to produce I^+ $\mathbf{\Delta}^1_1$ sets. The following example shows that some restrictions on the type of σ-ideals considered in Question 1.6 are necessary:

Let F_{ω_1} denote the countable admissible ordinal equivalence relation defined by $x \; F_{\omega_1} \; y$ if and only if $\omega_1^x = \omega_1^y$. F_{ω_1} is a thin $\mathbf{\Sigma}^1_1$ equivalence relation with all $\mathbf{\Delta}^1_1$ classes. Thin means that F_{ω_1} does not have a perfect set of pairwise F_{ω_1}-inequivalent elements. Let I be the σ-ideal which is σ-generated by the F_{ω_1}-classes. Suppose there is an I^+ $\mathbf{\Delta}^1_1$ set C so that $F_{\omega_1} \upharpoonright C$ is $\mathbf{\Delta}^1_1$. By definition of I, each F_{ω_1}-class is in I. So since C is I^+, C must intersect nontrivially uncountably many classes of F_{ω_1}. So $F_{\omega_1} \upharpoonright C$ has uncountable many classes. Since F_{ω_1} is thin, there is also no perfect set of $F_{\omega_1} \upharpoonright C$ inequivalent elements. This contradicts Silver's dichotomy [12] which states that every $\mathbf{\Delta}^1_1$ (even $\mathbf{\Pi}^1_1$) equivalence relation E on ${}^\omega\omega$ has countably many classes or there exists a perfect set of pairwise E-inequivalent elements.

In this example, I is not proper or even ω_1-preserving: Let $G \subseteq \mathbb{P}_I$ be a $\mathbb{P}_I$-generic filter over V. Fact 2.2 implies that $\dot{x}_{\text{gen}}[G]$ is not in any ground model coded $\mathbf{\Delta}^1_1$ set in I. $\omega_1^{\dot{x}_{\text{gen}}[G]}$ can not be a countable admissible ordinal of V since if it was countable then a theorem of Sacks shows that there is a $z \in ({}^\omega\omega)^V$ so that $\omega_1^z = \omega_1^{\dot{x}_{\text{gen}}[G]}$. Then $x \in [z]_{F_{\omega_1}}$. By definition of I, $[z]_{F_{\omega_1}}$ is a $\mathbf{\Delta}^1_1$ set coded in V that belongs to I. Hence $\omega_1^{\dot{x}_{\text{gen}}[G]}$ must be an uncountable admissible ordinal of V, but in $V[G]$, $\omega_1^{\dot{x}_{\text{gen}}[G]}$ is a countable admissible ordinal. Hence $\mathbb{P}_I$ collapses ω_1.

Definition 2.5. A measure μ on a set X is a nonprincipal ultrafilter on X. If κ is a cardinal, then μ is κ-complete if and only if for all $\beta < \kappa$ and sequences $(A_\alpha : \alpha < \beta)$ with each $A_\alpha \in \mu$, $\bigcap_{\alpha<\beta} A_\alpha \in \mu$. $\aleph_1$-completeness is often called countable completeness. Let $\text{meas}_\kappa(X)$ be the set of all κ-complete ultrafilters on X.

Suppose $\mu \in \text{meas}_{\aleph_1}({}^{<\omega}X)$. By countable completeness, there is a unique m so that ${}^mX \in \mu$. In this case, m is called the dimension of μ and this is denoted by $\dim(\mu) = m$.

Definition 2.6. Let X be a set and $m \leq n < \omega$. Let $\pi_{n,m} : {}^nX \to {}^mX$ be defined by $\pi_{n,m}(f) = f \restriction m$. Let ν be a measure of dimension m and μ be a measure of dimension n. μ is an extension of ν (or ν is a projection of μ) if and only if for all $A \in \nu$ with $A \subseteq {}^mX$, $\pi^{-1}_{n,m}[A] \in \mu$.

A tower of measures over X is a sequence $(\mu_n : n \in \omega)$ so that
(i) For all n, $\mu_n \in \text{meas}_{\aleph_1}({}^{<\omega}X)$ and $\dim(\mu_n) = n$.
(ii) For all $m \leq n < \omega$, μ_n is an extension of μ_m.

A tower of measures over X, $(\mu_n : n \in \omega)$, is countably complete if and only if for all sequence $(A_n : n \in \omega)$ with the property that for all $n \in \omega$, $A_n \in \mu_n$, there exists a $f : \omega \to X$ so that for all $n \in \omega$, $f \restriction n \in A_n$.

Definition 2.7. A tree T on X is a subset of ${}^{<\omega}X$ so that if $s \subseteq t$ and $t \in T$, then $s \in T$. If T is a tree on X, the body of T, denoted $[T]$, is the set of infinite paths through T, that is $[T] = \{f \in {}^\omega X : (\forall n \in \omega)(f \restriction n \in T)\}$.

If $s \in {}^n(X \times Y)$ where $n \in \omega$, then in a natural way, s be may be considered as a pair (s_0, s_1) with $s_0 \in {}^nX$ and $s_1 \in {}^nY$. Suppose T is a tree on $X \times Y$. For each $s \in {}^{<\omega}X$, define $T^s = \{t \in {}^{|s|}Y : (s,t) \in T\}$. If $f \in {}^\omega X$, then define $T^f = \bigcup_{n\in\omega} T^{f\restriction n}$. Define $p[T] = \{f \in {}^\omega X : T^f \text{ is ill-founded}\} = \{f \in {}^\omega X : [T^f] \neq \emptyset\}$.

Definition 2.8. For any $k \in \omega$, $A \subseteq {}^k({}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$ if and only if there exists a tree on ${}^k\omega \times \omega$ so that $A = p[T]$. $A \subseteq {}^k({}^\omega\omega)$ is $\mathbf{\Pi}^1_1$ if and only if $A = {}^k({}^\omega\omega) \setminus B$ for some $\mathbf{\Sigma}^1_1$ set $B \subseteq {}^k({}^\omega\omega)$. $A \subseteq {}^k({}^\omega\omega)$ is $\mathbf{\Delta}^1_1$ if and only if A is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

Definition 2.9. Let γ be an ordinal and $k \in \omega$. A tree T on ${}^k\omega \times \gamma$ is homogeneous if and only if there is a collection $(\mu_s : s \in {}^{<\omega}({}^k\omega))$ so that
(i) For each $s \in {}^{<\omega}({}^k\omega)$, $\mu_s \in \text{meas}_{\aleph_1}({}^{<\omega}\gamma)$ and concentrates on T^s (that is, $T^s \in \mu_s$).
(ii) For all $s, t \in {}^{<\omega}({}^k\omega)$, if $s \subseteq t$, then μ_t is an extension of μ_s.
(iii) For all $f \in p[T]$, $(u_{f\restriction n} : n \in \omega)$ is a countably complete tower of measures on γ.

A collection $(u_s : s \in {}^{<\omega}({}^k\omega))$ which witnesses the homogeneity of T is called a homogeneity system for T. Let κ be a cardinal. The homogeneous tree T is κ-homogeneous if and only if each μ_s is κ-complete.

Definition 2.10. For any $k \in \omega$, $A \subseteq {}^k({}^\omega\omega)$ is homogeneously Suslin if and only if there exists an ordinal γ and a homogeneous tree on ${}^k\omega \times \gamma$ so that $A = p[T]$. If the tree T is κ-homogeneous, then A is said to be κ-homogeneously Suslin.

Later, homogeneity systems will be used to show a certain player has a winning strategy in a particular game using techniques that are very similar to Martin's proof of $\mathbf{\Sigma}^1_1$ determinacy from a measurable cardinal.

Definition 2.11. Let X be some set. Let $A \subseteq {}^\omega X$. The game associated to A, denoted G_A, is the following: The game has two players, Player 1 and Player 2, who alternatingly take turns playing elements of X with Player 1 playing first. The picture below denotes a partial play where Player 1 plays the sequence $(a_i : i \in \omega)$ and Player 2 plays the sequence $(b_i : i \in \omega)$.

a_0		a_1		...	a_{k-1}	
	b_0		b_1	...		b_{k-1}

Player 2 is said to win this play of G_A if and only if the infinite sequence $(a_0 b_0 a_1 b_1 ...) \in A$. Otherwise Player 1 wins.

A function $\tau : {}^{<\omega}X \to X$ is a winning strategy for Player 1 if and only if for all sequence $(b_i : i \in \omega)$ played by Player 2, Player 1 wins by playing $(a_i : i \in \omega)$ where this sequence is defined recursively by $a_0 = \tau(\emptyset)$ and $a_{k+1} = \tau(a_0 b_0 ... a_k b_k)$. A winning stategy $\tau : {}^{<\omega}X \to X$ for Player 2 is defined similarly. The game G_A is determined if Player 1 or Player 2 has a winning strategy.

Let X be a set. ${}^{\omega}X$ is given the topology with basis $\{U_s : s \in {}^{<\omega}X\}$, where $U_s = \{f \in {}^{\omega}X : s \subseteq f\}$.

Fact 2.12. *([4], Gale-Stewart) If X is wellorderable and $A \subseteq {}^{\omega}X$ is open, then G_A is determined. Hence if A is closed, then G_A is also determined.*

3. The Game

Assume ZFC which includes the axiom of choice.

Definition 3.1. If R is a relation on $({}^{\omega}\omega)^2$, then let $R_x = \{y : (x, y) \in R\}$.

Let S be a homogeneous tree on $\omega \times \omega \times \gamma$, where γ is some ordinal. Let R_S denote $p[S]$. Let R^S denote $({}^{\omega}\omega \times {}^{\omega}\omega) \setminus p[S]$.

Definition 3.2. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$ for some ordinal γ. Let I be a σ-ideal on ${}^{\omega}\omega$ so that $\mathbb{P}_I$ is proper.

Let A assert that $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} \check{S}$ is a homogeneous tree.

Statement A just asserts that the tree S remains homogeneous in $\mathbb{P}_I$-generic extensions. Note that A is true since the completeness of countably complete measures is a measurable cardinal and $|\mathbb{P}_I|$ is always less than a measurable cardinal under AC.

Definition 3.3. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$ for some ordinal γ. Let I be a σ-ideal on ${}^{\omega}\omega$ such that $\mathbb{P}_I$ is a proper forcing.

Let D_Σ be the formula on ${}^{\omega}\omega \times {}^{\omega}\omega$ asserting:

$$D_\Sigma(x, T) \Leftrightarrow (T \text{ is tree on } \omega \times \omega) \wedge (\forall y)(R_S(x, y) \Leftrightarrow T^y \text{ is ill-founded})$$

Let D_Π be the formula on ${}^{\omega}\omega \times {}^{\omega}\omega$ asserting:

$$D_\Pi(x, T) \Leftrightarrow (T \text{ is a tree on } \omega \times \omega) \wedge (\forall y)(\neg(R^S(x, y)) \Leftrightarrow T^y \text{ is ill-founded})$$

If $D_\Sigma(x, T)$ holds, then T is a tree which witnesses $(R_S)_x$ is $\mathbf{\Sigma}^1_1$. Similarly, if $D_\Pi(x, T)$ holds, then T is a tree which witnesses ${}^{\omega}\omega \setminus (R^S)_x$ is $\mathbf{\Sigma}^1_1$, i.e. $(R^S)_x$ is $\mathbf{\Pi}^1_1$.

Definition 3.4. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$ for some ordinal γ. Let I be a σ-ideal on ${}^{\omega}\omega$ such that $\mathbb{P}_I$ is a proper forcing.

Let $\mathsf{B}_{\mathsf{\Sigma}}$ say: $(\forall x)(\exists T)D_\Sigma(x, T)$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} (\forall x)(\exists T)D_\Sigma(x, T)$.

Let $\mathsf{B}_{\mathsf{\Pi}}$ say: $(\forall x)(\exists T)D_\Pi(x, T)$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} (\forall x)(\exists T)D_\Pi(x, T)$.

$\mathsf{B}_{\mathsf{\Sigma}}$ states that all R_S sections are $\mathbf{\Sigma}^1_1$ and all R_S sections remain $\mathbf{\Sigma}^1_1$ in $\mathbb{P}_I$-generic extensions. Similarly, $\mathsf{B}_{\mathsf{\Pi}}$ states that all R^S sections are $\mathbf{\Pi}^1_1$ and all R^S sections remain $\mathbf{\Pi}^1_1$ in $\mathbb{P}_I$-generic extensions.

Definition 3.5. Let S be a homogeneously tree on $\omega \times \omega \times \gamma$ for some ordinal γ. Let I be a σ-ideal on ${}^{\omega}\omega$ such that $\mathbb{P}_I$ is a proper forcing.

Let $\mathsf{C}_{\mathsf{\Sigma}}$ state: There is an ordinal ϵ and a tree U on $\omega \times \omega \times \epsilon$ so that $p[U] = \{(x, T) : D_\Sigma(x, T)\}$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x, T) : D_\Sigma(x, T)\}$.

Let $\mathsf{C}_{\mathsf{\Pi}}$ state: There is an ordinal ϵ and a tree U on $\omega \times \omega \times \epsilon$ so that $p[U] = \{(x, T) : D_\Pi(x, T)\}$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x, T) : D_\Pi(x, T)\}$.

$\mathsf{C}_{\mathsf{\Sigma}}$ states that the set defined by D_Σ has a tree representation that continues to represent the formula D_Σ in $\mathbb{P}_I$-generic extensions. $\mathsf{C}_{\mathsf{\Pi}}$ is similar. The following game plays an important role the next theorem.

Definition 3.6. Let S be a tree on $\omega \times \omega \times \gamma$. Let T be a tree on $\omega \times \omega$. Let $g \in {}^{\omega}\omega$.

Consider the following game $G^{g,T}$:

$$\begin{array}{c|cccccccc} & m_0, n_0 & & m_1, n_1 & & ... & m_{k-1}, n_{k-1} & & \\ \hline & & \alpha_0 & & \alpha_1 & ... & & & \alpha_{k-1} \end{array}$$

The rules are:
(1) Player 1 plays $m_i, n_i \in \omega$. Player 2 plays $\alpha_i < \gamma$.
(2) $(m_0...m_{k-1}, n_0...n_{k-1}) \in T$
(3) $(g \restriction k, m_0...m_{k-1}, \alpha_0...\alpha_{k-1}) \in S$.

The first player to violate these rules loses. If the game continues forever, then Player 2 wins.

This game is open for Player 1 and hence closed for Player 2.

The following shows under certain assumptions a more general canonicalization property holds for relations. [2] defines this phenomenon as the rectangular canonization property.

Theorem 3.7. *Let γ be an ordinal. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$. Let I be a σ-ideal on ${}^{\omega}\omega$ so that $\mathbb{P}_I$ is proper. Assume A, B_Σ, and C_Σ hold for S and I. Then for any I^+ $\mathbf{\Delta}^1_1$ set $B \subseteq {}^{\omega}\omega$, there exists an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $R_S \cap (C \times {}^{\omega}\omega)$ is a $\mathbf{\Sigma}^1_1$ relation.*

Proof. Let U be the tree on $\omega \times \omega \times \epsilon$ witnessing C_Σ for S and I. Let $M \prec H_\Xi$ be a countable elementary substructure with Ξ sufficiently large and $B, I, \mathbb{P}_I, S, U \in M$.

<u>Claim 1</u> : Let g be $\mathbb{P}_I$-generic over M. If $x, T \in M[g]$ and $M[g] \models D_\Sigma(x,T)$, then $V \models D_\Sigma(x,T)$.

Proof of Claim 1: By C_Σ for S and I and the fact that $M \prec H_\Xi$, $M[g] \models D_\Sigma(x,T)$ implies $M[g] \models (x,T) \in p[U]$. There exists some $f \in M[g]$ with $f : \omega \to \epsilon$ so that $M[g] \models (x,T,f) \in [U]$. Hence for each $n \in \omega$, $M[g] \models (x \restriction n, T \restriction n, f \restriction n) \in U$. For each $n \in \omega$, $(x \restriction n, T \restriction n, f \restriction n) \in M$. So by absoluteness, $M \models (x \restriction n, T \restriction n, f \restriction n) \in U$. For all $n \in \omega$, $V \models (x \restriction n, T \restriction n, f \restriction n) \in U$. $V \models (x,T) \in p[U]$. $V \models D_\Sigma(x,T)$.

Now fix a $g \in {}^{\omega}\omega$ so that g is $\mathbb{P}_I$-generic over M. As $M \prec H_\Xi$, $M \models (\forall x)(\exists T) D_\Sigma(x,T)$. $M[g] \models (\forall x)(\exists T) D_\Sigma(x,T)$ by B_Σ and the fact that $M \prec H_\Xi$. So fix a tree T on $\omega \times \omega$ so that $M[g] \models D_\Sigma(g,T)$.

<u>Claim 2</u> : In $M[g]$, Player 2 has a winning strategy in the game $G^{g,T}$.

Proof of Claim 2: Work in $M[g]$: By an appropriate coding, $G^{g,T}$ is equivalent to a game G_A, where $A \subseteq {}^{\omega}\gamma$ is an open subset. Suppose Player 2 does not have a winning strategy. By Fact 2.12, Player 1 must have a winning strategy τ^*. By A, S is a homogeneous tree in $M[g]$. Let $(\mu_t : t \in {}^{<\omega}(\omega \times \omega))$ be a homogeneity system witnessing the homogeneity of S.

Now two sequences of natural numbers, $(a_i : i \in \omega)$ and $(b_i : i \in \omega)$, and a sequence $(A_n : n \in \omega)$ so that $A_n \subseteq {}^{n}\gamma$ will be constructed (in $M[g]$) by recursion:

Let $a_0, b_0 \in \omega$ so that $(a_0, b_0) = \tau^*(\emptyset)$. Let $A_0 = \{\emptyset\}$.

Suppose $a_0, ..., a_{k-1}$, $b_0, ..., b_{k-1}$, and $A_0, ..., A_{k-1}$ has been constructed. Let $h_k : S^{(g \restriction k, a_0 ... a_{k-1})} \to \omega \times \omega$ be defined by

$$h_k(\beta_0 ... \beta_{k-1}) = \tau^*(a_0, b_0, \beta_0, ..., a_{k-1}, b_{k-1}, \beta_{k-1})$$

$\mu_{(g \restriction k, a_0 ... a_{k-1})}$ concentrates on $S^{(g \restriction k, a_0 ... a_{k-1})}$ and is countably complete; therefore, there is a unique (a_k, b_k) so that $h_k^{-1}[\{(a_k, b_k)\}] \in \mu_{(g \restriction k, a_0 ... a_{k-1})}$. Let $A_k = h_k^{-1}[\{(a_k, b_k)\}]$.

This completes the construction of $(a_i : i \in \omega)$, $(b_i : i \in \omega)$, and $(A_i : i \in \omega)$.

Let $L \in {}^{\omega}(\omega \times \omega)$ be such that for all $i \in \omega$, $L(i) = (a_i, b_i)$. Note that $L \in [T]$. To see this, suppose not. Then there is some least $k \in \omega$ so that $L \restriction (k+1) = (a_0 ... a_k, b_0 ... b_k) \notin T$. For $i \leq k$, define $\mu_i = \mu_{g \restriction i, a_0 ... a_{i-1}}$. For $0 \leq i \leq j \leq k$, let $\pi_{j,i} : {}^{j}\gamma \to {}^{i}\gamma$ be defined by $\pi_{j,i}(s) = s \restriction i$. By definition of the homogeneity system for S, for $0 \leq i \leq j \leq k$, μ_j is an extension of μ_i. Hence for all $0 \leq i \leq k$, $\pi_{k,i}^{-1}[A_i] \in \mu_k$. By countable completeness of μ_k, $\bigcap_{0 \leq i \leq k} \pi_{k,i}^{-1}[A_i] \in \mu_k$. Let $(\beta_0 ... \beta_{k-1}) \in \bigcap_{0 \leq i \leq k} \pi_{k,i}^{-1}[A_i]$. Consider the following play of $G^{g,T}$ where player 1 uses the strategy τ^* and Player 2 plays $(\beta_0 ... \beta_{k-1})$:

$$\begin{array}{c|cccccccc} & a_0, b_0 & & a_1, b_1 & & ... & a_{k-1}, b_{k-1} & & a_k, b_k \\ \hline & & \beta_0 & & \beta_1 & ... & & \beta_{k-1} & \end{array}$$

Note that for all $0 \leq i \leq k$, $(\beta_0 ... \beta_{i-1}) \in A_i = h_i^{-1}[\{(a_i, b_i)\}] \subseteq S^{(g \restriction i, a_0 ... a_{i-1})}$. So rule (3) of the game $G^{g,T}$ is not violated by Player 2. However, $(a_0 ... a_k, b_0 ... b_k) = L \restriction (k+1) \notin T$. Player 1 violates rule (2) and is the first player to violate any rules. Player 1 loses this game. This contradicts the assumption that τ^* is a winning strategy for Player 1. So this completes the proof that $L \in [T]$.

Let $\mathfrak{a} = (a_i : i \in \omega)$. Since $L \in [T]$ and $D_\Sigma(g, T)$, this implies that $R_S(g, \mathfrak{a})$. Now let $J \in {}^\omega(\omega \times \omega)$ be such that for all $k \in \omega$, $J \restriction k = (g \restriction k, a_0 ... a_{k-1})$. Then by definition of S, $J \in p[S]$. Since S is a homogeneous tree via $(u_t : t \in {}^{<\omega}(\omega \times \omega))$, $(\mu_{J \restriction k} : k \in \omega)$ is a countably complete tower of measures.

Each $A_k \in \mu_{g \restriction k, a_0 ... a_{k-1}} = \mu_{J \restriction k}$. So by the countable completeness of the tower, there exists some $\Phi : \omega \to \gamma$ so that for all $k \in \omega$, $\Phi \restriction k \in A_k$. Now consider the play of $G^{g,T}$ where Player 1 uses its winning strategy τ^* and Player 2 plays Φ. By construction of the sequences $(a_i : i \in \omega)$, $(b_i, i \in \omega)$, and $(A_i : i \in \omega)$, a finite partial play of the game looks as follows:

$$\begin{array}{c|c|c|c|c|c|c|c} a_0, b_0 & & a_0, b_1 & & ... & a_{k-1}, b_{k-1} & \\ \hline & \Phi(0) & & \Phi(1) & ... & & \Phi(k-1) \end{array}$$

Neither player violates any rules in this play. Hence the game continues forever, and so Player 2 wins this play of $G^{g,T}$. This contradicts the fact that τ^* was a winning strategy for Player 1.

So Player 1 could not have had a winning strategy. Player 2 must have a winning strategy in $G^{g,T}$. This completes the proof of Claim 2.

By Claim 2, fix a Player 2 winning strategy $\tau \in M[g]$.

<u>Claim 3</u> : τ is a winning strategy for $G^{g,T}$ in V.

Proof of Claim 3: Suppose the following is a play of $G^{g,T}$ (in V) in which Player 2 uses τ and loses

$$\begin{array}{c|c|c|c|c|c|c} m_0, n_0 & & m_1, n_1 & & ... & m_{k-1}, n_{k-1} & \\ \hline & \alpha_0 & & \alpha_1 & ... & & \alpha_{k-1} \end{array}$$

Since $\tau \in M[g]$ and ${}^{<\omega}\omega \subseteq M[g]$, this entire finite play belongs to $M[g]$. So, Player 2 loses this game in $M[g]$, as well. This contradicts τ being a winning strategy in $M[g]$. This completes the proof of Claim 3.

<u>Claim 4</u> : For all $y \in {}^\omega\omega$, $R_S(g, y)$ if and only if $(S \cap M)^{(g,y)}$ is ill-founded.

Proof of Claim 4: By Claim 1, $M[g] \models D_\Sigma(g, T)$ implies $V \models D_\Sigma(g, T)$. Hence in V, T gives the $\mathbf{\Sigma}^1_1$ definition of $(R_S)_g$.

Suppose $R_S(g, y)$. Then T^y is ill-founded. Let $f \in [T^y]$. Consider the following play of the game $G^{g,T}$ where Player 1 plays y and f, and Player 2 responds using its winning strategy τ.

$$\begin{array}{c|c|c|c|c|c|c} y(0), f(0) & & y(1), f(1) & & ... & y(k-1), f(k-1) & \\ \hline & \alpha_0 & & \alpha_1 & ... & & \alpha_{k-1} \end{array}$$

Since $f \in [T^y]$, Player 1 can not lose. Since τ is a winning strategy for Player 2, Player 2 also does not lose at a finite stage. Hence Player 2 wins by having the game continue forever. Let $\Phi : \omega \to \gamma$ be the sequence coming from Player 2's response, i.e. for all k, $\Phi(k) = \alpha_k$.

Since $\tau \in M[g]$ and ${}^{<\omega}\omega \subseteq M[g]$, each finite partial play of $G^{g,T}$ above belongs to $M[g]$. Hence $\Phi \restriction k \in M[g]$ for all $k \in \omega$. As $\mathrm{On}^M = \mathrm{On}^{M[g]}$, $(g \restriction k, y \restriction k, \Phi \restriction k) \in (S \cap M)$ for all $k \in \omega$.

It has been shown that $R_S(g, y)$ implies $(S \cap M)^{(g,y)}$ is ill-founded. Of course, if $(S \cap M)^{(g,y)}$ is ill-founded, then $S^{(g,y)}$ is ill-founded. By definition, $R_S(g, y)$. This completes the proof of Claim 4.

Let $b : \omega \to \mathrm{On}^M$ be a bijection. Define a new tree S' on $\omega \times \omega \times \omega$ by $(s_1, s_2, s_3) \in S' \Leftrightarrow (s_1, s_2, b \circ s_3) \in S$. By Fact 2.4, let $C \subseteq B$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic reals over M inside B. If $g \in C$, then by Claim 4, for all $y \in {}^\omega\omega$, $R_S(g, y) \Leftrightarrow (S')^{(g,y)}$ is ill-founded. $R_S \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$. The proof of the theorem is complete. $\square$

Theorem 3.8. *Let γ be an ordinal. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Assume A, $\mathsf{B}_\sqcap$, and $\mathsf{C}_\sqcap$ hold for S and I. Then for any I^+ $\mathbf{\Delta}^1_1$ set $B \subseteq {}^\omega\omega$, there exists an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $R^S \cap (C \times {}^\omega\omega)$ is a $\mathbf{\Pi}^1_1$ relation.*

Proof. The proof of this is very similar to the proof of Theorem 3.7. $\square$

Theorem 3.9. *Let γ and ν be ordinals. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$. Let U be a homogeneous tree on $\omega \times \omega \times \nu$. Suppose $p[S] = ({}^\omega\omega \times {}^\omega\omega) \setminus p[U]$. Let $R = R_S = R^U$. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is a proper forcing. Suppose A, B_Σ, and C_Σ holds for S and I. Suppose A, B_Π, and C_Π holds for U and I. Then for any I^+ $\mathbf{\Delta}^1_1$ set $B \subseteq {}^\omega\omega$, there exists an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is a $\mathbf{\Delta}^1_1$ relation.*

Proof. By Theorem 3.7, there is some I^+ $\mathbf{\Delta}^1_1$ set $C' \subseteq B$ so that $R \cap (C' \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$. By Theorem 3.8, there is some I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq C'$ so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Pi}^1_1$. Therefore, $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Delta}^1_1$. $\square$

The follows are some applications: If the above assumptions holds and $R_S = E$ defines an equivalence relation with all $\mathbf{\Sigma}^1_1$ classes, then there is some I^+ $\mathbf{\Delta}^1_1$ set so that $E \restriction C$ is an $\mathbf{\Sigma}^1_1$ equivalence relation. Similarly, suppose $R_S = G$ is a graph on ${}^\omega\omega$. For each $x \in {}^\omega\omega$, let $G_x = \{y : x \; G \; y\}$ which is the set of neighbors of x. If G_x is $\mathbf{\Sigma}^1_1$ for all x, then there is an I^+ $\mathbf{\Delta}^1_1$ set C so that the induced subgraph $G \restriction C$ is an $\mathbf{\Sigma}^1_1$ graph.

4. Canonicalization for Relations in $L(\mathbb{R})$

The main results of this section are the following. The definition of some terms are stated further below.

Theorem 4.1. *Let λ be a limit of Woodin cardinals. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Let $R \in \mathrm{Hom}_{<\lambda}$ be a binary relation on ${}^\omega\omega$. If R has all $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$) sections, then for every I^+ $\mathbf{\Delta}^1_1$ set B, there is an I^+ $\mathbf{\Delta}^1_1$ $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$, respectively).*

Theorem 4.2. *Suppose there are infinitely many Woodin cardinals. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Let R be a projective binary relation on ${}^\omega\omega$. If R has all $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$) sections, then for every I^+ $\mathbf{\Delta}^1_1$ set B, there is an I^+ $\mathbf{\Delta}^1_1$ $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$, respectively).*

Theorem 4.3. *Suppose there is a measurable cardinal with infinitely many Woodin cardinals below it. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Let $R \in L(\mathbb{R})$ be a binary relation on ${}^\omega\omega$. If R has all $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$) sections, then for every I^+ $\mathbf{\Delta}^1_1$ set B, there is an I^+ $\mathbf{\Delta}^1_1$ $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$, respectively).*

This section will provide a brief description of the theory of tree representations of subsets of ${}^\omega\omega$ and absoluteness which will be used to indicate some circumstances in which the statements A, B_Σ, C_Σ, B_Π, and C_Π hold.

Definition 4.4. Let κ be a cardinal. A κ-weak homogeneity system with support some ordinal γ is a sequence of κ-complete measures on ${}^{<\omega}\gamma$, $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$, so that

(i) If $s \neq t$, then $\mu_s \neq \mu_t$.
(ii) $\dim(\mu_s) \leq |s|$.
(iii) If μ_s is an extension of some measure ν, then there exists some $k < |s|$ so that $\mu_{s \restriction k} = \nu$.

Define $W_{\bar{\mu}}$ by

$$W_{\bar{\mu}} = \{x \in {}^\omega\omega : (\exists f \in {}^\omega\omega)(f \text{ is an increasing sequence} \wedge (\mu_{x \restriction f(k)} : k \in \omega) \text{ is a countably complete tower})\}$$

A set $A \subseteq {}^\omega\omega$ is κ-weakly homogeneous if and only there is a κ-weak homogenity system $\bar{\mu}$ so that $A = W_{\bar{\mu}}$.

Definition 4.5. Let γ be an ordinal. A tree T on $\omega \times \gamma$ is κ-weakly homogeneous if and only there is some κ-weak homogeneity system $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ so that $p[T] = W_{\bar{\mu}}$ and for all $s \in {}^{<\omega}\omega$, there is some $k \leq |s|$ so that μ_s concentrates on $T^{s \restriction k}$. $A \subseteq {}^\omega\omega$ is κ-weakly homogeneously Suslin if and only if $A = p[T]$ for some tree T which is κ-weakly homogeneous.

Fact 4.6. *([13], Proposition 1.12.) If $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ is a κ-weak homogeneity system with support γ, then there is a tree T on $\omega \times \gamma$ so that $\bar{\mu}$ witnesses T is κ-weakly homogeneously Suslin. Hence a set is κ-weakly homogeneous if and only if it is κ-weakly homogeneously Suslin.*

Definition 4.7. Let μ be a countably complete measure on ${}^{<\omega}X$. Let M_μ be the Mostowski collapse of the ultrapower $\mathrm{Ult}(V, \mu)$. Let $j_\mu : V \to M_\mu$ be the composition of the ultrapower map and the Mostowski collapse map.

Suppose ν and μ are countably complete measures on ${}^{<\omega}X$. Suppose for some $m \leq n$, $\dim(\mu) = m$ and $\dim(\nu) = n$, and ν is an extension of μ. Define $\Lambda_{m,n} : {}^{{}^m X}V \to {}^{{}^n X}V$ by $\Lambda_{m,n}(f)(s) = f(s \restriction m)$ for each

$s \in {}^nX$. Define an elementary embedding $\mathrm{Ult}(V,\mu) \to \mathrm{Ult}(V,\nu)$ by $[f]_\mu \mapsto [\Lambda_{m,n}(f)]_\nu$. This induces an elementary embedding $j_{\mu,\nu} : M_\mu \to M_\nu$.

Definition 4.8. Let γ and θ be ordinals. Let $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ be a weak homogeneity system with support γ. The Martin-Solovay tree with respect to $\bar{\mu}$ below θ, denoted $\mathrm{MS}_\theta(\bar{\mu})$, is a tree on $\omega \times \theta$ defined by: for all $s \in {}^{<\omega}\omega$ and $h \in {}^{|s|}\theta$

$$(s,h) \in \mathrm{MS}_\theta(\bar{\mu}) \Leftrightarrow (\forall i < j < |s|)(\mu_{s\restriction j} \text{ is an extension of } \mu_{s\restriction i} \Rightarrow j_{\mu_{s\restriction i},\mu_{s\restriction j}}(h(i)) > h(j))$$

If $(\mu_n : n \in \omega)$ is a tower of measures, then the tower is countably complete if and only if the directed limit of the directed system $(M_{\mu_i} : j_{\mu_i,\mu_j} : i < j < \omega)$ is well-founded. If $(x, \Phi) \in [\mathrm{MS}_\theta(\bar{\mu})]$, then Φ witnesses in a continuous way that the directed limit model is ill-founded. This shows that $x \in p[\mathrm{MS}_\theta(\bar{\mu})]$ implies that $x \notin W_{\bar{\mu}}$. In fact, the converse is also true giving the following result:

Fact 4.9. *Let κ be a cardinal. Suppose $\bar{\mu}$ is a κ-weak homogeneity system with support γ. Then if $\theta > |\gamma|^+$, then $p[\mathrm{MS}_\theta(\bar{\mu})] = {}^\omega\omega \setminus W_{\bar{\mu}}$.*

Proof. See [13] Lemma 1.19, [7] Fact 1.3.12, or [5] Theorem 4.10. ☐

Let μ be a κ-complete ultrafilter on some set X. Let $\mathbb{P}$ be a forcing with $|\mathbb{P}| < \kappa$. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. In $V[G]$, define $\mu^* \subseteq \mathcal{P}(X)$ by $A \in \mu^*$ if and only there exists a $B \in \mu$ so that $B \subseteq A$. In $V[G]$, μ^* is a κ-complete ultrafilter on X. Suppose $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ is a κ-weak homogeneity system. Denote $\bar{\mu}^* = (\mu^*_s : s \in {}^{<\omega}\omega)$. $\bar{\mu}^*$ is a κ-weak homogeneity system. [13] Lemma 1.19 shows that $\mathrm{MS}_\theta(\bar{\mu})^V = \mathrm{MS}_\theta(\bar{\mu}^*)^{V[G]}$ if $\theta > |\gamma|^+$. Hence Fact 4.9 implies that $V[G] \models p[\mathrm{MS}_\theta(\bar{u})] = {}^\omega\omega \setminus W_{\bar{\mu}^*}$. (Also if one has a κ-homogeneous tree S with κ-homogeneity system $\bar{\mu}$ and $|\mathbb{P}| < \kappa$, then $\bar{\mu}^*$ is a κ-homogeneity system for S in $V[G]$. This shows A.)

Now suppose that T is κ-weakly homogeneous tree on $\omega \times \alpha$ witnessed by the κ-weak homogeneity system $\bar{\mu}$. This gives that $p[T] = W_{\bar{\mu}}$. One also has that $p[\mathrm{MS}_\theta(\bar{\mu})^V]$ continues to represent ${}^\omega\omega \setminus p[T]$ in $V[G]$. So in summary:

Fact 4.10. *($\mathsf{ZF} + \mathsf{DC}$) Let κ be a cardinal. Let T be a κ-weakly homogeneous tree on $\omega \times \gamma$, for some ordinal γ, with κ-weak homogeneity system $\bar{\mu}$. Let $\theta > |\gamma|^+$. Let $\mathbb{P}$ be a forcing with $|\mathbb{P}| < \kappa$ and $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. $V[G] \models \mathrm{MS}_\theta(\bar{\mu}^*) = \mathrm{MS}_\theta(\bar{\mu})^V$. $V[G] \models p[\mathrm{MS}_\theta(\bar{\mu})^V] = {}^\omega\omega \setminus p[T]$.*

Proof. See [13], Section 1 and especially Lemma 1.19. Also see [7], Section 1.3. ☐

So if T is κ-weakly homogeneous, an appropriate Martin-Solovay tree will continue to represent the complement of $p[T]$ in generic extensions by forcings of cardinality less than κ. The Martin-Solovay trees give the generically-correct tree representations for complements of κ-weakly homogeneously Suslin sets. However, the formulas D_Σ and D_Π involve more negations and quantifications over ${}^\omega\omega$. Multiple iterations of the Martin-Solovay construction will be needed. The following results are useful for continuing the Martin-Solovay construction of generically-correct tree representation for more complex sets. In addition, these results will also imply that these representations are also homogeneously Suslin.

Definition 4.11. If $B \subseteq {}^k({}^\omega\omega) \times {}^\omega\omega$, denote

$$\exists^{\mathbb{R}} B = \{x : (\exists y)((x,y) \in B)\} \text{ and } \forall^{\mathbb{R}} B = \{x : (\forall y)((x,y) \in B)\}$$

If $A \subseteq {}^k({}^\omega\omega)$, then denote

$$\neg A = {}^k({}^\omega\omega) \setminus A$$

Fact 4.12. *([13], Proposition 1.10) Let $A \subseteq {}^\omega\omega$. A is κ-weakly homogeneously Suslin if and only if there is a κ-homogeneously Suslin set $B \subseteq {}^\omega\omega \times {}^\omega\omega$ so that $A = \exists^{\mathbb{R}} B$.*

A Woodin cardinal is a technical large cardinal which has been very useful in descriptive set theory. (See [7], Section 1.5 for more information about Woodin cardinals.)

Fact 4.13. *([8]) Let δ be a Woodin cardinal. Let $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ be a δ^+-weak homogeneity system with support $\gamma \in \mathrm{ON}$. Then for sufficiently large θ, $\mathrm{MS}_\theta(\bar{\mu})$ is κ-homogeneous for all $\kappa < \delta$.*

Definition 4.14. If κ is a cardinal, then let Hom_κ be the collection of κ-homogeneously Suslin subsets of ${}^\omega\omega$. Let $\mathrm{Hom}_{<\kappa} = \bigcap_{\gamma<\kappa} \mathrm{Hom}_\gamma$.

Fact 4.15. *(Martin-Steel; [13] Section 2) Let λ be a limit of Woodin cardinals. Then* $\mathrm{Hom}_{<\lambda}$ *is closed under complements and* $\forall^{\mathbb{R}}$*.*

Fact 4.16. *(Martin; [9], Theorem 4.15) If κ is a measurable cardinal, then every $\mathbf{\Pi}^1_1$ set is κ-homogeneously Suslin.*

Fact 4.17. *(Martin-Steel) Let λ be a limit of Woodin cardinals, then all projective sets are in* $\mathrm{Hom}_{<\lambda}$*.*

Proof. Every Woodin cardinal has a stationary set of measurable cardinals below it. Hence every $\mathbf{\Pi}^1_1$ set is κ-homogeneously Suslin for all $\kappa < \lambda$. That is, all $\mathbf{\Pi}^1_1$ sets are in $\mathrm{Hom}_{<\lambda}$. Then by the closure properties given by Fact 4.15, all projective sets are in $\mathrm{Hom}_{<\lambda}$. □

Fact 4.18. *(Woodin) Suppose λ is a limit of Woodin cardinals and there is a measurable cardinal greater than λ. Then every subset of ${}^{\omega}\omega$ in $L(\mathbb{R})$ is in* $\mathrm{Hom}_{<\lambda}$*.*

Homogeneously Suslin sets were defined to be those sets that can be presented as projections of some trees satisfying certain properties. In the ground model, there could be many homogeneous trees representing the same homogeneously Suslin set A. For instance, suppose $\kappa_1 < \kappa_2$. In the ground model, suppose $A = p[T_1]$ where T_1 is a κ_1-homogeneous tree and $A = p[T_2]$ where T_2 is a κ_2-homogeneous tree. Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are two different forcings. Which tree should represent A in each forcing extension? What are the relations between $p[T_1]$ and $p[T_2]$ in various forcing extensions? Absolutely complemented trees and universal Baireness provide a way to interpret homogeneously Suslin sets in a way which is independent of the homogeneous tree representation in some sense:

Definition 4.19. (See [3]) Let κ be an ordinal. Let T be a tree on $\omega \times X$ and let U be a tree on $\omega \times Y$, for some sets X and Y. T and U are κ-absolute complements if and only if for all forcings $\mathbb{P} \in V_\kappa$ and all $G \subseteq \mathbb{P}$ which are $\mathbb{P}$-generic over V, $V[G] \models p[T] = {}^{\omega}\omega \setminus p[U]$.

A tree T on $\omega \times X$ is κ-absolutely complemented if and only if there exists some tree U on $\omega \times Y$ (for some set Y) so that T and U are κ-absolute complements.

A set $A \subseteq {}^{\omega}\omega$ is κ-universally Baire if and only if $A = p[T]$ for some tree T which is κ-absolutely complemented.

Fact 4.20. *Let T_1 and T_2 be trees on $\omega\times\gamma_1$ and $\omega\times\gamma_2$ which are κ-absolutely complemented and $p[T_1] = p[T_2]$. If $\mathbb{P} \in V_\kappa$ and $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V, then $V[G] \models p[T_1] = p[T_2]$.*

So if A is a κ-universally Baire set and if T_1 and T_2 are two κ-absolutely complemented trees so that $V \models A = p[T_1] = p[T_2]$, then either tree can be used to represent A in extensions by forcings in V_κ. As a matter of convention, if A is κ-universally Baire and $\mathbb{P} \in V_\kappa$, the set A will always refer to $p[T]$ for some and any κ-absolutely complemented tree $T \in V$ so that $V \models p[T] = A$.

Fact 4.21. *([13] Corollary 1.21) Let κ be a cardinal. κ-weakly homogenously Suslin sets are κ-universally Baire.*

In particular, κ-homogeneously Suslin sets can be interpreted unambiguously in $\mathbb{P}$-extensions whenever $\mathbb{P} \in V_\kappa$.

Let λ be a limit of Woodin cardinals. Let $\dot{A}$ be a new unary relation symbol. Let $A \subseteq ({}^{\omega}\omega)^n$ be such that $A \in \mathrm{Hom}_{<\lambda}$. Let $(H_{\aleph_1}, \in, A)$ be the $\{\dot{\in}, \dot{A}\}$-structure with domain $H_{\aleph_1}$ (the hereditarily countable sets) and with $\dot{A}$ interpreted as A. Now let $\mathbb{P} \in V_\lambda$ be some forcing and $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over V. $\mathbb{P} \in V_\kappa$ for some $\kappa < \lambda$. The structure $(H_{\aleph_1}^{V[G]}, \in, A^{V[G]})$ is understood in the following way: It is a structure with domain $H_{\aleph_1}^{V[G]}$ (the hereditarily countable subsets of $V[G]$) and $A^{V[G]}$ is $p[T]^{V[G]}$ for any γ-homogeneous tree T so that $V \models A = p[T]$ and $\gamma \geq \kappa$. By the above discussion, this is independent of which tree T is chosen. Actually, in the proof of the fact below, depending on the quantifier complexity of a particular formula φ involving $\dot{A}$, A will be considered as $p[T]$ for a sufficiently homogeneous tree T so that after the appropriate number of applications of the Martin-Solovay tree construction, the resulting tree representation of φ will be at least κ-homogeneous.

Using ideas very similar to the proof of Fact 4.15, one has the following absoluteness result:

Fact 4.22. *(Woodin; [13], Theorem 2.6) Let λ be a limit of Woodin cardinals. Let $A \in Hom_{<\lambda}$. Let $\mathbb{P} \in V_\lambda$ and $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Then $(H^V_{\aleph_1}, \in, A)$ and $(H_{\aleph_1}^{V[G]}, \in, A^{V[G]})$ are elementarily equivalent.*

In this setting, V and $V[G]$ satisfy the same formulas involving $\dot{A}$ and quantifications over the reals with the above intended interpretation. In particular, V and $V[G]$ satisfy the same projective formulas.

Now, the above discussion will be applied to indicate when A, B_Σ, C_Σ, B_Π, and C_Π hold. First, consider the setting of Theorem 4.1. Let λ be a limit of Woodin cardinals. Let $R \in \mathrm{Hom}_{<\lambda}$ with all sections $\mathbf{\Sigma}^1_1$ and fix a σ-ideal I on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper.

Let S be a sufficiently homogeneous tree representation for R. The tree S remains homogeneous in the $\mathbb{P}_I$-extension by the remark mentioned before Fact 4.10. This shows that A holds for S and I. R having all $\mathbf{\Sigma}^1_1$ sections can be expressed as a formula using some real quantifiers over the relation $R \in \mathrm{Hom}_{<\lambda}$. Fact 4.22 implies that these statements are absolute to the $\mathbb{P}_I$-extension. This shows that B_Σ holds for S and I. The formula D_Σ and D_Π both involve complements and real quantification over the homogeneously Suslin set R. By Fact 4.15, $D_\Sigma, D_\Pi \in \mathrm{Hom}_{<\lambda}$. Starting with an appropriate weakly homogeneous tree representation S for R, the construction used in the proof of Fact 4.15 produces a tree U representing D_Σ that is generically correct for $\mathbb{P}_I$, in the sense that $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x,T) : D_\Sigma(x,T)\}$. So C_Σ holds for S and I.

This discussion verifies that using a suitably homogeneous tree S, statements A, B_Σ, and C_Σ holds for S and I. Thus Theorem 3.7 yields Theorem 4.1 in the case that R has all $\mathbf{\Sigma}^1_1$ section.

By Fact 4.15, if $R \in \mathrm{Hom}_{<\lambda}$, then ${}^\omega\omega \setminus R \in \mathrm{Hom}_{<\lambda}$. Given a κ-weakly homogeneous tree representation S for R, the associated Martin-Solovay tree will be a sufficiently homogeneous tree representation for ${}^\omega\omega \setminus R$ by Fact 4.13. Hence in this setting using the notation from Definition 3.1, $R_S = R^T$, where T is the appropriate Martin-Solovay tree using the homogeneity system on S. To prove Theorem 4.1 in the case that R has all $\mathbf{\Pi}^1_1$ classes, the argument should be modified to establish A, B_Π, C_Π for T and I.

By Fact 4.17, if λ is a limit of Woodin cardinals, then all projective sets belong to $\mathrm{Hom}_{<\lambda}$. Moreover, if there is a measurable cardinal above λ, then Fact 4.18 implies every subset of ${}^\omega\omega$ in $L(\mathbb{R})$ belong to $\mathrm{Hom}_{<\lambda}$. The discussion given above to establish Theorem 4.1 applies to give Theorem 4.2 and Theorem 4.3.

With the appropriate assumptions, even more sets of reals are homogeneously Suslin and these canonicalization results would hold for relations in those classes. For example, Chang's model $L({}^\omega\mathrm{ON}) = \bigcup_{\alpha \in \mathrm{ON}} L({}^\omega\alpha)$ is the smallest inner model of ZF containing all the countable sequences of ordinals of V. Woodin has shown that with a proper class of Woodin cardinals every set of reals in $L({}^\omega\mathrm{ON})$ is ∞-homogeneously Suslin. Hence under this assumption, the above result would hold for binary relations in $L({}^\omega\mathrm{ON})$ with all $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, or $\mathbf{\Delta}^1_1$ sections.

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