Topology Homework 1

This homework is due by the end of Wednesday, January 30, 2019, that is before 12:00 AM Thursday, January 31, 2019. Submit the homework at GAB 418A. Put the homework underneath the door if no one is in the office.

Exercise 1: (10 pt) Consider the integer grid on the plane (an infinite checkerboard). There are four possible directions: up, down, left, and right.

There is a rabbit that starts on some square, moves forever in only one direction, and moves one square in that direction each day. For example, the rabbit always moves up one square each day. A hunter is trying to catch the rabbit. Each day the hunter can guess one square and check if the rabbit is there.

Show the hunter can catch the rabbit.

Exercise 2: (a) (5 pt) Show that \( N \) injects into \( 2^{\omega} \), where \( 10 = \{0,1,...,9\} \).

(b) (10 pt) Produce an explicit bijection between the closed interval \([0,1]\) and the open interval \((0,1)\).

Describe the function. You may not use any abstract cardinality result like Cantor-Schröder-Bernstein.

Hint: A full train with countably infinite many seats pulls up to a station. One new passager gets on the train and no one gets off. How is this possible?

Exercise 3: (15 pt) Suppose \( A, B, \) and \( C \) are sets so that \( A \subseteq B \subseteq C \) and \( A \approx C \). Show that \( A \approx B \approx C \).

(You can not use the Cantor-Schröder-Bernstein since this exercise was used in class to prove the Cantor-Schröder-Bernstein.) Recall that if \( X \) and \( Y \) are two sets, then \( X^Y \) is the set of all functions from \( X \) into \( Y \). So \( 2^{\omega} \) is the set of all functions from \( \omega \) into \( 2 = \{0,1\} \), i.e. the set of infinite binary sequences. (You may use the Cantor-Schröder-Bernstein Theorem.)

Hint: Consider reals numbers by its decimal representation. Show \( 2^{\omega} \) inject into \( R \) which injects into \( 10 \) which injects into \( 2^{\omega} \).

(a) (5 pt) Show that \( F \in \mathcal{E} \).

(b) (10 pt) Let \( \Phi : C \to A \) be a bijection. Let \( D = B \setminus \Phi[C] \). Let \( \mathcal{E} = \{ X \subseteq C : D \cup \Phi[X] \subseteq X \} \). Let \( \mathcal{F} = \bigcap \mathcal{E} \).

Show that \( \Phi[C] \setminus \Phi[F] \) are disjoint. (f) Finish by producing a bijection \( \Psi : C \to B \).

Exercise 4: (10 pt) Let \( \mathcal{U} \) be a nonprincipal ultrafilter on \( \omega \), then there is a nonprincipal countably complete ultrafilter on \( \omega \) that is needed for this exercise.

(a) (5 pt) Consider \( \mathcal{U} \) be an ultrafilter on \( N \). Let \( \langle x_n : n \in N \rangle \) be a sequence of real numbers and \( x \in \mathbb{R} \). Define \( \lim_{\mathcal{U}} x_n = x \) if and only if for all \( \epsilon > 0 \), \( \{ n \in N : |x_n - x| < \epsilon \} \in \mathcal{U} \). In this case, one says that the sequence \( \langle x_n : n \in N \rangle \) converges according to \( \mathcal{U} \) and \( x \) is its limit.

(c) (10 pt) Show that this sequence converges according to \( \mathcal{U} \) to a unique limit.

Exercise 5: (10 pt) Show that if there is a countably complete nonprincipal ultrafilter on \( \omega_1 \), then there is no uncountable wellordered set of real numbers.

(Recall that \( \omega_1 \) is the least uncountable ordinal. The only fact about \( \omega_1 \) that is needed for this exercise is that \( \omega_1 \) injects into any uncountable wellordering.) This exercise shows that such ultrafilters on \( \omega_1 \) is incompatible with the axiom of choice.

Hint: Since \( |\mathbb{R}| = |2^{\omega}| \), work with \( 2^{\omega} \) instead of \( \mathbb{R} \). So \( \mathcal{U} \) is an uncountable wellorderable subset of \( 2^{\omega} \) that is needed for this exercise. Suppose \( \mathcal{U} \) is a nonprincipal countably complete ultrafilter on \( \omega_1 \). If there was an uncountable wellorderable subset of \( 2^{\omega} \), then there is an \( \omega_1 \)-length sequence of distinct elements of \( 2^{\omega} \). Let
\( \langle f_\alpha : \alpha < \omega_1 \rangle \) be this sequence of elements from \( \mathbb{N}^2 \) with the property that if \( \alpha \neq \beta \), then \( f_\alpha \neq f_\beta \). Let \( A^0_n = \{ \alpha \in \omega_1 : f_\alpha(n) = 0 \} \) and \( A^1_n = \{ \alpha \in \omega_1 : f_\alpha(n) = 1 \} \). Observe that \( A^0_n \cup A^1_n = \omega_1 \) and \( A^0_n \cap A^1_n = \emptyset \). Since \( \mathcal{U} \) is an ultrafilter, exactly one of \( A^0_n \) or \( A^1_n \) belongs to \( \mathcal{U} \). Do this for all \( n \in \mathbb{N} \). Argue that \( \langle f_\alpha : \alpha \in \omega_1 \rangle \) is not a sequence of distinct elements to obtain the contradiction.

As a reminder, the following are some definitions associated to filters and ultrafilters.

Let \( X \) be a set. A set \( F \subseteq \mathcal{P}(X) \) is a filter on \( X \) if and only if

(a) \( \emptyset \notin F \) and \( X \in F \).
(b) (Closure under finite intersection): If \( A, B \in F \), then \( A \cap B \in F \).
(c) (Upward closure) Suppose \( A, B \) are two subsets of \( X \) with \( A \subseteq B \). If \( A \in F \), then \( B \in F \).

A filter on \( X \) is principal if and only if there is some \( x \in X \) so that \( F = \{ A \in \mathcal{P}(X) : x \in A \} \). Otherwise, \( F \) is said to be nonprincipal.

A filter \( F \) on \( X \) is countably complete if and only if it is closed under countable intersection: Suppose \( \langle X_n : n \in \mathbb{N} \rangle \) is such that \( X_n \in \mathcal{U} \) for each \( n \in \mathbb{N} \). Then \( \bigcap_{n \in \mathbb{N}} X_n \in \mathcal{U} \).

A filter \( F \) is an ultrafilter if and only if for all \( A \subseteq X \), either \( A \in \mathcal{U} \) or \( X \setminus A \in \mathcal{U} \).